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Brief paper

Global nonlinear output regulation: Convergence-based controller design ☆

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Abstract

In this paper we present output-feedback controllers solving the global output regulation problem for a class of nonlinear systems. The proposed controllers are based on the notion of convergent systems. The presented solution extends well-established results on the linear output regulation problem and the local nonlinear output regulation problem to the global case. For Lur'e systems, which are not necessarily in the output-feedback form, the proposed controllers can be found by solving the regulator equations and certain linear matrix inequalities. For systems in the output-feedback form with uncertain parameters and uncertain nonlinearities we provide a robust regulator that does not rely on the minimum phaseness assumption on the system, which is crucial in the previous regulator designs for output-feedback systems. The results are illustrated by examples.

Keywords: Output regulation; Nonlinear control systems; Convergent systems; Non-minimum phase systems; Lur'e systems

1. Introduction

In this paper we address the output regulation problem (ORP) that includes the problems of tracking reference signals and rejecting disturbances generated by an exosystem. For nonlinear systems this problem has received significant attention in the control community, see e.g. the monographs (Byrnes, Delli Priscoli, & Isidori, 1997; Huang, 2004; Isidori, Marconi, & Serrani, 2003; Pavlov, van de Wouw, & Nijmeijer, 2005b) and references therein. Most of existing results correspond to the local and semiglobal variants of the ORP. Only a few global results exist and these are mostly limited to lower-triangular systems (Huang & Chen, 2004; Marconi & Serrani, 2002) and minimum phase systems in the output-feedback form, see e.g. Serrani and Isidori (2000), Ding (2001) and Lin and Dong (2003). A method of converting the global ORP into a global robust stabilization problem has been proposed in Huang and Chen (2005).

In this paper, we give a solution to the global ORP for a class of nonlinear systems. This solution is based on the notion of convergent systems, see e.g. Pavlov, Pogromsky, van de Wouw, and Nijmeijer (2004), Pavlov, van de Wouw, and Nijmeijer (2005a), Demidovich (1967). Roughly speaking, a system is called convergent if for any bounded input it has a bounded globally asymptotically stable steady-state solution. Similar notions are incremental stability, incremental input-to-state stability, contraction analysis. A Lyapunov approach to incremental stability and incremental input-to-state stability has been presented in Angeli (2002). Incremental stability for systems given in an operator form has been studied in Fromion, Scorletti, and Ferreres (1999). Contraction analysis with applications has been considered in Lohmiller and Slotine (1998) and Jouffroy and Slotine (2004). These notions, along with the notion of convergent systems, prove to be very convenient for non-equilibrium stability analysis of non-autonomous nonlinear systems. At the moment there is a good share of overlap between these notions and one can say that this area of systems theory is still far from a "steady state". For our analysis we choose the notion of convergent systems since it is coordinate

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independent, it is less restrictive and more appropriate for our purposes than the other notions mentioned above.

For the nonlinear ORP, the notion of convergent systems allows one to develop a solvability theory that in a natural way extends the solvability theory of the local nonlinear ORP to the global case (Pavlov et al., 2005b). With a result from this solvability theory and the machinery of convergent systems as a starting point, we design output-feedback controllers solving the global ORP for a class of nonlinear systems. This solution extends the well-known solutions of the linear and local nonlinear ORP to the global case. When applied to the class of Lur'e systems, this result allows one to obtain solutions to the global ORP that have not been reported before. In particular, for Lur'e systems that are not necessarily in the output-feedback form we obtain relatively simple design criteria given in terms of solvability of certain linear matrix inequalities (LMI). Moreover, for Lur'e systems in the output-feedback form with parametric as well as *functional* uncertainties, we provide a robust regulator design that does not rely on the minimum phaseness assumption on the system. Notice that previous results for Lur'e systems deal with *minimum phase systems* in the *output-feedback form* (Ding, 2001; Lin & Dong, 2003; Serrani & Isidori, 2000). Moreover, the global ORP for systems with functional uncertainties has not been considered before. This paper is an extended version of the conference paper (Pavlov, van de Wouw, & Nijmeijer, 2004).

The paper is organized as follows. In Section 2 the global ORP is formulated and preliminaries are provided. A solution to this problem is presented in Section 3 and illustrated with examples in Section 4. Section 5 contains conclusions.

2. Problem statement and preliminaries

Consider a system modeled by

$$\dot{x} = f(x, u, w), \quad e = h(x, w), \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$ and output $e \in \mathbb{R}^l$. The exogenous signal $w(t)$ is generated by the exosystem

$$\dot{w} = s(w), \quad w(0) \in \mathscr{W}, \tag{2}$$

where $\mathscr{W} \subset \mathbb{R}^m$ is a given set of initial conditions. It is assumed that the set $\mathscr{W}$ is compact and positively invariant with respect to the exosystem dynamics. The vectorfields/functions $f(x, u, w)$, $h(x, w)$ and $s(w)$ are assumed to be continuously differentiable.

The global ORP is formulated in the following way: *find, if possible, a feedback of the form*

$$\dot{\xi} = \eta(\xi, e), \quad u = \theta(\xi, e), \quad \xi \in \mathbb{R}^q, \tag{3}$$

for some $q \geqslant 0$, *such that for all initial conditions* $(x(0), \xi(0), w(0)) \in \mathbb{R}^{n+q} \times \mathscr{W}$ *all solutions of the closed-loop system* (1), (3) *and exosystem* (2) *are bounded for* $t \geqslant 0$ *and satisfy* $e(t) \to 0$ *as* $t \to \infty$.

In this paper we will seek solutions to the global ORP based on the notion of convergent systems. Below we give relevant definitions of convergent systems, see Pavlov et al. (2004, 2005a, 2005b). Consider the system

$$\dot{z} = F(z, w), \tag{4}$$

with state $z \in \mathbb{R}^d$, input $w \in \mathbb{R}^m$ and continuous vectorfield $F(z, w)$. The input $w(t)$ is assumed to belong to the class $\overline{\mathbb{PC}}_m$ of piecewise continuous functions $w(t) : \mathbb{R} \to \mathbb{R}^m$ that are bounded on $\mathbb{R}$.

Definition 1. System (4) is said to be uniformly convergent for inputs $w \in \overline{\mathbb{PC}}_m$ if for any $w \in \overline{\mathbb{PC}}_m$ there exists a unique solution $\bar{z}_w(t)$ defined and bounded on $\mathbb{R}$ and this solution is uniformly globally asymptotically stable.[1] The solution $\bar{z}_w(t)$ is called a *steady-state solution*.

Definition 2. System (4) is said to be input-to-state convergent if it is uniformly convergent for inputs $w \in \overline{\mathbb{PC}}_m$ and for every $w \in \overline{\mathbb{PC}}_m$ the system

$$\dot{\tilde{z}} = F(\bar{z}_w(t) + \tilde{z}, w(t) + \tilde{w}) - F(\bar{z}_w(t), w(t))$$

with $\tilde{w}$ as input is input-to-state stable (ISS).

The notions of convergent systems and input-to-state convergent systems are closely related to incrementally stable systems (Angeli, 2002; Fromion et al., 1999), incrementally ISS systems (Angeli, 2002) and to contraction analysis (Lohmiller & Slotine, 1998). Convergence is a less restrictive property since it requires asymptotic stability of only *one* solution corresponding to an input rather than of *all* solutions as in incremental stability and contraction analysis. Moreover, it is coordinate independent, which is not the case for incrementally stable and incrementally ISS systems. For details on convergent systems the reader is referred to Pavlov et al. (2004, 2005a, 2005b).

Further we introduce the following notations. Let $w(t, w_0)$ denote a solution of exosystem (2) starting in $w(0, w_0) = w_0$. By $\Omega(w_0)$ we denote the ω-limit set of the trajectory $w(t, w_0)$. For trajectories starting in the set $\mathscr{W} \subset \mathbb{R}^m$, the notation $\Omega(\mathscr{W})$ denotes $\Omega(\mathscr{W}) := \bigcup_{w_0 \in \mathscr{W}} \Omega(w_0)$. The set $\Omega(\mathscr{W})$ is invariant and, since $\mathscr{W}$ is compact and positively invariant, $\Omega(\mathscr{W}) \subset \mathscr{W}$. To solve the global ORP we need the following assumption:

A1. There exist continuous mappings $\pi : \mathbb{R}^m \to \mathbb{R}^n$ and $c : \mathbb{R}^m \to \mathbb{R}^p$ that satisfy the regulator equations:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \quad \forall t \in \mathbb{R}, \\ &0 = h(\pi(w(t)), w(t)), \end{aligned} \tag{5}$$

for any solution of the exosystem $w(t)$ lying in $\Omega(\mathscr{W})$.

Notice that $\pi(w)$ and $c(w)$ must satisfy the regulator equations only in the set $\Omega(\mathscr{W})$. Outside of this set they can be arbitrarily, yet continuously, extended onto $\mathbb{R}^m$. Assumption A1

[1] The solution $\bar{z}_w(t)$ is called uniformly globally asymptotically stable (UGAS) if the time-varying system $\dot{\tilde{z}} = F(\bar{z}_w(t) + \tilde{z}, w(t)) - F(\bar{z}_w(t), w(t))$ is UGAS at the origin, see Khalil (1996). Uniformity corresponds to the initial time instant t_0.

is a less restrictive counterpart of the standard assumption on the solvability of the regulator equations given in the form of PDEs (Byrnes et al., 1997). It is known (Isidori & Byrnes, 2003; Pavlov et al., 2005b) that for non-local variants of the ORP, necessary conditions for the solvability of the problem cannot be, in general, formulated in terms of PDEs like in the standard regulator equations.

Controller designs presented further in this paper are based on the following result (see Pavlov et al., 2005b, Theorem 4.20, Property 2.19).

Theorem 1. *Suppose Assumption* A1 *is satisfied and* (i) *controller* (3) *makes the corresponding closed-loop system with w as input input-to-state convergent. Controller* (3) *solves the global ORP if and only if* (ii) *for any solution of the exosystem $w(t)$ lying in $\Omega(\mathscr{W})$ there exists a solution $\bar{\xi}_w(t)$ of system* (3) *with $e \equiv 0$ such that $\bar{\xi}_w(t)$ is bounded on $\mathbb{R}$ and the corresponding output equals $\theta(\bar{\xi}_w(t), 0) \equiv c(w(t))$.*

A controller satisfying the conditions of Theorem 1 guarantees that the closed-loop system is input-to-state convergent and that for any solution of the exosystem $w(t)$ lying in $\Omega(\mathscr{W})$ the closed-loop system has a bounded solution $(x(t), \xi(t)) = (\pi(w(t)), \bar{\xi}_w(t))$ along which the regulated output is identically zero.

Based on Theorem 1, we will design controllers that satisfy condition (ii) of Theorem 1 and make the corresponding closed-loop system input-to-state convergent. The main tool for ensuring input-to-state convergence of the closed-loop system is the notion of quadratic stability of a matrix function. A matrix function $\mathscr{A}(\chi) \in \mathbb{R}^{d\times d}$, depending on some parameter $\chi \in \mathscr{X}$, where $\mathscr{X}$ is some set, is called *quadratically stable* if there exist matrices $\mathscr{P} = \mathscr{P}^{\mathrm{T}} > 0$ and $\mathscr{Q} = \mathscr{Q}^{\mathrm{T}} > 0$ such that $\mathscr{P}\mathscr{A}(\chi) + \mathscr{A}^{\mathrm{T}}(\chi)\mathscr{P} \leqslant -\mathscr{Q}$ for all $\chi \in \mathscr{X}$. Notice that if $\mathscr{A}(\chi)$ is constant, then quadratic stability is equivalent to $\mathscr{A}$ being Hurwitz. The next theorem links quadratic stability with the input-to-state convergence property of a system (Pavlov et al., 2005a, 2005b).

Theorem 2. *Consider system* (4) *with the function F being C^1 with respect to z and continuous with respect to w. If $(\partial F/\partial z)(z, w)$ is quadratically stable then system* (4) *is input-to-state convergent. Moreover, there exist constants $\alpha > 0$ and $C > 0$ such that any two solutions $z_1(t)$ and $z_2(t)$ of system* (4) *corresponding to the same input $w(t)$ satisfy*

$$|z_1(t) - z_2(t)| \leqslant C\mathrm{e}^{-\alpha(t-t_0)}|z_1(t_0) - z_2(t_0)|. \tag{6}$$

3. Main results

The results in this section are formulated using the following notations: $\chi := (x, u, w)$, $\mathscr{A}(\chi) := \partial f/\partial x(\chi)$, $\mathscr{B}(\chi) := \partial f/\partial u(\chi)$, $\mathscr{E}(\chi) := \partial f/\partial w(\chi)$, $\mathscr{C}(\chi) := \partial h/\partial x(\chi)$, $\mathscr{H}(\chi) := \partial h/\partial w(\chi)$, $\mathscr{S}(\chi) := \partial s/\partial w(\chi)$.

Theorem 3. *Consider system* (1) *and exosystem* (2). *Suppose Assumption* A1 *holds. If there exist matrices $K \in \mathbb{R}^{p\times n}$ and $L = (L_x^{\mathrm{T}}, L_w^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{(n+m)\times l}$ such that the matrices $\mathscr{A}(\chi) + \mathscr{B}(\chi)K$ and*

$$\begin{bmatrix} \mathscr{A}(\chi) & \mathscr{E}(\chi) \\ 0 & \mathscr{S}(\chi) \end{bmatrix} + L[\mathscr{C}(\chi) \;\; \mathscr{H}(\chi)] \tag{7}$$

are quadratically stable, then the controller

$$u = c(\hat{w}) + K(\hat{x} - \pi(\hat{w})), \tag{8}$$
$$\dot{\hat{x}} = f(\hat{x}, u, \hat{w}) + L_x(h(\hat{x}, \hat{w}) - e), \tag{9}$$
$$\dot{\hat{w}} = s(\hat{w}) + L_w(h(\hat{x}, \hat{w}) - e) \tag{10}$$

solves the global ORP.

Proof. First, we show that the closed-loop system with w as input is input-to-state convergent. System (1) in closed loop with (8) can be written as

$$\dot{x} = f(x, Kx + K\tilde{x} + c(w + \tilde{w}) - K\pi(w + \tilde{w}), w), \tag{11}$$

where $\tilde{x} := \hat{x} - x$ and $\tilde{w} := \hat{w} - w$ are observer errors. The Jacobian of the right-hand side of (11) with respect to x equals $\mathscr{A}(\chi) + \mathscr{B}(\chi)K$, which is quadratically stable. Hence, by Theorem 2 system (11) with $\tilde{x}$, $\tilde{w}$ and w as inputs is input-to-state convergent.

Next, we consider the observer error dynamics. Notice that the Jacobian of the right-hand side of (9), (10) equals the matrix given in (7) with $\chi = (\hat{x}, u, \hat{w})$, which is quadratically stable. Therefore, by Theorem 2 any two solutions of observer (9), (10) corresponding to the same input u, e converge exponentially to each other, see (6). Since $x(t)$, $w(t)$—a solution of system (1) and exosystem (2)—is also a solution of the observer (9), (10), we conclude that for any other solution $(\hat{x}(t), \hat{w}(t))$, the observer error $\tilde{z} := (\tilde{x}^{\mathrm{T}}, \tilde{w}^{\mathrm{T}})^{\mathrm{T}}$ satisfies

$$|\tilde{z}(t)| \leqslant C\mathrm{e}^{-\alpha(t-t_0)}|\tilde{z}(t_0)|, \tag{12}$$

for some $C > 0$ and $\alpha > 0$. Therefore, the closed-loop system can be considered as an interconnection of the input-to-state convergent system (11) with the inputs $\tilde{x}$, $\tilde{w}$ and w, and the observer error dynamics satisfying (12). Such an interconnected system with w as input is input-to-state convergent, see Pavlov et al. (2005b), Property 2.28.

Since the mappings $\pi(w)$ and $c(w)$ satisfy the regulator equations (5) and because of the structure of controller (8)–(10), one can routinely check that condition (ii) of Theorem 1 is satisfied with $\bar{\xi}_w(t) := (\pi(w(t))^{\mathrm{T}}, w(t)^{\mathrm{T}})^{\mathrm{T}}$. By Theorem 1 controller (8)–(10) solves the global ORP. □

The result of Theorem 3 can be readily extended to the case when the measured output is different from the regulated output. The corresponding counterpart of Theorem 1, which lies in the foundation of the proof, can be found in Pavlov et al. (2005b), while the proof of input-to-state convergence of the closed-loop systems is almost the same.

Notice that for the case of linear systems, the matrices $\mathscr{A}(\chi) + \mathscr{B}(\chi)K$ and (7) are constant and the requirement of quadratic stability is equivalent to the requirement that these matrices are Hurwitz. Therefore, the existence of the controller

and observer gains K and L satisfying the conditions of the theorem is equivalent to stabilizability and detectability conditions on the appropriate system matrices. These conditions together with the assumption on solvability of the regulator equations constitute standard solvability conditions for the linear ORP. Similar conditions arise in the local nonlinear ORP, see e.g. Isidori and Byrnes (1990). Therefore Theorem 3 serves as a natural extension of the standard solutions of the linear ORP and local nonlinear ORP to the case of global output regulation for nonlinear systems. When applied to particular classes of nonlinear systems, the conditions of Theorem 3 simplify and in some cases can be checked by solving LMI, as will be illustrated in the next subsection for the class of Lur'e systems.

3.1. Global output regulation for Lur'e systems

Consider the system

$$\begin{aligned}\dot{x} &= Ax + Bu + D\varphi(\zeta) + Ew,\\ e &= Cx + Hw, \quad \zeta = C_\zeta x + H_\zeta w.\end{aligned} \tag{13}$$

The nonlinearity $\varphi(\zeta)$ is a scalar function of scalar argument ζ satisfying the inequality

$$\left|\frac{\partial\varphi}{\partial\zeta}(\zeta)\right| \leqslant \gamma, \quad \zeta \in \mathbb{R} \tag{14}$$

for some $\gamma > 0$. We assume that the exosystem is linear

$$\dot{w} = Sw, \tag{15}$$

where S is such that all its eigenvalues are simple and lie on the imaginary axis.

Denote $\mathscr{C}_\mathrm{o} := [C \;\; H]$ and define matrices $A^+_{\mathrm{c}\gamma}$, $A^-_{\mathrm{c}\gamma}$, $A^+_{\mathrm{o}\gamma}$ and $A^-_{\mathrm{o}\gamma}$ according to the formulas $A^\pm_{\mathrm{c}\gamma} := A \pm \gamma DC_\zeta$,

$$A^\pm_{\mathrm{o}\gamma} := \begin{bmatrix} A \pm \gamma DC_\zeta & E \pm \gamma DH_\zeta \\ 0 & S \end{bmatrix}.$$

The following theorem provides conditions for output-feedback controller design for Lur'e systems.

Theorem 4. *Consider system* (13) *and exosystem* (15). *Under Assumption* A1, *if there exist* $\mathscr{P}_\mathrm{c} = \mathscr{P}_\mathrm{c}^\mathrm{T} > 0$, $\mathscr{P}_\mathrm{o} = \mathscr{P}_\mathrm{o}^\mathrm{T} > 0$, $\mathscr{Y}$ and $\mathscr{Z}$ *satisfying the LMIs*

$$\begin{aligned}A^+_{\mathrm{c}\gamma}\mathscr{P}_\mathrm{c} + \mathscr{P}_\mathrm{c}(A^+_{\mathrm{c}\gamma})^\mathrm{T} + B\mathscr{Y} + \mathscr{Y}^\mathrm{T}B^\mathrm{T} < 0,\\ A^-_{\mathrm{c}\gamma}\mathscr{P}_\mathrm{c} + \mathscr{P}_\mathrm{c}(A^-_{\mathrm{c}\gamma})^\mathrm{T} + B\mathscr{Y} + \mathscr{Y}^\mathrm{T}B^\mathrm{T} < 0,\end{aligned} \tag{16}$$

$$\begin{aligned}\mathscr{P}_\mathrm{o}A^+_{\mathrm{o}\gamma} + (A^+_{\mathrm{o}\gamma})^\mathrm{T}\mathscr{P}_\mathrm{o} + \mathscr{Z}\mathscr{C}_\mathrm{o} + \mathscr{C}_\mathrm{o}^\mathrm{T}\mathscr{Z}^\mathrm{T} < 0,\\ \mathscr{P}_\mathrm{o}A^-_{\mathrm{o}\gamma} + (A^-_{\mathrm{o}\gamma})^\mathrm{T}\mathscr{P}_\mathrm{o} + \mathscr{Z}\mathscr{C}_\mathrm{o} + \mathscr{C}_\mathrm{o}^\mathrm{T}\mathscr{Z}^\mathrm{T} < 0,\end{aligned} \tag{17}$$

then controller (8)–(10) *with* $f(x, u, w)$, $h(x, w)$ *and* $s(w)$ *corresponding to* (13) *and* (15) *and with the gains* $K = \mathscr{Y}\mathscr{P}_\mathrm{c}^{-1}$, $L = [L_x^\mathrm{T},\; L_w^\mathrm{T}]^\mathrm{T} = \mathscr{P}_\mathrm{o}^{-1}\mathscr{Z}$ *solves the global ORP.*

Proof. We will show that the gains K and L satisfy the conditions of Theorem 3. For system (13), the matrix $\mathscr{A}(\chi) + \mathscr{B}(\chi)K$ defined in Theorem 3 equals $J(\zeta) := A + DC_\zeta(\partial\varphi/\partial\zeta)(\zeta) + BK$. Due to condition (14), it satisfies $J(\zeta) \in \mathrm{co}\{A^-_{\mathrm{c}\gamma} + BK, A^+_{\mathrm{c}\gamma} + BK\}$, where $\mathrm{co}\{\cdot\}$ denotes the convex hull. Therefore, $J(\zeta)$ is quadratically stable if there exists $P = P^\mathrm{T} > 0$ such that

$$\begin{aligned}P(A^-_{\mathrm{c}\gamma} + BK) + (A^-_{\mathrm{c}\gamma} + BK)^\mathrm{T}P < 0,\\ P(A^+_{\mathrm{c}\gamma} + BK) + (A^+_{\mathrm{c}\gamma} + BK)^\mathrm{T}P < 0.\end{aligned} \tag{18}$$

Multiplying (16) from the left and from the right by $\mathscr{P}_\mathrm{c}^{-1}$, we conclude that (18) holds for $K = \mathscr{Y}\mathscr{P}_\mathrm{c}^{-1}$ and $P = \mathscr{P}_\mathrm{c}^{-1}$. Therefore, the matrix $\mathscr{A}(\chi) + B(\chi)K$ is quadratically stable. Quadratic stability of the matrix (7) can be shown in the same way. Applying Theorem 3, we obtain the statement of the theorem. □

Remark 1. For Lur'e systems, instead of the observer (9), (10) one can use other observers that guarantee exponential stability of the observer error dynamics, e.g. Arcak and Kokotovic (2001).

Remark 2. The result of Theorem 4 allows one to solve the global ORP for Lur'e systems that are not necessarily in the output-feedback form. To the best of our knowledge the global ORP for this class of systems has not been considered in the literature so far.

Remark 3. Although in Theorem 4 we deal with Lur'e systems with one nonlinearity, similar results can be obtained for systems with multiple nonlinearities. In this case, the general conditions of Theorem 3 can be reformulated in terms of solvability of certain LMIs, if the corresponding Jacobian matrices of the system and exosystem dynamics (see the beginning of Section 3) lie in the convex hull of a finite set of matrices.

Controller (8)–(10) requires the knowledge of the exact system model and the mappings $\pi(w)$ and $c(w)$, which are, in general, difficult to compute (so far a systematic way of finding these mappings exists only for some classes of lower-triangular systems). To bypass this problem, an appropriate internal model can be included in the controller. Although the result of Theorem 4 cannot be directly applied in this case, the notion of convergence can still facilitate the robust controller design, as will be illustrated in the next subsection.

3.2. Robust global output regulation for Lur'e systems

In this subsection we consider the global ORP for exosystem (15) and system (13) with uncertain parameters as well as with an uncertain nonlinearity $\varphi(\zeta)$. We will tackle this more difficult problem under the following assumptions:

B1. There exists a matrix $\Psi \in \mathbb{R}^{1\times l}$ such that $\zeta = \Psi e$. The output e and control u are of the same dimension, i.e. $l = p$. The nonlinearity $\varphi(\zeta)$ belongs to the class $\mathscr{F}_\gamma$ of nonlinearities satisfying (14) and $\varphi(0) = 0$.

B2. The nominal system matrices A°, B°, C° and D° are such that the pair (A°, B°) is stabilizable, the pair (A°, C°) is de-

tectable and for every λ being an eigenvalue of the matrix S the following matrix has full rank:

$$\begin{pmatrix} A^\circ - \lambda I & B^\circ \\ C^\circ & 0 \end{pmatrix}.$$

The design of a robust regulator closely follows the design of a regulator for the linear robust ORP (see e.g. Byrnes et al., 1997). Let $S_{\min}$ be an $r \times r$ matrix whose characteristic polynomial coincides with the minimal polynomial of S. Construct a block-diagonal $lr \times lr$ matrix Φ which has l blocks $S_{\min}$ on its diagonal, where l is the number of outputs. First, choose an $lr \times l$ matrix Γ and an $l \times lr$ matrix N such that (Φ, Γ) is controllable and (Φ, N) is observable. Consider the augmented system

$$\begin{aligned} \dot{x} &= A^\circ x + B^\circ \Gamma \xi_1 + B^\circ v + D^\circ \varphi, \\ \dot{\xi}_1 &= \Phi \xi_1 + N C^\circ x, \end{aligned} \tag{19}$$

with the output $\zeta = \Psi C^\circ x$. Secondly, find a controller

$$\dot{\xi}_2 = \Lambda \xi_2 + L C^\circ x, \quad v = M \xi_2 + R C^\circ x, \tag{20}$$

such that system (19) in closed loop with this controller is asymptotically stable for $\varphi = 0$ and the transfer function $W^\circ_{\zeta\varphi}(s)$ from input φ to output ζ satisfies $\|W^\circ_{\zeta\varphi}\|_\infty < 1/\gamma$. The existence of such a controller is a standard problem in H_∞ optimization, which can be numerically solved with efficient solvers available, for example, in MATLAB. We proceed with the construction of the overall regulator given by

$$\begin{aligned} &\dot{\xi}_1 = \Phi \xi_1 + N e, \quad \dot{\xi}_2 = \Lambda \xi_2 + L e, \\ &u = \Gamma \xi_1 + M \xi_2 + R e. \end{aligned} \tag{21}$$

The next theorem provides properties of this controller.

Theorem 5. *Under Assumptions* B1 *and* B2, *controller* (21) *constructed above solves the global ORP for system* (13) *and exosystem* (15) *for all matrices* $E \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{l\times m}$, *for all nonlinearities* $\varphi \in \mathscr{F}_\gamma$, *and for all matrices* A, B, C *and* D *from some neighborhood of the nominal matrices* A°, B°, C°, D°.

Proof. Let us first consider system (13) *without* nonlinearity $\varphi(\zeta)$ (or, equivalently, with $\varphi = 0$). This is a linear system. It is known (Byrnes et al., 1997) that under Assumption B2 for any A, B and C being close enough to the nominal ones and for arbitrary E and H, the regulator equations for system (13) with $\varphi = 0$ are solvable with $\pi(w) = \Pi w$ and $c(w) = \Upsilon w$ for some matrices Π and Υ. Moreover, controller (21) with $e = 0$ has a bounded solution $\bar{\xi}_w(t)$ with the corresponding control u being equal to $c(w(t))$. Since $\varphi(\zeta) = \varphi(\Psi e) = 0$ for $e = 0$, we conclude that the same $\pi(w)$ and $c(w)$ satisfy the regulator equations for system (13) *with* the nonlinearity $\varphi(\zeta)$ and controller (21) with $e = 0$ generates the corresponding steady-state control $c(w(t))$ along the bounded solution $\bar{\xi}_w(t)$. Therefore, condition (ii) of Theorem 1 is satisfied for all A, B and C being close enough to the nominal ones, for arbitrary E and H and for any nonlinearity $\varphi \in \mathscr{F}_\gamma$.

It remains to show that the closed-loop system is input-to-state convergent. System (13) in closed loop with (21) is a Lur'e system of the form

$$\dot{z} = \tilde{A} z + \tilde{D} \varphi(\zeta) + f_1(w), \quad \zeta = \tilde{C} z + f_2(w) \tag{22}$$

with $z := (x^{\mathrm{T}}, \xi_1^{\mathrm{T}}, \xi_2^{\mathrm{T}})^{\mathrm{T}}$, $f_2(w) = \Psi H w$, $f_1(w) := ((Ew + BRHw)^{\mathrm{T}}, (NHw)^{\mathrm{T}}, (LHw)^{\mathrm{T}})^{\mathrm{T}}$,

$$\tilde{A} := \begin{bmatrix} A + BRC & B\Gamma & BM \\ NC & \Phi & 0 \\ LC & 0 & \Lambda \end{bmatrix}, \quad \tilde{D} := \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

and $\tilde{C} = \Psi[C \ 0 \ 0]$. Notice, that the transfer function $W^\circ_{\zeta\varphi}(s)$ defined at the stage of controller design equals $W^\circ_{\zeta\varphi}(s) = \tilde{C}^\circ (sI - \tilde{A}^\circ)^{-1} \tilde{D}^\circ$, where $\tilde{A}^\circ$, $\tilde{C}^\circ$ and $\tilde{D}^\circ$ equal $\tilde{A}$, $\tilde{C}$ and $\tilde{D}$ defined for the nominal system parameters. Moreover, by construction of the controller (21), the matrix $\tilde{A}^\circ$ is Hurwitz. Since $\sup_{\omega\in\mathbb{R}} |\tilde{C}^\circ (\mathrm{i}\omega I - \tilde{A}^\circ)^{-1} \tilde{D}^\circ| = \|W^\circ_{\zeta\varphi}\|_\infty < 1/\gamma$, by continuity we obtain

$$\sup_{\omega\in\mathbb{R}} |\tilde{C} (\mathrm{i}\omega I - \tilde{A})^{-1} \tilde{D}| < 1/\gamma, \tag{24}$$

for all $\tilde{A}$, $\tilde{D}$ and $\tilde{C}$ from some neighborhood of the nominal $\tilde{A}^\circ$, $\tilde{D}^\circ$ and $\tilde{C}^\circ$. Since the nonlinearity $\varphi(\zeta)$ satisfies (14), the Jacobian of the right-hand side of (22) satisfies

$$\tilde{A} + \tilde{D}\tilde{C} \frac{\partial \varphi}{\partial \zeta}(\zeta) \in \mathrm{co}\{\tilde{A} - \gamma \tilde{D}\tilde{C}, \tilde{A} + \gamma \tilde{D}\tilde{C}\}, \quad \forall \zeta \in \mathbb{R}. \tag{25}$$

At the same time, since $\tilde{A}$ is Hurwitz (because it is close enough to the Hurwitz matrix $\tilde{A}^\circ$), condition (24) guarantees (see e.g. Yakubovich, 1964) that there exists a matrix $P = P^{\mathrm{T}} > 0$ satisfying the LMI $P(\tilde{A} - \gamma\tilde{D}\tilde{C}) + (\tilde{A} - \gamma\tilde{D}\tilde{C})^{\mathrm{T}} P < 0$, $P(\tilde{A} + \gamma\tilde{D}\tilde{C}) + (\tilde{A} + \gamma\tilde{D}\tilde{C})^{\mathrm{T}} P < 0$. This fact, together with (25) implies that $\tilde{A} + \tilde{D}\tilde{C}(\partial\varphi/\partial\zeta)(\zeta)$—the Jacobian of the right-hand side of the closed-loop system (22)—is quadratically stable. By Theorem 2 we conclude that system (22) is input-to-state convergent for all $\varphi \in \mathscr{F}_\gamma$, for all matrices E and H and for all $\tilde{A}$, $\tilde{D}$ and $\tilde{C}$ close enough to $\tilde{A}^\circ$, $\tilde{D}^\circ$ and $\tilde{C}^\circ$, which is so if the original system matrices A, B, C and D are close enough to their nominal values. Finally, the application of Theorem 1 proves the statement of the theorem. □

Remark 1. Systems in the output-feedback form, like the one considered in this section, have been studied in multiple publications on the global ORP (Ding, 2001; Lin & Dong, 2003; Serrani & Isidori, 2000). At the same time, these works rely on the crucial assumption that the system is minimum phase. This assumption is, in general, not necessary for the controller design presented above, as will be illustrated with an example in the next section.

Remark 2. Theorem 5 deals with the case of systems with *functional* uncertainties. It is more challenging than the case of only parametric uncertainties, which is mostly considered in the literature. One may argue that the assumption that the uncertain nonlinearity φ vanishes for $e = 0$ is too strong. Below, with

an example regarding a specific system, we demonstrate the idea that this condition can be necessary for the solvability of the ORP for systems with uncertainties from an infinite dimensional set. Consider the scalar system $\dot{x} = \varphi(x) + u$, $e = x - \kappa w$, where κ is some constant matrix, $w(t)$ is a solution of the exosystem (15) and $\varphi \in \mathscr{F}_\gamma$ is an uncertain nonlinearity. Here we see that the nonlinearities do not vanish for $e = 0$. Suppose controller (3) solves the global ORP for this system for any $\varphi \in \mathscr{F}_\gamma$. According to the internal model principle, see e.g. Pavlov et al. (2005b), Isidori and Byrnes (2003), Byrnes et al. (1997), for any solution of the exosystem $w(t)$ and for any $\varphi \in \mathscr{F}_\gamma$ this controller (3) with $e = 0$ must have a solution along which $u(t)$ equals the steady-state control $c(w(t)) = \kappa S w(t) - \varphi(\kappa w(t))$, which is obtained from the corresponding regulator equations. Choose some $w(t)$ such that $\kappa w(t)$ is not constant. From the expression for $c(w(t))$ one can verify that the set of all steady-state controls corresponding to this $w(t)$ and various $\varphi \in \mathscr{F}_\gamma$ cannot be characterized by any finite dimensional set, since $\mathscr{F}_\gamma$ is an infinite dimensional set. At the same time, all possible outputs of controller (3) with $e = 0$ are parameterized by the initial conditions $\xi(0) \in \mathbb{R}^q$, which constitute a finite dimensional set. Thus we come to a contradiction. Although here this reasoning is performed only for a particular system, it motivates the necessity of strong assumptions on the functional uncertainties for the solvability of the ORP.

4. Examples

Consider the mass-spring system with friction

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -kx_1 - \psi(x_2) + u, \tag{26}$$

where x_1 is the position of the unit mass, u is a control force and $k = 400$ is a spring stiffness. The nonlinear term $\psi(x_2)$ is due to dry friction and viscous damping acting on the mass. In this example we consider a smooth friction model $\psi(x_2)$ given by the formula

$$\psi(x_2) = \alpha x_2 \mathrm{e}^{-x_2^2/\delta} + \beta(1 - \mathrm{e}^{-x_2^2/\delta})\,\mathrm{sign}(x_2) + \delta x_2,$$

with the parameters $\alpha = 200$, $\beta = 7$ and $\delta = 3$. The graph of this $\psi(x_2)$ is shown in Fig. 1. This smooth friction model differs from conventional friction models often found in the literature. However, since for this friction model $\mathrm{d}\psi/\mathrm{d}x_2$ is bounded, we can apply the theory developed in Section 3.1. The regulated

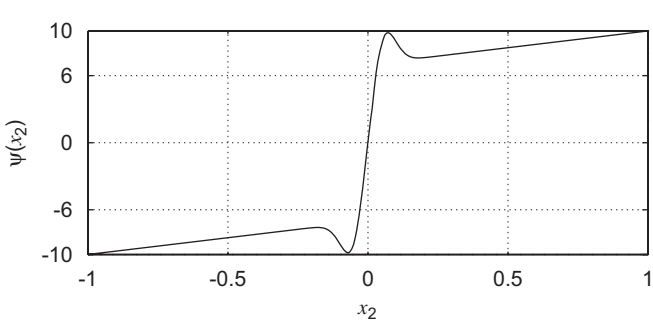

Fig. 1. Graph of the nonlinearity $\psi(x_2)$.

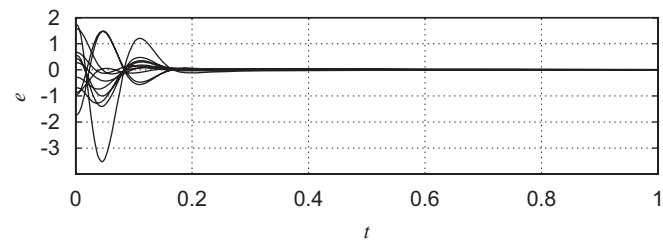

Fig. 2. Regulated output $e(t)$ for various initial conditions of the closed-loop system and the exosystem.

output is given by $e = x_1 - w_1$, where the external signal is generated by the linear exosystem

$$\dot{w}_1 = w_2, \quad \dot{w}_2 = -w_1. \tag{27}$$

System (26) is a Lur'e system and it is not in the output-feedback form often considered in the literature on the global ORP (Ding, 2001; Lin & Dong, 2003; Serrani & Isidori, 2000). The derivative of the nonlinear term $\psi(x_2)$ lies within the bounds $[-40.3, 218.8]$. In order to make these bounds symmetrical with respect to the origin, we decompose $\psi(x_2)$ into a sum of the linear term λx_2 with $\lambda = 89.3$ and the remaining nonlinearity $\varphi(x_2) := \psi(x_2) - \lambda x_2$. With such a decomposition of $\psi(x_2)$, system (26) is a Lur'e system of form (13) with the nonlinearity satisfying (14) with $\gamma = 129.6$. Hence, we can apply Theorem 4 in order to solve the global ORP.

First, we find solutions to the regulator equations: $\pi(w) = (w_1, w_2)^{\mathrm{T}}$ and $c(w) = (-1+k)w_1 + \lambda w_2 + \varphi(w_2)$. The mappings $\pi(w)$ and $c(w)$ are globally defined continuous mappings, i.e. Assumption A1 is satisfied. Secondly, we find solutions to the LMIs (16) and (17). For the given system parameters, these LMIs are solvable. The corresponding controller and observer gains defined in Theorem 4 equal $K = [-800, -40.75]$ and $L = [2050, -13\,636, 2351, 3422]^{\mathrm{T}}$. Hence, by Theorem 4 controller (8)–(10) with $f(x, u, w)$, $h(x, w)$ and $s(w)$ corresponding to (26), (27), $e = x_1 - w_1$, and with the gains K and L specified above solves the global ORP. Simulations for various initial conditions of the closed-loop system and the exosystem have been performed. The regulated output $e(t)$ corresponding to these simulations is shown in Fig. 2.

In order to illustrate controller design for the global robust ORP presented in Section 3.2, we consider system (13) with the nominal matrices $A^\circ = [1, -2, 0; 40, 3, 4; 1, 0, 5]$, $B^\circ = [0, 3, 1]^{\mathrm{T}}$, $D^\circ = [1, 1, 0]^{\mathrm{T}}$ and $C^\circ = [1, 0, 0]$. The exosignal w is generated by exosystem (27). The outputs of the system are equal: $e = \zeta = Cx + Hw$. The matrices E and H can be chosen arbitrarily. The value γ for the bound in (14) is chosen $\gamma = 0.1$. With such a choice of system matrices, Assumptions B1, B2 hold. Notice that with these matrices our system is not minimum phase, since it has an unstable linear zero dynamics corresponding to the zero $\lambda = 3.667$. Therefore, the existing results for systems in the output-feedback form (Ding, 2001; Lin & Dong, 2003; Serrani & Isidori, 2000) are not applicable, since they rely on the minimum phaseness assumption. Following the design procedure given in Section 3.2, we set $\Phi = [0, 1; -1, 0]$, $N = [1, 0]^{\mathrm{T}}$, $\Gamma = [1, 0]$. Next, we search for a controller (20) that

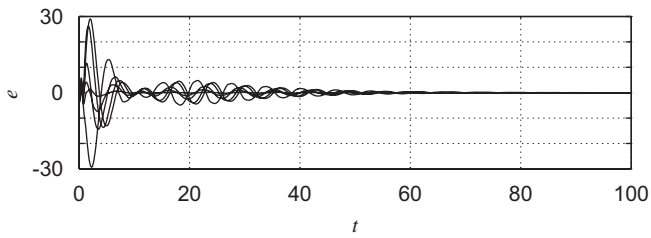

Fig. 3. Simulation results for various initial conditions and for various matrices E and H.

would satisfy the inequality $\|W^{\circ}_{\zeta\varphi}\| < 1/\gamma$. Such a controller is found using the MATLAB routine `hinflmi`. The obtained controller is validated by means of simulations. In the simulations the matrices A, B, C and D are taken equal to their nominal values and the nonlinearity is chosen $\varphi(\zeta) = \gamma \sin(\zeta)$. Results of several simulations with randomly chosen matrices E and H and random initial conditions for the closed-loop system and exosystem are given in Fig. 3.

5. Conclusions

We have presented output-feedback controllers solving the global ORP for a class of nonlinear systems. The presented solution is based on the notion of convergent systems. The obtained regulators can be found by solving the regulator equations and finding linear gains for the controller and observer, which constitute the regulator, that make the closed-loop system input-to-state convergent. The existence of the above mentioned controller and observer gains can in some cases be checked by solving certain LMIs, as has been illustrated for the case of Lur'e systems. The obtained result allows one to solve the global ORP for Lur'e systems that are not in the output-feedback form. To the best of our knowledge there are no publications on the global ORP for this class of systems. In the case when parameters of a Lur'e system are not known exactly and the nonlinearity can be arbitrary from a given class, it has been shown that the robust global ORP can be reduced to certain linear H_∞ optimization problem, which can be solved numerically. Global ORP for systems with *functional* uncertainties has not been addressed in the literature so far. The obtained results extend in a natural way the well-known solutions of the linear ORP and the local nonlinear ORP to the case of global ORP for nonlinear systems.

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