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Brief paper

Analysis of undercompensation and overcompensation of friction in 1DOF mechanical systems☆

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Abstract

This paper investigates the effects of undercompensation and overcompensation of friction in PD controlled 1DOF mechanical systems. The friction force that is acting on the mechanical system and the friction compensation term in the feedback loop are described by a class of discontinuous friction models consisting of static, Coulomb and viscous friction, and including the Stribeck effect. Lyapunov's stability theorem and LaSalle's invariance principle are applied to prove that undercompensation of friction leads to steady-state errors and the properties of the ω-limit set of trajectories of a two-dimensional autonomous differential inclusion are used to show that overcompensation of friction may induce limit cycling. Furthermore, the analysis also indicates that the limit cycling effect can be eliminated by tuning the PD controller gains.

Keywords: Friction compensation; Equilibrium set; Limit cycles; Friction problems in mechanics; Discontinuous systems

1. Introduction

Friction occurs in many positioning systems and it can deteriorate performance of those controlled systems in terms of large steady-state errors and limit cycling, see for example (Armstrong-Hélouvry, Dupont, & Canudas de Wit, 1994). Friction compensation is, therefore, needed in order to improve the system performance. Satisfactory friction compensation can be obtained if a good friction model is available. However, friction is a highly nonlinear phenomenon, which is difficult to be described by a simple model (Armstrong-Hélouvry et al., 1994; Olsson, Åström, Canudas de Wit, Gäfvert, & Lischinsky, 1998). Because of such modeling errors and parameter estimation errors, inexact friction compensation is inevitable.

The limit cycling effect that is induced by the overcompensation of friction in PD and PID controlled one-degree-of-freedom (1DOF) systems has been analyzed in Canudas de Wit (1993) by means of the describing function method. Papadopoulos and Chasparis (2002) validate the predicted limit cycle on an experimental setup but, at the same time, they also show that the prediction of the describing function is not always accurate. The effects of undercompensation and overcompensation of friction in an observer-based controlled 1DOF robot has been investigated in Mallon, van de Wouw, Putra, and Nijmeijer (2006) using a scaling rule. The numerical and the experimental results of Mallon et al. (2006) also indicate that overcompensation of friction induces limit cycling and that undercompensation of friction leads to steady-state errors.

This manuscript is intended to provide a rigorous mathematical analysis of the observations reported in Mallon et al. (2006) for a class of discontinuous friction models and for more general cases of undercompensation and overcompensation of friction. The analysis is based on LaSalle's invariance principle (Adly & Goeleven, 2004; Alvarez, Orlov, & Acho, 2000; van de Wouw & Leine, 2004) and the properties of the ω-limit set of trajectories of a two-dimensional differential inclusion (Filippov, 1988).

In this study, we focus on friction compensation in PD controlled 1DOF systems with state-feedback. We omit the

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inclusion of an observer, commonly used to reconstruct velocity from position measurements, in order to study the effect of undercompensation and overcompensation of friction on the control performance in an isolated fashion. Note in this respect that the use of an observer may itself induce limit cycling and steady-state errors even for the case of exact friction compensation (Putra & Nijmeijer, 2004). However, note that in Mallon et al. (2006) and Putra, Moreau, and Nijmeijer (2004) observer-based friction compensation schemes have been successfully applied both in simulations and experiments.

This paper is organized as follows. Section 2 explains the model of the controlled system with friction compensation. The effects of undercompensation and overcompensation of friction on the performance of the controlled system are studied in Sections 3 and 4, respectively. Section 5 provides numerical demonstrations of the analytical results. Finally, conclusions are drawn in Section 6.

2. Controlled 1DOF frictional systems

We consider 1DOF frictional mechanical systems that can be described by

$$\dot{x} = y, \tag{1a}$$

$$\dot{y} = -\frac{F_v}{J}y - \frac{1}{J}F(y,u) + \frac{1}{J}u, \tag{1b}$$

where x, y and J are the position, the velocity and the inertia of the mechanical system, respectively, $F_v > 0$ is the linear viscous friction damping, u is the input force and $F(y,u)$ is the nonlinear friction force given by (Hensen, Van de Molengraft, & Steinbuch, 2003; Olsson et al., 1998)

$$F(y,u) = \begin{cases} g(y)\,\mathrm{sign}(y) & \text{if } y \neq 0, \\ \min\{|u|, F_s\}\,\mathrm{sign}(u) & \text{if } y = 0, \end{cases} \tag{2}$$

with $F_s > 0$ the static friction level and $g(y)$ a continuous Stribeck function, which represents the continuous decay[1] of the friction curve from F_s to a Coulomb friction level $F_c > 0$. Note that in the friction law (2) knowledge on the dynamics in which it is embedded is used by realizing that in the stiction mode, i.e. $y = 0$ and $|u| \leqslant F_s$, the friction force and the input force counteract each other exactly. This formulation is useful for the 1DOF case but is not suitable for MDOF systems with multiple frictional contacts. Alternatively, the dependency of static friction on the input force can be removed by using a set-valued friction law, e.g. see (Glocker, 2001; van de Wouw & Leine, 2004). Note that similar results as in the present paper can be obtained by using the set-valued friction models but for this specific case of a single frictional element it results in more complex representation of the closed-loop system (Putra, 2004, Chapter 7).

Model (1), (2) describes a controlled inertia subject to viscous plus Coulomb friction if $g(y) = F_s = F_c$. Other commonly used Stribeck functions in the control literature (Armstrong-Hélouvry et al., 1994; Olsson et al., 1998) are of the forms

$$g(y) = F_c + (F_s - F_c)\,\mathrm{e}^{-(|y|/v_s)^{\delta}} \tag{3}$$

and

$$g(y) = F_c + (F_s - F_c)\frac{1}{1 + (|y|/v_s)^{\delta}}, \tag{4}$$

where $v_s > 0$ is called the Stribeck velocity and $\delta > 0$ is the shaping parameter of the Stribeck curve. The combination of a linear viscous friction and the nonlinear friction F as considered in (1b) is able to represent a rather general class of static friction models (Armstrong-Hélouvry et al., 1994; Olsson et al., 1998). However, the nonlinear friction model (2) excludes the friction model with a discontinuous drop of the friction curve from F_s to F_c, which is shown in Armstrong-Hélouvry and Amin (1996) to be inadequate for describing the possible disappearance of the friction-induced stick-slip phenomenon.

Here we opted for a static friction model since we focus on the effect of friction on the global dynamics; however, when the behavior for very small velocities is particularly important one could opt for a dynamic friction model, see e.g. (Bliman & Sorine, 1995; Canudas de Wit, Olsson, Aström, & Lischinsky, 1995).

In order to regulate the frictional mechanical system (1) towards a setpoint x_s, we consider a PD controller with friction compensation of the form

$$u = K_p(x_s - x) + K_d(0 - y) + \widetilde{F}(y, \bar{u}), \tag{5}$$

where $K_p > 0$ is the proportional gain, $K_d > 0$ is the derivative gain, and $\widetilde{F}(y, \bar{u})$ is a friction compensation term given by

$$\widetilde{F}(y, \bar{u}) = \begin{cases} \widetilde{g}(y)\,\mathrm{sign}(y) & \text{if } y \neq 0, \\ \widetilde{F}_s\,\mathrm{sign}(\bar{u}) & \text{if } y = 0, \end{cases} \tag{6}$$

where $\bar{u} = K_p(x_s - x)$.

Without loss of generality, we assume that the setpoint is the origin, i.e. $x_s = 0$, such that the input u becomes

$$u = -K_p x - K_d y + \widetilde{F}(y, -K_p x). \tag{7}$$

Substitution of the feedback (7) into the system (1) results in the closed-loop system

$$\dot{x} = y, \tag{8a}$$

$$\dot{y} = -\frac{K_p}{J}x - \frac{(K_d + F_v)}{J}y + \frac{1}{J}\Delta F(y, u_0), \tag{8b}$$

where $\Delta F(y, u_0) = \widetilde{F} - F$ is the friction compensation error given by

$$\Delta F(y, u_0) = \begin{cases} (\widetilde{g}(y) - g(y))\,\mathrm{sign}(y) & \text{if } y \neq 0, \\ (F_s - \widetilde{F}_s)\,\mathrm{sign}(x) & \text{if } y = 0 \wedge |u_0| > F_s, \\ K_p x & \text{otherwise}, \end{cases} \tag{9}$$

with $u_0 = -K_p x - \widetilde{F}_s\,\mathrm{sign}(x)$ the input force at zero velocity.

[1] The continuous decay implies $g(0) = F_s$, $\lim_{y\downarrow 0} F(y,u) = F_s$ and $\lim_{y\uparrow 0} F(y,u) = -F_s$.

The closed-loop system (8) is, likewise the open-loop system (1), a system of ordinary differential equations with discontinuous right-hand side. Here, we adopt the solution concept of Filippov to define solutions of the discontinuous system, see e.g. (Filippov, 1988; Leine & Nijmeijer, 2004). In Filippov's solution concept, the discontinuous system is interpreted as a differential inclusion that is obtained through convexification of the discontinuous right-hand side. Existence of solutions in the sense of Filippov is guaranteed but uniqueness of solutions is not automatically ensured.

In our case, the convexification of the right-hand side of the closed-loop system (8) yields the differential inclusion

$$\dot{x} = y, \tag{10a}$$

$$\dot{y} \in -\frac{K_\mathrm{p}}{J}x - \frac{(K_\mathrm{d} + F_\mathrm{v})}{J}y + \frac{1}{J}\overline{\Delta F}(y), \tag{10b}$$

where

$$\overline{\Delta F}(y) \in \begin{cases} \{(\widetilde{g}(y) - g(y))\,\mathrm{sign}(y)\} & \text{if } y \neq 0, \\ [-|\widetilde{F}_\mathrm{s} - F_\mathrm{s}|, |\widetilde{F}_\mathrm{s} - F_\mathrm{s}|] & \text{if } y = 0 \end{cases} \tag{11}$$

is the closed convex hull of ΔF. Following Filippov's solution concept, from now on, we study the dynamics of the closed-loop system (10) instead of (8). Equilibria of the controlled system (10) are given by the set

$$S_\mathrm{E} = \left\{ (x, y) \in \mathbb{R}^2 : |x| \leqslant \frac{|\widetilde{F}_\mathrm{s} - F_\mathrm{s}|}{K_\mathrm{p}}, y = 0 \right\}. \tag{12}$$

Obviously, the equilibrium set S_E contains the origin, which is the setpoint of the controlled system (10).

Observe that in the case of exact friction compensation, i.e. $\widetilde{F}_\mathrm{s} = F_\mathrm{s}$ and $\widetilde{g}(y) = g(y)$, (9) yields $\Delta F(y, u_0) = 0$. Namely, the only possible non-zero value of ΔF is given by $\Delta F(y, u_0) = K_\mathrm{p}x$, which holds if $y = 0$ and $|-K_\mathrm{p}x - F_\mathrm{s}\,\mathrm{sign}(x)| \leqslant F_\mathrm{s}$ that is true only for $x = 0$, thus also results in $\Delta F(y, u_0) = 0$. Consequently, (11) gives $\overline{\Delta F} \in \{0\}$. Hence the controlled system (10) becomes linear and the equilibrium set S_E is reduced to a single equilibrium point at the origin, which is globally exponentially stable since $J, F_\mathrm{v}, K_\mathrm{p}, K_\mathrm{d} > 0$. This fact agrees with the intuition that exact friction compensation linearizes the closed-loop system (10) and it allows to assess the effects of inexact friction compensation.

In the following, we give definitions of undercompensation and overcompensation of friction in a controlled system. The friction force F is said to be undercompensated if both the static friction level and the level of the Stribeck curve of the friction compensation term $\widetilde{F}$ is smaller than those of the friction force F, i.e.

$$\widetilde{F}_\mathrm{s} - F_\mathrm{s} < 0 \quad \text{and} \quad \widetilde{g}(y) - g(y) < 0, \ \forall y \neq 0, \tag{13}$$

and the friction force F is said to be overcompensated if

$$\widetilde{F}_\mathrm{s} - F_\mathrm{s} > 0 \quad \text{and} \quad \widetilde{g}(y) - g(y) > 0, \ \forall y \neq 0. \tag{14}$$

A schematic representation of (13) and (14) is depicted in Fig. 1. The effects of undercompensation and overcompensation of friction to the dynamics of the closed-loop system (10) will be analyzed separately in the next two sections.

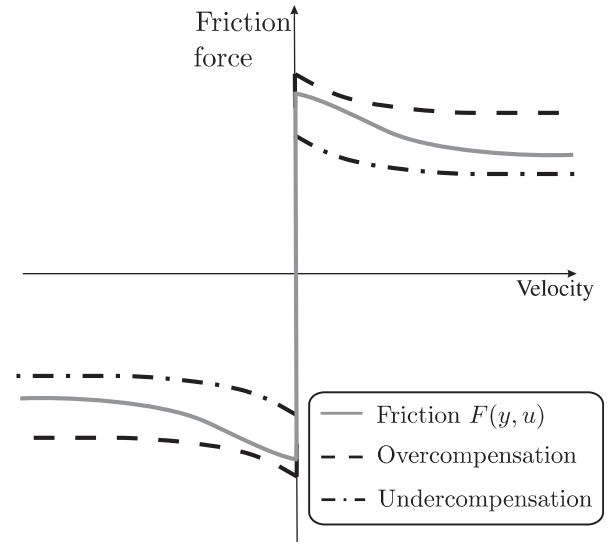

Fig. 1. Schematic representation of undercompensation, i.e. (13), and overcompensation, i.e. (14).

3. Undercompensation of friction

In this section, it will be proven that in the case of undercompensation the origin of the closed-loop system (10) is stable and the equilibrium set S_E is globally attractive. For this purpose, we apply Lyapunov's stability theorem (Shevitz & Paden, 1994) and LaSalle's invariance principle (Adly & Goeleven, 2004; Alvarez et al., 2000).

The invariance principle requires uniqueness of solutions in forward time (Alvarez et al., 2000, Theorem 1). Following Filippov's solution concept, existence of solutions of the differential inclusion (10) is guaranteed but uniqueness of solutions depends on the dynamics near the discontinuity manifold of the vector field (Filippov, 1988; Leine & Nijmeijer, 2004). The discontinuity manifold of (10) is $S = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ with $n = [0 \ \ 1]^\mathrm{T}$ is the corresponding normal vector. The manifold S partitions the state space into $G^- = \{(x, y) \in \mathbb{R}^2 : y < 0\}$ and $G^+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. The projections of the vector field in G^+ and G^- on the normal vector n at the discontinuity manifold S are given by

$$n^\mathrm{T} f^+(x, y) = -\frac{K_\mathrm{p}}{J}x + \frac{\widetilde{F}_\mathrm{s} - F_\mathrm{s}}{J}, \quad \forall (x, y) \in S$$

and

$$n^\mathrm{T} f^-(x, y) = -\frac{K_\mathrm{p}}{J}x - \frac{\widetilde{F}_\mathrm{s} - F_\mathrm{s}}{J}, \quad \forall (x, y) \in S,$$

respectively.

Solutions of the system (10) cross the discontinuity manifold S transversally if and only if

$$n^\mathrm{T} f^+(x, y) \cdot n^\mathrm{T} f^-(x, y) > 0 \Longrightarrow K_\mathrm{p}|x| > |\widetilde{F}_\mathrm{s} - F_\mathrm{s}|.$$

Therefore, these transversal intersections occur at

$$S_\mathrm{T} = \left\{ (x, y) \in \mathbb{R}^2 : |x| > \frac{|\widetilde{F}_\mathrm{s} - F_\mathrm{s}|}{K_\mathrm{p}}, \ y = 0 \right\}. \tag{15}$$

Note that $S_T = S \backslash S_E$. Repulsive sliding modes occur at the discontinuity manifold S if $n^T f^+(x, y) > 0$ and $n^T f^-(x, y) < 0$, which results in

$$K_p|x| < (\widetilde{F}_s - F_s). \tag{16}$$

Since $K_p > 0$ and in the undercompensation case $\widetilde{F}_s - F_s < 0$, the inequality (16) never holds and thus repulsive sliding modes never occur. The absence of repulsive sliding modes and the fact that $S = S_E \cup S_T$ guarantee uniqueness of solutions of the closed-loop system (10) on the discontinuity manifold S (Leine & Nijmeijer, 2004, Section 3.3). Hence, we can conclude that the closed-loop system (10) has unique solutions in forward time. Now, we are ready to state a result on the global attractivity of the equilibrium set S_E.

Theorem 1. *The origin of the closed-loop system* (10) *is globally stable and the equilibrium set S_E given by* (12) *is globally attractive if the friction force is undercompensated, i.e. condition* (13) *holds.*

Proof. Consider the Lyapunov function candidate

$$V(x, y) = \frac{K_p}{2}x^2 + \frac{J}{2}y^2. \tag{17}$$

The time-derivative of $V(x, y)$ along trajectories of the closed-loop system (10) is given by

$$\dot{V}(x, y) = -(K_d + F_v)y^2 + \overline{\Delta F}(y)y. \tag{18}$$

From the condition (13), it can be shown that

$$\overline{\Delta F}(y)y < 0, \quad \forall y \neq 0 \quad \text{and} \quad \overline{\Delta F}(y)y = 0 \quad \text{iff } y = 0. \tag{19}$$

Substitution of (19) into (18), yields

$$\dot{V}(x, y) \leqslant -(K_d + F_v)y^2. \tag{20}$$

The existence of the Lyapunov function (17) with its time-derivative satisfying (20) proves that the origin is globally stable (Shevitz & Paden, 1994). Furthermore, $\dot{V}(x, y) = 0$ only in the set S and the equilibrium set S_E is the largest invariant set of (10) contained in the set S. Because the controlled system (10) has unique solutions in forward time, the invariance principle (Alvarez et al., 2000, Theorem 1) can be applied to conclude that all trajectories of the system (10) converge to the equilibrium set S_E. Hence, the equilibrium set S_E is globally attractive. □

Theorem 1 indicates that undercompensation of friction leads to steady-state errors, which are bounded by $|\widetilde{F}_s - F_s|/K_p$ due to the size of the equilibrium set S_E. Limit cycling, however, never occurs. This result on the undercompensation case can be extended to a multi-degree of freedom system with multiple friction forces because in this case the equilibrium set is due to the remaining friction forces. By choosing an appropriate Lyapunov function, for example as proposed in van de Wouw and Leine (2004), and applying LaSalle's invariant principle a similar result can be obtained.

4. Overcompensation of friction

The objective of this section is to provide a rigorous analysis showing that overcompensation of friction in the controlled system (10) may provoke limit cycling around the setpoint. The analysis is based on the properties of the ω-limit set of trajectories of a two-dimensional differential inclusion. Here, we adopt the definition of ω-limit sets given in Filippov (1988, p. 129). In order to prove that the system (10) exhibits limit cycling, it is sufficient to show that the ω-limit set of trajectories of (10) contains an isolated closed orbit. The following theorem, which is proven in Filippov (1988, Theorem 3, p. 137), is used to achieve this goal.

Theorem 2. *Consider a two-dimensional autonomous differential inclusion*

$$\dot{z} \in F(z) \tag{21}$$

with $F(z)$ a set-valued function that is closed, convex and bounded for all $z \in \mathbb{R}^2$ and the function F is upper semi-continuous. Suppose that uniqueness of solutions in forward time holds at any point on a trajectory $\Gamma = \{z \in \mathbb{R}^2 : z = \varphi(t), t \in [0, \infty)\}$ of (21). *If the ω-limit set of Γ is bounded and contains no equilibrium points then it consists of one closed orbit.*

The right-hand side of the closed-loop system (10) satisfies the conditions of Theorem 2 because it is obtained from Filippov's convexification method. Next, we state a result on boundedness of trajectories of the system (10).

Proposition 3. *The ω-limit set of all trajectories of the closed-loop system* (10) *is bounded if the friction force is overcompensated, i.e. condition* (14) *holds.*

Proof. Consider the positive definite function

$$V(x, y) = \tfrac{1}{2}(F_v x + Jy)^2 + \tfrac{1}{2}(K_p J + K_d F_v)x^2 \tag{22}$$

that is radially unbounded. Its time-derivative along trajectories of (10) is given by

$$\dot{V}(x, y) \in -K_p F_v x^2 + F_v \overline{\Delta F}(y)x - K_d J y^2 + J\overline{\Delta F}(y)y. \tag{23}$$

Using the property of the Stribeck curve, $F_c \leqslant g(y) \leqslant F_s$ and (11), it can be shown that

$$\overline{\Delta F}(y)x \leqslant (\widetilde{F}_s - F_c)|x| \quad \text{and} \quad \overline{\Delta F}(y)y \leqslant (\widetilde{F}_s - F_c)|y|. \tag{24}$$

Substitution of (24) into (23) yields

$$\dot{V}(x, y) \leqslant -K_p F_v x^2 + F_v(\widetilde{F}_s - F_c)|x| - K_d J y^2 + J(\widetilde{F}_s - F_c)|y|. \tag{25}$$

Following (25), $\dot{V}(x, y) < 0$ if

$$K_p F_v x^2 + K_d J y^2 > (\widetilde{F}_s - F_c)\,(F_v|x| + J|y|)\,. \tag{26}$$

Since $\widetilde{F}_s - F_c > 0$, inequality (26) holds for all pairs (x, y) that are sufficiently separated from the origin because the left-hand

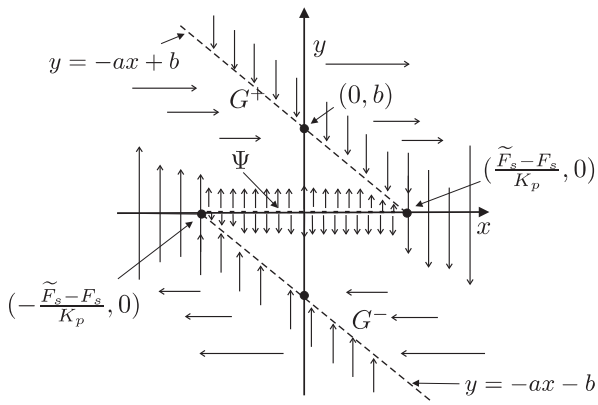

Fig. 2. The vector field of (10) in the case of overcompensation of friction, where $a = K_p/(K_d + F_v)$ and $b = (\widetilde{F}_s - F_s)/(K_d + F_v)$.

side of the inequality is a quadratic function of x and y while the right-hand side is a linear function of the absolute values of x and y. Therefore, trajectories of the closed-loop system (10) cannot grow unbounded in forward time. □

In the following, we find the conditions on the closed-loop system (10) such that the ω-limit set of its trajectories does not contain any equilibrium points. It has been shown in the previous section that repulsive sliding modes occur at the discontinuity manifold S if the inequality (16) holds. Since for the overcompensation case $\widetilde{F}_s - F_s > 0$, following (16) repulsive sliding modes occur at the segment

$$\Psi = \left\{ (x, y) \in \mathbb{R}^2 : |x| < \frac{|\widetilde{F}_s - F_s|}{K_p}, \quad y = 0 \right\} \tag{27}$$

of the discontinuity manifold S. Note that the equilibrium set S_E can be stated as

$$S_E = \left(-\frac{\widetilde{F}_s - F_s}{K_p}, 0 \right) \cup \Psi \cup \left(\frac{\widetilde{F}_s - F_s}{K_p}, 0 \right). \tag{28}$$

Therefore, the set Ψ is an unstable equilibrium set of (10) and it can be concluded that the ω-limit set of all trajectories of (10) starting at $(x_0, y_0) \in \mathbb{R}^2 \backslash S_E$ does not contain the set Ψ but it may contain one or both of the extremal equilibrium points $(-(\widetilde{F}_s - F_s)/K_p, 0)$ and $((\widetilde{F}_s - F_s)/K_p, 0)$.

Next, possible convergence of trajectories to those two extremal equilibrium points is investigated through a phase-plane analysis as depicted in Fig. 2. The projection of the vector field of (10) on the normal $m = [1 \;\; 0]^T$ to the y-axis is

$$m^T f(x, y) = y \tag{29}$$

such that $m^T f(x, y) < 0$ for all $(x, y) \in G^-$ and $m^T f(x, y) > 0$ for all $(x, y) \in G^+$. The projection of the vector field on the normal $n = [0 \;\; 1]^T$ to the x-axis is given by

$$n^T f(x, y) \in -\frac{K_p}{J} x - \frac{(K_d + F_v)}{J} y + \frac{1}{J} \overline{\Delta F}(y). \tag{30}$$

Since in the overcompensation case we have that $\overline{\Delta F}(y) \leqslant \widetilde{F}_s - F_c$, $\forall y \geqslant 0$ and $\overline{\Delta F}(y) \geqslant -(\widetilde{F}_s - F_c)$, $\forall y \leqslant 0$, (30) yields

$$n^T f(x, y) < 0 \quad \text{if } y > \frac{-K_p}{K_d + F_v} x + \frac{\widetilde{F}_s - F_c}{K_d + F_v} \quad \text{and} \quad y \geqslant 0,$$

$$n^T f(x, y) > 0 \quad \text{if } y < \frac{-K_p}{K_d + F_v} x - \frac{\widetilde{F}_s - F_c}{K_d + F_v} \quad \text{and} \quad y \leqslant 0.$$

In the previous section, we have shown that trajectories of the closed-loop system (10) cross the x-axis transversally at S_T given by (15).

The phase-plane analysis shows that the extremal equilibrium points $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ and $(-\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ can only be reached from G^+ and G^-, respectively. The dynamics of (10) in G^+ reduces to

$$\dot{x} = y, \tag{31a}$$

$$\dot{y} = -\frac{K_p}{J} x - \frac{K_d + F_v}{J} y + \frac{\Delta g(y)}{J}, \tag{31b}$$

with $\Delta g(y) = \widetilde{g}(y) - g(y)$. Let us approximate the Stribeck functions $\widetilde{g}(y)$ and $g(y)$ by a Taylor expansion such that $\Delta g(y)$ can be approximated by

$$\Delta g(y) = \widetilde{g}(0) - g(0) + (\widetilde{g}'(0) - g'(0))y + \text{h.o.t.}, \tag{32}$$

where $\widetilde{g}(0) = \widetilde{F}_s$, $g(0) = F_s$, $\widetilde{g}'(y) = \frac{\partial \widetilde{g}(y)}{\partial y}$ and $g'(y) = \frac{\partial g(y)}{\partial y}$. Note that for the Stribeck functions (3) and (4) this approximation is possible only for $\delta \geqslant 1$ because $g'(0) = 0$ if $\delta > 1$, $g'(0) = -\frac{F_s - F_c}{v_s}$ if $\delta = 1$ and $g'(0)$ is not defined if $\delta < 1$. Hence, for the case where $\widetilde{g}'(0)$ and $g'(0)$ are well-defined, the system (31) around $y \gtrapprox 0$ can be approximated by the linear system

$$\dot{x} = y, \tag{33a}$$

$$\dot{y} = -\frac{K_p}{J} x - \frac{K_d + F_v - \widetilde{g}'(0) + g'(0)}{J} y + \frac{\widetilde{F}_s - F_s}{J}. \tag{33b}$$

Note that the extremal equilibrium point $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ coincides with the equilibrium point of the linear system (33). Because of the symmetry of the vector field about the y-axis a similar linear approximation also holds for the extremal equilibrium point $(-\frac{\widetilde{F}_s - F_s}{K_p}, 0)$. The linear approximation allows to investigate the possible convergence of trajectories of the closed-loop system (10) to the extremal equilibrium points such that the following result can be concluded.

Theorem 4. *Consider the closed-loop system* (10) *in the case of overcompensation of friction, i.e. condition* (14) *holds, and assume that* $\widetilde{g}'(0) = \frac{\partial \widetilde{g}(y)}{\partial y}|_{y=0}$ *and* $g'(0) = \frac{\partial g(y)}{\partial y}|_{y=0}$ *are well-defined. The* ω*-limit set of any trajectory of the closed-loop system* (10), *starting away from the equilibrium set* S_E, *consists of one closed orbit that encircles the equilibrium set* S_E *if the inequality*

$$K_d + F_v - \widetilde{g}'(0) + g'(0) > 0 \tag{34}$$

is violated or if both (34) *and*

$$(K_d + F_v - \widetilde{g}'(0) + g'(0))^2 < 4K_p J \tag{35}$$

hold. But if only (35) *is violated, the* ω*-limit set does not contain such a closed orbit but it may consist of one of the extremal equilibrium points* $(-\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ *or* $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$.

Proof. Since $J, K_p > 0$, applying the Hurwitz condition, the linear approximation (33) is stable if and only if the inequality (34) holds. Thus, if (34) is violated the linear system (33) becomes unstable and following the phase plane analysis trajectories of the closed-loop system (10) will not converge to the extremal equilibrium points. Consequently the ω-limit set of any trajectory of (10), starting outside the equilibrium set S_E, does not contain any equilibrium points. Applying Theorem 2 and Proposition 3 the ω-limit set consists of one closed orbit if uniqueness of solutions holds any point along those trajectories. It has been shown in Section 3 that transversal intersections occur at the segment S_T of the discontinuity manifold S and uniqueness of solutions in forward time holds at any point on S_T. Since $S = S_E \cup S_T$ and the trajectories do not contain any point in S_E, we can conclude that uniqueness of solutions in forward time holds at any point along those trajectories and the first part of the theorem is proven.

If the Hurwitz condition (34) holds, trajectories in G^+ may eventually converge to the extremal equilibrium point $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$. However, if the inequality (35) holds the dynamics of the linear system (33) are undercritically damped (it has a pair of complex eigenvalues) such that those trajectories will oscillate before converging to the equilibrium point. Note that the dynamics (33) hold only in G^+ and once a trajectory crosses the x-axis it will move away from the x-axis towards the region G^- as depicted in Fig. 2. Because the vector field in G^- and in G^+ are symmetric the same scenario takes place and the cycle repeats such that the two extremal equilibrium points cannot be reached neither in finite time nor in infinite time. Hence, the ω-limit set of those trajectories does not contain any equilibrium points. By applying the same reasoning as in the first part, we can conclude the second part of the theorem.

If only the inequality (35) is violated, the dynamics of the linear system (33) become supercritically damped, i.e. it has two real eigenvalues, such that trajectories in G^+ converge exponentially fast to the extremal equilibrium point $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ without oscillation. Therefore, the extremal equilibrium point can be reached in infinite time. This result also holds for the other extremal equilibrium point due to the symmetry of the vector field. Hence, the ω-limit set of trajectories of the closed-loop system (10) may consist of one of those two extremal equilibrium points. □

Theorem 4 indicates that overcompensation of friction may provoke limit cycling and that the limit cycling effect can be eliminated by tuning the gains of the PD controller, i.e. choose K_p and K_d satisfying the Hurwitz condition (34) and violating the inequality (35). This limit cycling result cannot be extended to a multi-degree of freedom system because it is based on Theorem 2, which is valid only for two-dimensional systems. However, the result on boundedness of the ω-limit set, Proposition 3, can possibly be extended to a multi-degree of freedom system by using a similar approach. The sliding-mode analysis of the discontinuous manifold, see for example (Leine & Nijmeijer, 2004), and the local stability analysis of the extremal equilibrium points can also be applied to investigate possible convergence of trajectories of a multi-degree of freedom frictional system to an equilibrium point. Such extended analysis can predict whether trajectories of a controlled system converge to an attractor—not necessarily a closed orbit—or to an equilibrium point as the result of overcompensation of friction.

5. A numerical example

This section provides numerical illustrations of the theoretical results obtained in the previous two sections. For this purpose we consider the 1DOF mechanical system studied in Putra and Nijmeijer (2004). The dynamics of the system can be described by (1) with $g(y) = F_c + (F_s - F_c)e^{-(y/v_s)^2}$ and the parameter values: $J = 0.0260\,\text{kg m}^2$, $F_v = 0.0710\,\text{Nm s/rad}$, $F_c = 0.4195\,\text{Nm}$, $F_s = 0.5005\,\text{Nm}$ and $v_s = 0.15\,\text{rad/s}$. The friction compensation is given by (6) with $\widetilde{F}_s = \alpha F_s$ and $\widetilde{g}(y) = \alpha g(y)$, where $\alpha > 0$ is a scaling factor. Following the definitions in Section 2, we have undercompensation case if $\alpha < 1$ and overcompensation case if $\alpha > 1$.

Solutions of the closed-loop system are obtained numerically using the so-called switch-model approximation for the dynamics around the discontinuity manifold S, see e.g. (Leine, van Campen, de Kraker, & van den Steen, 1998; Putra & Nijmeijer, 2004). A phase portrait showing an attracting equilibrium set of the controlled system (10), with $\alpha = 0.8$ (20% undercompensation) and the PD controller gains set to $K_p = 0.1$ and $K_d = 0.1$, is depicted in Fig. 3. This simulation result agrees with Theorem 1. Fig. 4(a) depicts a phase portrait showing an asymptotically stable closed orbit of the closed-loop system (10) with $\alpha = 1.2$ (20% overcompensation) in the undercritically damped

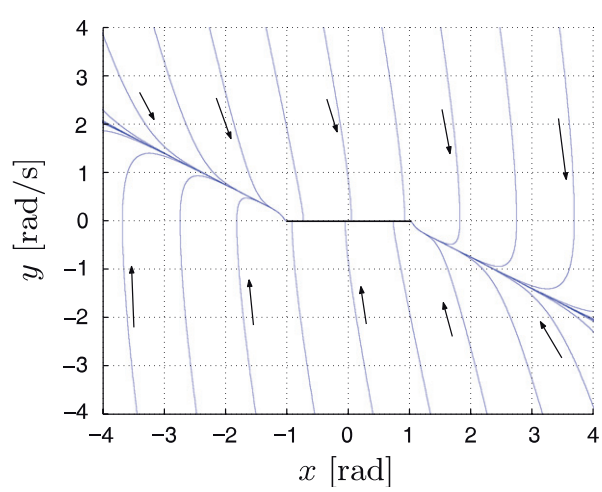

Fig. 3. Phase portrait of the controlled system (10) with $\alpha = 0.8$ (undercompensation), $K_p = 0.1$ and $K_d = 0.1$.

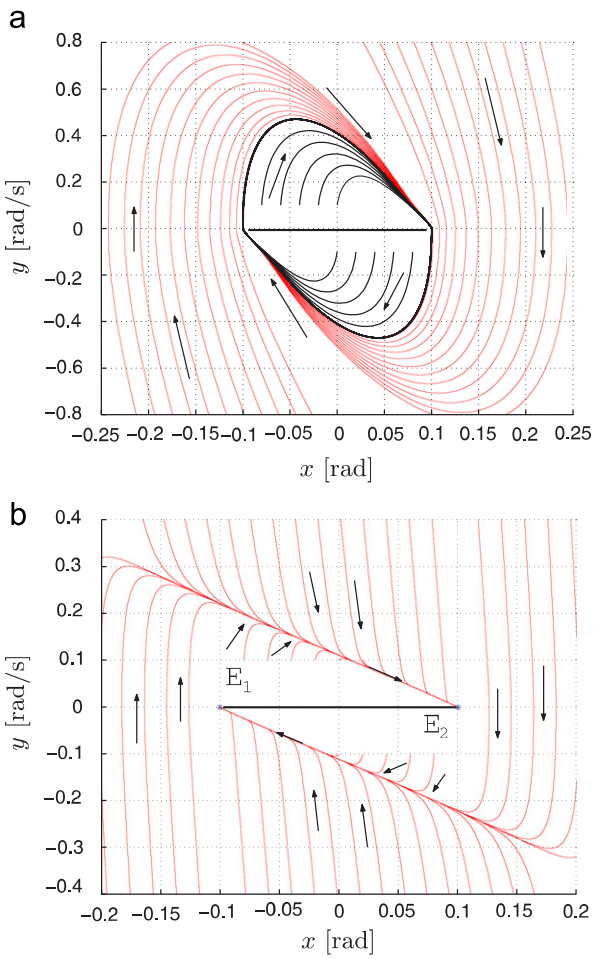

Fig. 4. Phase portrait of the system (10) with $\alpha = 1.2$ (overcompensation): (a) the undercritically damped case with $K_p = 1$ and $K_d = 0.2$, and (b) the supercritically damped case with $K_p = 1$ and $K_d = 0.8$, E_1 and E_2 are the extremal equilibrium points.

case, with $K_p = 1$ and $K_d = 0.2$. The closed orbit comes closer to the extremal points of the equilibrium set S_E as it is about to cross the y-axis but does not hit them such that the closed orbit encircles the equilibrium set S_E. On the other hand, Fig. 4(b) depicts a phase portrait of (10) with the same value of α in the supercritically damped case, with $K_p = 1$ and $K_d = 0.8$. The phase portrait shows two attracting extremal equilibrium points of the equilibrium set S_E. The last two simulation results confirm the prediction of Theorem 4.

6. Conclusions

We have investigated the negative effects of undercompensation and overcompensation of friction in PD controlled 1DOF mechanical systems for a class of discontinuous friction models consisting of static, Coulomb and viscous friction, and including the Stribeck effect. It has been proven that undercompensation of friction in 1DOF controlled mechanical systems results in a globally attracting equilibrium set containing the setpoint, which is globally stable. This result indicates that the controlled system may exhibit steady-state errors and that limit cycling effect never occurs. The steady-state error is bounded by the size of the equilibrium set, which can be influenced by tuning the proportional gain of the PD controller.

It also has been rigorously proven that overcompensation of friction in the same controlled mechanical systems provokes limit cycling in case the linearized dynamics of the controlled systems around the extremal equilibrium points are undercritically damped. However, such a limit cycling effect disappears if the PD controller gains are tuned such that the linearized dynamics become supercritically damped. Since the analysis involves the linearized dynamics around the extremal equilibrium points, this result is valid only for discontinuous friction models whose the first partial derivative of the Stribeck function is well-defined locally at zero velocity. The predictions of the theoretical results have been demonstrated by a numerical example. Furthermore, possible extensions of the results to multi-degree of freedom systems are also indicated.

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