Model-based piecewise affine variable-gain controller synthesis

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Abstract— In this paper, we introduce piecewise affine variable-gain controllers as a means to improve performance compared to linear controllers. Variable-gain controllers can improve upon the tradeoff between low-frequency tracking and sensitivity to high-frequency disturbances. However, performance-based tuning of the variable-gain controllers, is far from trivial. The piecewise affine control structure introduced in this paper allows to synthesize the shape of the variable-gain controller, by means of model-based optimization of a certain performance objective. Subsequently, this allows the controller design to be tuned for the disturbance situation at hand while optimizing performance. The proposed nonlinear performancebased controller synthesis strategy is applied to a model of a wafer stage of a wafer scanner.

I. INTRODUCTION

The control of industrial motion systems is mostly done using linear controllers of the proportional-integral-derivative (PID) type. However, it is well-known that many linear control loops suffer from certain inherent performance tradeoffs such as the waterbed effect [14], [3]: an increase of lowfrequency (below the bandwidth) disturbance suppression automatically yields an increase of noise amplification at high frequencies (above the bandwidth). Given this tradeoff, linear motion controllers are designed to balance between low-frequency tracking and sensitivity to high-frequency disturbances. In the controller design this is often achieved by frequency-domain loop shaping.

To balance this tradeoff (induced by the waterbed-effect) in a more desirable manner, it has been shown that variablegain control (also called N-PID control) can be effective [1], [18], [6], [16], [10], [9]. In these references, it has been shown that the variable-gain controllers have the capability of outperforming linear controllers. Although the underlying linear controller part can be based on well-known performance-based loop-shaping arguments, the performance-based tuning of the variable-gain control part is far from trivial. Typically, the design of this variable-gain control part is based on heuristic rules and depends on the specific application and disturbances at hand. Moreover, the type of nonlinearity is typically chosen a priori, e.g. a deadzone characteristic [6], [8], [16] or a saturation characteristic [7]. To facilitate a more constructive design of the variablegain part, a data-based tuning method has been used in [5] to tune the parameters of a fixed dead-zone like characteristic.

In this paper, we would like to develop a true synthesis approach for a variable-gain controller, instead of a mere tuning procedure. We propose to use a general piecewise affine variable-gain characteristic, thus without a priori fixing

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the type of nonlinearity. This means that the controller design and tuning tailors the design of the shape of the variable-gain element to the disturbance situation at hand. By increasing the number of segments of the piecewise affine structure, arbitrarily shaped characteristics can be constructed, paving the way for a general performance-based nonlinear controller design.

Two approaches can be followed in the synthesis of the piecewise affine variable-gain controllers, a data-based approach or a model-based approach. A data-based machinein-the-loop approach is presented in companion paper [4]. Here, we will focus on an efficient model-based approach to synthesize the controllers, which is beneficial in a designphase where no machine is available yet, in situations where performing many experiments on a machine becomes prohibitive and in performing parameter studies of the closedloop system.

Summarizing, the contribution of this paper is the development of an efficient model-based synthesis method for performance-optimal variable-gain controllers applied to linear motion systems, which is tuned for the disturbance situation at hand. This is achieved without making an a priori heuristic choice for the type of nonlinearity which will be illustrated by simulations on a model of a wafer stage of a wafer scanner.

The remainder of the paper is organized as follows. In Section II, we introduce the piecewise affine variable-gain control strategy with stability conditions. The controller synthesis method will be discussed in Section III and will be applied to a model of a short-stroke of a wafer stage in Section IV. Conclusions and recommendations will be presented in Section V.

II. PIECEWISE LINEAR VARIABLE-GAIN CONTROL

Consider the variable-gain control structure depicted in Fig. 1. The linear closed-loop part consists of the linear plant $P(s), s \in \mathbb{C}$, linear controller C(s), and linear shaping-filter F(s). The variable-gain part of the controller is represented by the nonlinearity $\varphi(e)$, which is a function of the tracking error e in time-domain.

Many linear closed-loop systems suffer from the waterbedeffect [14], [3]: an increase in low-frequency (below the



Fig. 1. Closed-loop variable-gain control scheme.

This research is financially supported by the Dutch Technology Foundation STW.



Fig. 2. Piecewise linear variable-gain element $\varphi(e)$.

bandwidth) performance results in a deterioration of performance for high frequencies (above the bandwidth). It is known that variable-gain control can balance this waterbed effect in a more desirable manner than linear controllers; moreover, it can outperform linear controllers by a suitable design of the nonlinear characteristics [1], [6].

Usually, the form of the nonlinearity φ is chosen heuristically based on a certain specific disturbance situation. For example, in [8], a typical dead-zone characteristic is chosen for the nonlinearity $\varphi(e)$. For the motion system considered [8], high-frequency small-amplitude disturbances are not amplified since they stay within the dead-zone length. Contrarily, low-frequency large-amplitude disturbances are additionally suppressed by the extra gain of the dead-zone nonlinearity.

It is known that other disturbance situations may require other shapes for φ , see e.g. [7]. We would like to avoid making such a heuristic a priori choice for the shape of the nonlinearity φ and synthesize a controller that is tuned for the disturbance situation at hand. To facilitate such a general controller synthesis approach, we do not specify the type of nonlinearity φ but construct it on the basis of piecewise affine segments as in Fig. 2. The point-symmetric continuous nonlinearity $\varphi(e)$ consists of N segments with slopes α_i , which are defined as

$$\alpha_i = \frac{\partial \varphi}{\partial e} \quad \forall \delta_{i-1} < |e| < \delta_i, \tag{1}$$

with $i \in \{1, 2, ..., N\}$, $\delta_0 = 0$ and $\delta_N = \infty$. The maximum slope α_{max} is defined as

$$\alpha_{max} = \max_{i} \alpha_i, \quad i \in \{1, 2, ..., N\}.$$
 (2)

Note that with this type of piecewise affine construction, by choosing N large enough, it is possible to generate (approximate) arbitrary (point-symmetric) nonlinearities.

With the variable-gain element $\varphi(e)$, stability of the closed-loop system can be assessed through circle-criterion arguments. The closed-loop dynamics in Fig. 1 can be written as a Lur'e-type system of the following form:

$$\dot{x} = Ax + Bu + B_w w(t) \tag{3}$$

$$e = Cx + D_w w(t) \tag{4}$$

$$\mu = -\varphi(e). \tag{5}$$

with state $x \in \mathbb{R}^n$ and external inputs $w(t) \in \mathbb{R}^m$, which typically consist of the reference r and force disturbance d. The linear dynamics from input $u \in \mathbb{R}$ to output $e \in \mathbb{R}$ is denoted by $G_{eu}(s)$ and can be expressed as

$$G_{eu}(s) = C(sI - A)^{-1}B = \frac{P(s)C(s)F(s)}{1 + P(s)C(s)}.$$
 (6)



Fig. 3. Process sensitivity showing the waterbed-effect. The disturbance frequencies used in Section IV-B are also indicated by dashed lines.

The following theorem provides conditions under which system (3)-(5), excited by a bounded T-periodic input w(t), has a uniquely defined T-periodic globally exponentially stable steady-state solution.

Theorem II.1 [16], [17] Consider system (3)-(5). Suppose A1 The matrix A is Hurwitz;

A2 The nonlinearity $\varphi(e)$ satisfies the incremental sector condition:

$$0 \le \frac{\varphi(e_2) - \varphi(e_1)}{e_2 - e_1} \le \alpha_{max},\tag{7}$$

for all $e_1, e_2 \in \mathbb{R}$, $e_1 \neq e_2$;

A3 The transfer function $G_{eu}(s)$ given by (6) satisfies

$$\sup_{\omega \in \mathbb{R}} |G_{eu}(i\omega)| > -\frac{1}{\alpha_{max}}.$$
(8)

Then for any bounded T-periodic piecewise continuous input w(t), system (3)-(5) has a unique T-periodic solution $\bar{x}_w(t)$, which is globally exponentially stable and bounded for all $t \in \mathbb{R}$.

The proof follows from circle-criterion-type arguments [16], [17]. We will denote $\bar{x}_w(t)$ as the steady-state solution. Systems with such a uniquely defined globally exponentially stable steady-state solution (for arbitrary bounded inputs w(t)) are called exponentially convergent, see e.g. [2], [13].

III. CONTROLLER SYNTHESIS METHOD

In this section the performance-optimal variable-gain controller synthesis method will be discussed. The performance quantification will be discussed in the Section III-A followed by the gradient-based optimization strategy in Section III-B.

A. Performance quantification

As mentioned in Section II, the waterbed-effect describes the effect of many linear closed-loop systems, where an increase of performance for low frequencies results in a decrease of performance for high frequencies. This is for example visible in the process sensitivity (numerical specifics will be given in Section IV), denoting the transfer from force disturbance d to closed-loop error e, see Fig. 3.

In order to quantify the effect of low-frequency (i.e. below the bandwidth) performance improvement and high-frequency (i.e. above the bandwidth) performance degradation when increasing the controller gain, we propose to use the following \mathcal{H}_2 steady-state performance indicator:

$$J = c_{lf}J_{lf} + c_{hf}J_{hf}, (9)$$



Fig. 4. Filtering of the low-frequency and high-frequency part of the error.

where

$$J_{lf} = \int_0^T \bar{e}_{lf}^2(t) dt, \qquad J_{hf} = \int_0^T \bar{e}_{hf}^2(t) dt, \qquad (10)$$

where T is the period of the disturbance w(t), and $\bar{e}(t) = \bar{e}_{lp}(t) + \bar{e}_{hp}(t) = C\bar{x}_w(t) + D_ww(t)$. The low-frequency part of the error steady-state $\bar{e}_{lf}(t)$ is obtained by low-pass filtering¹ the error signal $\bar{e}(t)$ with a low-pass filter with a cut-off frequency ω_b around the bandwidth of the system:

$$F_{lp}(s) = \frac{\omega_b^2}{s^2 + 2\beta\omega_b + \omega_b^2}.$$
(11)

The high-frequency part of the tracking error \bar{e}_{hf} is simply obtained by calculating $\bar{e}_{hp}(t) = \bar{e}(t) - \bar{e}_{lp}(t)$. A performance criterion as in (9) allows to weigh the low-frequency and high-frequency parts of the error signal $\bar{e}(t)$ separately. An illustration of the filtering operation is shown in Fig. 4.

The coefficients $c_{lf} \ge 0$ and $c_{hf} \ge 0$ in (9) can be used to balance the importance of the low-frequency and highfrequency part of the error depending on the application under study.

B. Optimization algorithm

The performance of the piecewise affine variable-gain controllers can now be uniquely quantified using the performance measure J in (9), (10). Next, we aim to minimize J by tuning the parameters δ_i and α_i , thereby constructively shaping the nonlinearity $\varphi(e)$ for the disturbance w(t) at hand. Here, for a disturbance w(t), we pursue a modelbased optimization in order to find the performance-optimal piecewise-linear variable-gain element, see Fig. 2.

We will use a second-order gradient-based Quasi-Newton algorithm, see Fig. 5, to minimize the performance objective J in (9). Given the disturbance w(t), the T-periodic steadystate error $\bar{e}(t)$ is calculated using the numerically efficient Mixed-Time-Frequency (MTF) algorithm [12]. Suppose the optimization parameters (the α_i 's and δ_i 's) at iteration k are collected in the vector $\theta_k = [\alpha_1, \dots, \alpha_N, \delta_1, \dots, \delta_{N-1}]$. The following update is used in the Quasi-Newton algorithm [11]:

$$\theta_{k+1} = \theta_k - H_k^{-1} \left(\frac{\partial J}{\partial \theta}(\theta_k)\right)^T, \qquad (12)$$

¹Since we do this off-line, the filtering can be applied in both forward and recursive direction, to avoid phase distortion [15]



Fig. 5. Schematic of the gradient-based Quasi-Newton optimization algorithm.

where θ_0 is the initial parameter setting. The gradients $\frac{\partial J}{\partial \theta}(\theta_k)$ are computed using finite-difference approximations. The Hessian estimate H_k is obtained by using subsequent gradient information in a Broyden–Fletcher–Goldfarb–Shanno (BFGS) update:

$$H_{k+1} = H_k + \frac{q^T q}{q^T s} - \frac{H_k^T s^T s H_k}{s^T H_k s},$$
 (13)

where $q = \partial J/\partial \theta(\theta_{k+1}) - \partial J/\partial \theta(\theta_k)$, $s = \theta_{k+1} - \theta_k$, and the initial Hessian estimate H_0 is the identity matrix, see [11] for more details.

If $J(\theta_{k+1})$ is smaller than $J(\theta_k)$ and θ_{k+1} lies within a predefined region, see Fig. 5, the point θ_{k+1} is accepted as the new point. Otherwise, a line-search is performed in the direction $H_k^{-1}(\partial J/\partial \theta(\theta_k))^T$ until the new point does satisfy the above conditions. Note that Theorem II.1 gives a lowerand upper-bound for the α_i 's to be used as constraints in the optimization problem. Sensible constraints for the δ_i 's will be given in Section IV-C.

IV. NUMERICAL EXAMPLE

In this section, the variable-gain controller synthesis method will be applied to a model of a wafer stage of a wafer scanner which is used to produce integrated circuits.

A. Model specification

The plant dynamics are modeled by the following fourthorder transfer function

$$P(s) = \frac{m_1 s^2 + bs + k}{s^2 (m_1 m_2 s^2 + b(m_1 + m_2)s + k(m_1 + m_2))},$$
(14)

 $s \in \mathbb{C}$, where the following numerical values are used for the plant model [6]: $m_1 = 5 \text{ kg}$, $m_2 = 17.5 \text{ kg}$, $k = 7.5 \cdot 10^7 \text{ N/m}$, b = 90 Ns/m.

The nominal low-gain controller (corresponding to $\varphi(e) = 0$) $C(s) = C_{PID}(s)C_{lp}(s)C_n(s)$ is designed by loopshaping arguments for a bandwidth of 150 Hz and consists of a PID controller $C_{PID}(s)$, a second-order low-pass filter $C_{lp}(s)$ and a notch filter $C_n(s)$, the latter is added to deal with the plant resonance. The filters are given by: $C_{PID}(s) = (k_p(s^2 + (\omega_i + \omega_d)s + \omega_i\omega_d))/(\omega_d s)$, where $k_p = 6.9 \cdot 10^6$ N/m, $\omega_d = 3.8 \cdot 10^2$ rad/s, and $\omega_i =$ $3.14 \cdot 10^2$ rad/s; $C_{lp}(s) = \omega_{lp}^2/(s^2 + 2\beta_{lp}\omega_{lp}s + \omega_{lp}^2)$, where $\omega_{lp} = 3.04 \cdot 10^3$ rad/s, and $\beta_{lp} = 0.08$; $C_n(s) =$ $(\omega_p/\omega_z)^2(s^2 + 2\beta_z\omega_z s + \omega_z^2)/(s^2 + 2\beta_p\omega_p s + \omega_p^2)$, where $\omega_p = 5.03 \cdot 10^3$ rad/s, $\beta_p = 0.88$, $\omega_z = 4.39 \cdot 10^3$ rad/s, and $\beta_z = 2.7 \cdot 10^{-3}$. The loop-shaping filter F(s) is given by $F(s) = (\omega_{p,F}/\omega_{z,F})^2(s^2 + 2\beta_z, \omega_z, Fs + \omega_z^2, F)/(s^2 +$



Fig. 6. Nyquist plot of $G_{eu}(s)$ in (6).

 $2\beta_{p,F}\omega_{p,F}s + \omega_{p,F}^2$, with $\omega_{p,F} = \omega_{z,F} = 2000$ rad/s, $\beta_{p,F} = 4.8$, and $\beta_{z,F} = 0.6$. Note that these filters define the transfer function $G_{eu}(s)$ in (6), the low-gain (i.e. $\varphi(e) = 0$) process sensitivity -P(s)/(1 + P(s)C(s)), and the highgain (i.e. $\varphi(e) = \alpha_{max}e$) process sensitivity $-P(s)/(1 + (1 + \alpha_{max}F(s))P(s)C(s))$ in Fig. 3.

With these transfer functions defined, we can verify the stability conditions of Theorem II.1. By design (we use a stabilizing controller C(s) and a stable shaping-filter F(s)), the matrix A, or equivalently the transfer function $G_{eu}(s)$ in (6), is Hurwitz, such that condition A1 is satisfied. From the Nyquist plot in Fig. 6 we can graphically verify that for $\alpha_{max} = 3$, the frequency-domain condition A3 is also satisfied. Note that Fig. 6 illustrates the reason for using the filter F(s). Finally, by constraining the optimization to values between 0 and 3 for all α_i 's, we guarantee that condition A2 is satisfied. Since all conditions of Theorem II.1 are satisfied, system (3)-(5) exhibits a unique bounded globally exponentially stable T-periodic steady-state solution. Note that the δ_i 's are completely stability-invariant and only influence performance. This makes the controller synthesis mainly performance relevant.

B. Disturbance specification

To illustrate the fact that we are able to synthesize a nonlinear controller that depends on the specific disturbance situation at hand, we consider two different periodic forcedisturbance situations (r = 0), both parameterized by:

$$d(t) = d_{lf}(t)W(t) + d_{hf}(t)W(t - T/2),$$
(15)

with $d_{lf}(t) = A_{lf} \sin(200t)$ a disturbance below the bandwidth of 150 Hz and $d_{hf}(t) = A_{hf} \sin(2600t)$ a disturbance above the bandwidth, see Fig. 3. Disturbance d_{lf} can be suppressed by additional controller gain, in contrast to d_{hf} , whose effect on the closed-loop error signals blows up due to additional controller gain. W(t) is a window specified by:

$$W(t) = \begin{cases} 0.5(1 - \cos(4\pi t/T)) & \text{if } t \in [0, T/2] + kT, \\ 0 & \text{otherwise,} \end{cases}$$
(16)

with k an integer. The period time T = 0.1 s results in a Tperiodic disturbance signal d(t) starting with a low-frequency part and smoothly making a transition to a high-frequency part. The two disturbance situations considered differ only in the amplitude A_{hf} of the high-frequency disturbance which is equal to 2 in case of disturbance situation 1, and equal to 12 in case of disturbance situation 2, see Fig. 7. The amplitude A_{lf} of the low-frequency disturbance equals 1 in both situations.



Fig. 7. One period of disturbance d(t) for disturbance situation 1 and disturbance situation 2.

C. Application of controller synthesis strategy

In this section, we will optimize the piecewise affine variable-gain controllers, using the gradient-based algorithm discussed in Section III-B, for a piecewise affine $\varphi(e)$ consisting of N = 2 elements, see Fig. 2.

For both disturbance situations, the nominal linear lowgain controller (i.e. with $\varphi(e) = 0$), is normalized to a performance of J = 1: with the linear low-gain controller, the steady-state error \bar{e} is computed. Subsequently, the filtering discussed in Section III-A is used to compute the low-frequency part $\bar{e}_{lf}(t)$ and high-frequency part $\bar{e}_{hf}(t)$. The low-frequency and high-frequency errors are given equal importance by selecting c_{lf} and c_{hf} in such a way that $c_{lf}J_{lf} = c_{hf}J_{hf} = 0.5$ such that J = 1 in (9) for the lowgain controller settings. The coefficients of the low-pass filter in (11) are chosen as $\beta = 0.7$ and $\omega_b = 1000$ rad/s, i.e. near the bandwidth. As mentioned in Section IV-A, the values for α_1 and α_2 will be constrained to the range $0 \le \alpha_i \le 3$, i = 1, 2. Moreover, δ_1 is constrained to $\delta_1 \ge 0$.

For the initial parameters of the optimizations of disturbance situations 1, all possible combinations between $\delta_1 \in \{10, 30, 50\}$ nm and $\alpha_i \in \{0.1, 1.5, 2.9\}$ are used, leading to 27 different initial starting points. For disturbance situation 2, all possible combinations between $\delta_1 \in \{5, 70, 150\}$ nm and $\alpha_i \in \{0.1, 1.5, 2.9\}$, i = 1, 2, are used.

The result of these optimizations² is shown in Fig. 8 for disturbance situation 1 (left) and disturbance situation 2 (right). In the upper part of the figure the steady-state error is shown for the linear low-gain controller ($\varphi(e) = 0$), the linear high-gain controller ($\varphi(e) = \alpha_{max}e$), and the optimal synthesized variable-gain controller. In the lower part of the figure the optimal piecewise affine elements $\varphi(e)$ are shown. *Disturbance situation 1*

From Fig. 7 we conclude that the optimal³ piecewise affine variable-gain controller synthesized is a dead-zone controller with $\alpha_1 = 0$, $\alpha_2 = 3$ and $\delta_1 = 18.5$ nm with corresponding performance J = 0.71. The reasoning behind this nonlinearity result can be understood as follows. When comparing to the low- and high-gain controller in Fig. 7, the variable-gain controller does not apply any additional gain if a high-frequency small-amplitude disturbance is present, thereby performing equally well as the low-gain controller.

²In the optimizations, a dummy variable $\tilde{\delta} = \delta \cdot 10^8$ is introduced that is optimized, such that $\tilde{\delta}$ is of the same order of magnitude as the α 's, which improves the numerical conditioning of the optimization problem.

³From brute-force simulations it is verified that for this disturbance situation this is indeed the global optimum.



Fig. 8. Error signals for low-gain, high-gain, and optimal variable-gain controllers (top) and optimal synthesized nonlinearities $\varphi(e)$ (bottom).



Fig. 9. Performance objective J for a grid of dead-zone shaped nonlinearities (left) for disturbance situation 1 and $\alpha_1 = 0$, and for a grid of saturation shaped nonlinearities (right) for disturbance situation 2 and $\alpha_2 = 0$. The arrows indicate the optimal piecewise affine variable-gain controller.

However, when a low-frequency large-amplitude disturbance is present, additional gain is applied such that the lowfrequency disturbance suppression is improved compared to the case of low-gain linear control.

A plot of the performance objective J for a grid of deadzone nonlinearities (i.e. all controllers with $\alpha_1 = 0$) is shown in Fig. 9 (left plot). Note that for $\alpha_2 = 0$ and for $\delta_1 >$ 60 nm (where $|e| < \delta_1$) the low-gain controller is active, which is normalized to a performance of J = 1. The highgain controller can be found at $\delta_1 = 0$ and $\alpha_2 = 3$ with performance J = 1.33. Note that values of J > 1 have been omitted from the plot for clarity of presentation. The optimal piecewise affine variable-gain controller, with J = 0.71, is also indicated in Fig. 9 with an arrow.

From the two-parameter space plot in Fig. 9 (although non-convex) one might be tempted to think that it is easy to find the global optimum in the three-parameter space without getting stuck in local minima. However, from Fig. 10, which shows the iteration history for a few optimizations corresponding to different initial starting points, we conclude differently. From this figure, it is clear that some optimizations do converge to the global optimum (black) but that others converge to local optima (gray). The optimal piecewise affine variable-gain controller is indicated by the dashed red lines, see also Fig. 9. In fact, 12 out of 27 optimizations converged to a different optimum than the global one. The local optimum resembles in all cases the same linear



Fig. 10. Iteration history for disturbance situation 1 for different initial starting points, illustrating that certain optimizations do (black) and certain optimizations do not (gray) find the global optimum (dashed red).

controller, with $\alpha_1 = 0.872$ and $\delta_1 > \max(|\bar{e}|)$ (such that α_2 and δ_1 are irrelevant, hence $\partial J/\partial \alpha_2 = \partial J/\partial \delta_1 = 0$), or a controller with $\alpha_2 = 0.872$ and $\delta_1 = 0$ (such that α_1 is irrelevant, hence $\partial J/\partial \alpha_1 = 0$), resulting in a non-optimal performance of J = 0.84, see Fig. 10.

Remark IV.1 By changing the constraints on δ_1 in our optimization routine, we can prevent the cases where either α_1 is irrelevant (for $\delta_1 = 0$) or α_2 is irrelevant (for $\delta_1 > \max(|\bar{e}|)$), since this hampers the finding of the global optimum. By selecting a small but strictly positive lowerbound for δ equal to 0.1 nm and an upper-bound for δ equal to 0.95 max($|\bar{e}|$) (for the error \bar{e} at the current iteration) all 27 initial conditions converge to the global optimum.

Disturbance situation 2

The optimal piecewise affine variable-gain controller in case of disturbance situation 2 consists of a saturation nonlinearity with $\alpha_1 = 3$, $\alpha_2 = 0$ and $\delta_1 = 17.8$ nm with corresponding performance J = 0.65, see the right plots in Fig. 8. This result can be understood, when comparing the optimal



Fig. 11. Iteration history for disturbance situation 2 for different initial starting points, illustrating that certain optimizations do (black) and certain optimizations do not (gray) find the global optimum (dashed red).

controller to the low-gain and high-gain controller limits. The additional gain within the saturation band achieves equal low-frequency disturbance suppression to the high-gain controller. However, by limiting the amount of additional gain for $|\bar{e}| > \delta_1$, the high-frequency disturbance amplification is kept to a minimum, being almost equal to the amount of the low-gain controller. A plot of the performance J is shown in the right part of Fig. 9, now for a grid of saturation nonlinearities (i.e. all controllers with $\alpha_2 = 0$). Note that for $\alpha_1 = 0$ and for $\delta_1 = 0$ nm the low-gain controller is active, which is normalized to a performance of J = 1. The highgain controller can be found at $\delta_1 > 200$ nm and $\alpha_2 = 3$ with performance J = 1.33. Again, values of J > 1 have been omitted from the plot for clarity of presentation. The optimal piecewise affine variable-gain controller is also indicated in Fig. 9.

Considering the convergence of the optimizations in case of disturbance situation 2, 21 out of 27 initial starting points converged to the global optimum. The local optimum resembles in all cases the same linear controller with $\alpha_1 = 0.870$ and $\delta_1 > \max(|\bar{e}|)$ (such that α_2 and δ_1 are irrelevant, hence $\partial J/\partial \alpha_2 = \partial J/\partial \delta_1 = 0$) or with $\alpha_2 = 0.870$ and $\delta_1 = 0$ (such that α_1 is irrelevant, hence $\partial J/\partial \alpha_1 = 0$), resulting in a non-optimal performance of J = 0.84, see Fig. 11. Again, by changing the constraints on δ_1 as in Remark IV.1, all of the 27 initial conditions converge to the global optimum.

Remark IV.2 When the number of segments of the piecewise affine nonlinearity is enlarged to 3 or 4, the same deadzone and saturation type controllers result from the controller synthesis. Therefore, in this application, performance is not improved by considering more complex nonlinearities.

Remark IV.3 The gradients used in the optimizations are obtained by finite-difference approximations using simulations, which is acceptable for the considered model-based implementation. An alternative method is based on using a sensitivity model for the dynamics with respect to the optimization parameters, which will be more suitable in an

experimental implementation, since this reduces the number of experiments and simplifies obtaining the gradients [5], [4].

V. CONCLUSIONS AND RECOMMENDATIONS

In this paper, we have proposed a piecewise affine nonlinearity in a variable-gain motion controller with the aim to improve the performance compared to linear controllers. By not fixing the shape of the nonlinearity a priori, we developed a strategy to synthesize a variable-gain controller which is tuned for the disturbance situation at hand. We have illustrated the controller synthesis approach on a model of a wafer stage. One disturbance situation led to an optimal dead-zone characteristic, and one disturbance situation led to an optimal saturation characteristic. The piecewise affine description allows for general nonlinearities to be synthesized, paving the way to general performance-based nonlinear controller design.

REFERENCES

- B.S.R. Armstrong, J.A. Guitierrez, B.A. Wade, and R. Joseph. Stability of phase-based gain modulation with designer-chosen switch functions. *The International Journal of Robotics Research*, 25:781– 796, 2006.
- [2] B.P. Demidovich. Dissipativity of a nonlinear system of differential equations. Ser. Mat. Mekh., Part I-6:19–27, 1961.
- [3] J. Freudenberg, R. Middleton, and A. Stefanopoulou. A survey of inherent design limitations. In *Proc. of the American Control Conference*, pages 2987–3001, Chicago, Illinois, 2000.
- [4] M.F. Heertjes, B.G.B. Hunnekens, N. van de Wouw, and H. Nijmeijer. An iterative learning approach towards data-driven variable-gain controller synthesis. In *Proc. American Control Conference, Washington*, USA, accepted, 2013.
- [5] M.F. Heertjes and H. Nijmeijer. Self-tuning of a switching controller for scanning motion systems. *Mechatronics*, 22:310–319, 2012.
- [6] M.F. Heertjes, X.G.P. Schuurbiers, and H. Nijmeijer. Performanceimproved design of N-PID controlled motion systems with application to wafer stages. In *IEEE Transactions on Industrial Electronics*, volume 56 (5), pages 1347–1355, 2009.
- [7] M.F. Heertjes and N. van de Wouw. Variable control design and its applications to wafer scanners. In *Proc. of the 45th IEEE Conference* on Decision and Control (CDC2006), pages 3765–3770, San Diego, USA, 2006.
- [8] M.F. Heertjes, B.P.T. van Goch, and H. Nijmeijer. Optimal switching control of motion stages. In *In Proc. 5th IFAC Symposium on Mechatronic Systems, Cambridge, Massachusetts, USA*,, pages 111– 116, 2010.
- [9] J.P. Hespanha and A.Stephen Morse. Switching between stabilizing controllers. *Automatica*, 38(11):1905–1917, 2002.
- [10] B.G.B. Hunnekens, N. van de Wouw, and H. Nijmeijer. Variable gain motion control for transient performance improvement. In *Proc. American Control Conference, Montréal, Canada*, pages 2467–2472, 2012.
- [11] P.Y. Papalambros and D.J. Wilde. Principles of Optimal Design: modeling and optimization. Cambridge University Press, 2000.
- [12] A. Pavlov and N. Van De Wouw. Fast computation of frequency response functions for a class of nonlinear systems. In 47th IEEE Conference on Decision and Control, CDC 2008, pages 1180–1186, Cancun, Mexico, 2008.
- [13] A. Pavlov, N. van de Wouw, and H. Nijmeijer. Uniform output regulation of nonlinear systems: a convergent dynamics approach. Boston: Birkhäuser, 2005.
- [14] M.M. Seron, J.H. Braslavsky, and G.C. Goodwin. Fundamental Limitations in Filtering and Control. Berlin: Springer, 1997.
- [15] Masayoshi Tomizuka. Zero phase error tracking algorithm for digital control. J Dyn Syst Meas Control Trans ASME, 190(1):65–68, 1987.
- [16] N. van de Wouw, H.A. Pastink, M.F. Heertjes, A.V. Pavlov, and H. Nijmeijer. Performance of convergence-based variable-gain control of optical storage drives. *Automatica*, 44:15–27, 2008.
- [17] V.A. Yakubovich. The matrix inequality method in the theory of the stability of nonlinear control systems - I. The absolute stability of forced vibrations. *Automation and Remote Control*, 7:905–917, 1964.
- [18] J. Zheng, G. Guo, and Y. Wang. Nonlinear PID control of linear plants for improved disturbance rejection. *16th IFAC World Congress (IFAC 2005)*, Prague, 2005.