Automatica 46 (2010) 615–619

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

On polytopic inclusions as a modeling framework for systems with time-varying delays[☆]

R.H. Gielen[a,∗], S. Olaru[b], M. Lazar[a], W.P.M.H. Heemels[a], N. van de Wouw[a], S.-I. Niculescu[c]

[a] Eindhoven University of Technology, Eindhoven, The Netherlands

[b] Département Automatique, Supélec, Gif-sur-Yvette, France

[c] Laboratoire des Signaux et Systèmes, CNRS-Supélec, Gif-sur-Yvette, France

ARTICLE INFO

Article history:
Received 12 September 2008
Received in revised form
20 October 2009
Accepted 23 November 2009
Available online 6 February 2010

Keywords:
Time-varying input delay
Polytopic inclusions
Networked control systems

ABSTRACT

One of the important issues in networked control systems is the appropriate handling of the nonlinearities arising from uncertain time-varying delays. In this paper, using the Cayley–Hamilton theorem, we develop a novel method for creating discrete-time models of linear systems with time-varying input delays based on polytopic inclusions. The proposed method is compared with existing approaches in terms of conservativeness, scalability and suitability for controller synthesis.

1. Introduction

Surveys on future directions in control, e.g. Murray (2002), identify networked control systems (NCS) to be one of the emerging key topics in control. The distinguishing feature of NCS is that the connection between plant and controller is made through a communication network. As discussed in the comprehensive NCS overviews (Hespanha, Naghshtabrizi & Xu, 2007; Tipsuwan & Chow, 2003), this brings specific, additional challenges to controller design, such as the presence of uncertain time-varying delays, timing jitter and packet dropouts.

This paper focusses on the problem of obtaining a discrete-time model of a linear system affected by time-varying input delays that is suitable for stabilizing controller synthesis. Interestingly, in Cloosterman, van de Wouw, Heemels and Nijmeijer (2006) it was shown that time-varying input delays may destabilize systems that cannot be destabilized by constant delays of a similar size. This indicates that time-varying input delays must be taken into account in the controller synthesis procedure. Unfortunately, even for linear system dynamics, closing the control loop via a communication network subject to time-varying input delays yields a nonlinear appearance of the delay in the closed-loop discrete-time dynamics, which hampers the implementation of most classical robust controller synthesis methods.

Recently, an increasingly popular solution to this problem was obtained by over-approximating the delay-induced nonlinearity with a polytopic inclusion, see Cloosterman et al. (2006); Cloosterman, van de Wouw, Heemels and Nijmeijer (2007, 2009), Hetel, Daafouz, and Iung (2006) and Olaru and Niculescu (2008). The advantage of this approach is that the resulting closed-loop dynamics becomes a polytopic difference inclusion for which efficient stabilizing controller design techniques exist. This technical communique proposes a novel approach for finding a polytopic over-approximation of the delay-induced nonlinearity arising from time-varying delays, based on the Cayley–Hamilton theorem. The novel method is compared with existing ones in terms of conservativeness and scalability. Furthermore, suitability of the resulting models for controller synthesis is illustrated using the control scheme of Kothare, Balakrishnan and Morari (1996), which can handle polytopic difference inclusions.

☆ The authors would like to acknowledge the financial support from STW-NWO Veni grant 10230, Van Gogh grant "Predictive control over networks" and the thematic European Union programme FP7-ICT-2007-2 under the WIDE-224168 project. The material in this paper was partially presented at International workshop on Assessment and Future Directions of Nonlinear Model Predictive Control, Pavia, Italy, September 5–9, 2008. This paper was recommended for publication in revised form by Associate Editor Mikael Johansson under the direction of Editor André L. Tits. We are grateful to the associate editor and the reviewers for their helpful comments. We are grateful in particular to one of the anonymous reviewers whose suggestions led to the results presented in Section 3.2.

∗ Corresponding address: Eindhoven University of Technology, P.O. Box 513, 5600MB Eindhoven, The Netherlands. Tel.: +31 40 247 3142; fax: +31 40 243 4582.

E-mail addresses: r.h.gielen@tue.nl (R.H. Gielen), sorin.olaru@supelec.fr (S. Olaru), m.lazar@tue.nl (M. Lazar), w.p.m.h.heemels@tue.nl (W.P.M.H. Heemels), n.v.d.wouw@tue.nl (N. van de Wouw), silviu.niculescu@lss.supelec.fr (S.-I. Niculescu).

0005-1098/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2010.01.002

1.1. Basic notation and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{C}$ denote the field of real numbers, the set of non-negative reals, the set of integers, the set of non-negative integers and the field of complex numbers, respectively. For every subset Π of $\mathbb{R}$ we define $\mathbb{R}_\Pi := \{k \in \mathbb{R} \mid k \in \Pi\}$ and $\mathbb{Z}_\Pi := \{k \in \mathbb{Z} \mid k \in \Pi\}$. A polyhedron, or a polyhedral set, in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A polytope is a closed and bounded polyhedron. Let $\mathrm{Co}(\cdot)$ denote the convex hull. For two arbitrary matrices $A, B \in \mathbb{R}^{n\times n}$, let $\mathrm{Span}(A, B) := \{\mu_1 A + \mu_2 B \mid \mu_1, \mu_2 \in \mathbb{R}\}$ and let $[A]_{i,j}$ denote the ijth entry of A. For two arbitrary sets $\mathcal{S}, \mathcal{P} \subseteq \mathbb{R}^n$, let $\mathcal{S} \oplus \mathcal{P} := \{x + y | x \in \mathcal{S}, y \in \mathcal{P}\}$ denote their *Minkowski addition* and let $\mathcal{S} \times \mathcal{P} := \{(x, y) | x \in \mathcal{S}, y \in \mathcal{P}\}$ denote their *Cartesian product*.

2. Problem formulation

Consider the continuous-time system with input delay

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c u(t), \\ u(t) &= u_k, \quad \forall t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1}), \end{aligned} \tag{1}$$

where $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$, $x(t) \in \mathbb{R}^n$ is the system state and $u(t) \in \mathbb{R}^m$ is the system input at time $t \in \mathbb{R}_+$. $u_k \in \mathbb{R}^m$ is the control action generated at time $t = t_k$ and we assume that $u(t) \in \mathbb{R}^m$ is known for all $t \in [0, \tau_0)$. Furthermore, $t_k = kT_s$, $k \in \mathbb{Z}_+$, are the sampling instants, $T_s \in \mathbb{R}_+$ is the sampling period, $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$ denotes the delay induced by the network at sample time t_k and $\underline{\tau} \in \mathbb{R}_{[0,\overline{\tau}]}$, $\overline{\tau} \in \mathbb{R}_{[\underline{\tau},T_s]}$ are the minimal and maximal possible delays, respectively. Observe that for clarity of exposition we considered the case when the delay is smaller than or equal to the sampling time. However, the techniques developed in this paper apply straightforwardly to NCS models involving large delays, packet dropouts and timing jitter, at the expense of an increase in complexity. Next, consider the exact discretization of (1), i.e.

$$\begin{aligned} x_{k+1} = {} & \mathrm{e}^{A_c T_s} x_k + \int_0^{\tau_k} \mathrm{e}^{A_c(T_s-\theta)} \mathrm{d}\theta B_c u_{k-1} \\ & + \int_{\tau_k}^{T_s} \mathrm{e}^{A_c(T_s-\theta)} \mathrm{d}\theta B_c u_k \end{aligned} \tag{2}$$

and let

$$\Delta(\tau_k) := \int_0^{\tau_k} \mathrm{e}^{A_c(T_s-\theta)} \mathrm{d}\theta B_c, \quad k \in \mathbb{Z}_+. \tag{3}$$

Furthermore, by manipulating (2) and introducing a new augmented state vector, i.e. $\xi_k := [x_k^\top u_{k-1}^\top]^\top$, we obtain

$$\xi_{k+1} = A(\Delta(\tau_k))\xi_k + B(\Delta(\tau_k))u_k, \tag{4}$$

with $A(\Delta(\tau_k)) := \begin{bmatrix} A_d & \Delta(\tau_k) \\ 0 & 0 \end{bmatrix}$, $B(\Delta(\tau_k)) := \begin{bmatrix} B_d - \Delta(\tau_k) \\ I_m \end{bmatrix}$, $A_d = \mathrm{e}^{A_c T_s}$ and $B_d = \int_0^{T_s} \mathrm{e}^{A_c(T_s-\theta)} \mathrm{d}\theta B_c$. Now (4) is a difference inclusion with $\Delta(\tau_k)$ lying in some non-convex subset of $\mathbb{R}^{n\times m}$. The goal is to construct a set of matrices

$$\Delta := \mathrm{Co}(\{\overline{\Delta}_l\}_{l\in\mathbb{Z}_{[1,L]}}), \quad \overline{\Delta}_l \in \mathbb{R}^{n\times m}, \tag{5}$$

such that $\Delta(\tau_k) \in \Delta$ for all $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$. Notice that $\overline{\Delta}_l$ are the vertices (generators) of the convex parameter set Δ and $L \in \mathbb{Z}_+$ is the number of generators.

2.1. Existing solutions

We present a brief overview of four existing methods for finding the generators of the set in (5). In Cloosterman et al. (2006) an elementwise minimization–maximization is proposed (referred to by EMM) leading to matrices $\overline{\Delta}_l$ that contain all possible combinations of maxima and minima for all entries of $\Delta(\tau_k)$. In Cloosterman et al. (2007, 2009) and Olaru and Niculescu (2008), methods based on the Jordan normal form (JNF) of the matrix A_c are proposed. The generators of (5) are obtained by solving algebraic expressions involving the delay. The foremost difference between Cloosterman et al. (2007, 2009) (referred to by JNF1) and Olaru and Niculescu (2008) (referred to by JNF2) is that in the latter a method is proposed to reduce the number of generators at the cost of a larger polytope. The fourth method was proposed in Hetel et al. (2006) (referred to by TA) and makes use of a Taylor series expansion of (3). The generators of (5) are obtained by solving a linear system of equalities. The infinite sum of the Taylor series expansion is approximated by a finite number of terms p, which is also the number of generators for Δ, i.e. $L = p$. The remaining terms of the Taylor series expansion are dealt with via an additional uncertainty block in the model.

It is worth to mention that another class of over-approximation methods that can be applied to NCS, but is not considered in this paper, is based on gridding and norm bounding; see, e.g. Suh (2008).

3. Main result

Consider the Cayley–Hamilton theorem.

Theorem 1 (*Cayley (1857)*). *Let the characteristic polynomial of a matrix $A \in \mathbb{R}^{n\times n}$ be denoted by $q(\lambda) := \det(\lambda I_n - A)$. Then $q(A) = 0$.* □

The polynomial $q_m(\lambda)$ of lowest degree that satisfies $q_m(A) = 0$ is called the minimal polynomial. Clearly, based on Theorem 1, the degree of the minimal polynomial is smaller than or equal to n. Based on the Jordan form, the minimal polynomial can be derived Usmani (1987) as $q_m(\lambda) = \prod_{r=1}^{p}(\lambda - \lambda_r)^{n_r}$, where $p \in \mathbb{Z}_+$ is the number of *distinct* eigenvalues of A, $\lambda_r \in \mathbb{C}$ are the distinct eigenvalues of A, and $n_r \in \mathbb{Z}_+$ is the maximal order of a Jordan block corresponding to eigenvalue λ_r. Note that $\nu := \sum_{r=1}^{p} n_r \le n$. As for the minimal polynomial it holds that $q_m(A) = 0$, it is possible to express all powers of A of order ν and higher as a combination of the first ν powers of A, i.e. for all $i \in \mathbb{Z}_{\ge\nu}$

$$A^i = c_{i,0} I + \cdots + c_{i,\nu-1} A^{\nu-1}, \tag{6}$$

for some $c_{i,j} \in \mathbb{R}$ and $j \in \mathbb{Z}_{[0,\nu-1]}$. Furthermore, for all $i \in \mathbb{Z}_{[0,\nu-1]}$ and $j \in \mathbb{Z}_{[0,\nu-1]}$ let $c_{i,j} = 1$ when $i = j$ and $c_{i,j} = 0$ when $i \ne j$. To obtain a simplified expression for (3), let

$$f_j(T_s - \theta) := \sum_{i=0}^{\infty} \frac{c_{i,j}}{i!}(T_s - \theta)^i, \quad j \in \mathbb{Z}_{[0,\nu-1]}. \tag{7}$$

Lemma 2. *Let $g_j(\tau_k) := \int_0^{\tau_k} f_j(T_s - \theta)\mathrm{d}\theta$ and $f_j(T_s - \theta)$ as defined in (7) for all $j \in \mathbb{Z}_{[0,\nu-1]}$. Then*

$$\Delta(\tau_k) = \sum_{j=0}^{\nu-1} g_j(\tau_k) A_c^j B_c. \tag{8}$$

Proof. Starting from (3) and using $e^{A_c s} := \sum_{i=0}^{\infty} \frac{(A_c s)^i}{i!}$ yields $\Delta(\tau_k) = \int_0^{\tau_k} \sum_{i=0}^{\infty} \frac{(T_s-\theta)^i}{i!} A_c^i B_c \mathrm{d}\theta$. Applying (6) and reordering yield

$$\Delta(\tau_k) = \int_0^{\tau_k} \sum_{i=0}^{\infty} \frac{(T_s-\theta)^i}{i!} \sum_{j=0}^{\nu-1} c_{i,j} A_c^j B_c \mathrm{d}\theta = \sum_{j=0}^{\nu-1} \int_0^{\tau_k} \sum_{i=0}^{\infty} \frac{c_{i,j}}{i!}(T_s-\theta)^i \mathrm{d}\theta A_c^j B_c. \quad \square$$

Given $\underline{g}_j \in \mathbb{R}$ and $\overline{g}_j \in \mathbb{R}$ such that $\underline{g}_j \leq g_j(\tau_k) \leq \overline{g}_j$ for all $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$, one can use (8) to write (3) as a convex combination of a finite number of matrices $\overline{\Delta}_l$. Let the set $\mathcal{V}_j := \{\underline{g}_j, \overline{g}_j\}$ for all $j \in \mathbb{Z}_{[0,\nu-1]}$ and let $\mathcal{V} := \mathcal{V}_0 \times \cdots \times \mathcal{V}_{\nu-1} \in \mathbb{R}^\nu$. Furthermore, let $v_i \in \mathcal{V}$ denote the ith element of the set $\mathcal{V}$ and let $[v_i]_j$ denote the jth element of the vector $v_i \in \mathcal{V}$.

Theorem 3. *For any $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$, $\Delta(\tau_k)$ satisfies*

$$\Delta(\tau_k) \in Co(\overline{\Delta}_0, \ldots, \overline{\Delta}_{2^\nu-1}), \tag{9}$$

where $\overline{\Delta}_i := \sum_{j=0}^{\nu-1} [v_i]_{j+1} A_c^j B_c$ for all $i \in \mathbb{Z}_{[0,2^\nu-1]}$. Furthermore, for any $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$, $\Delta(\tau_k)$ also satisfies

$$\Delta(\tau_k) \in Co(\nu\tilde{\Delta}_0, \ldots, \nu\tilde{\Delta}_{2\nu-1}), \tag{10}$$

where $\tilde{\Delta}_j := \underline{g}_j A_c^j B_c$, $\tilde{\Delta}_{j+\nu} := \overline{g}_j A_c^j B_c$, $\forall j \in \mathbb{Z}_{[0,\nu-1]}$.

Proof. Let $\underline{g}_j$ and $\overline{g}_j$ be given such that $\underline{g}_j \leq g_j(\tau_k) \leq \overline{g}_j$ for all $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$ and $j \in \mathbb{Z}_{[0,\nu-1]}$. From (8) it then follows that for all $j \in \mathbb{Z}_{[0,\nu-1]}$ there exist $\mu_j \in \mathbb{R}_{[0,1]}$ so that $\Delta(\tau_k) = \sum_{j=0}^{\nu-1} \left(\mu_j \underline{g}_j + (1-\mu_j)\overline{g}_j\right) A_c^j B_c$. Because $\mu_j \in \mathbb{R}_{[0,1]}$ is itself a convex combination of 0 and 1 the generators can be obtained for $\mu_j \in \{0, 1\}$ for all $j \in \mathbb{Z}_{[0,\nu-1]}$. This leads to the 2^ν possible combinations defined as the generators of the set in (9), thus proving (9). To prove (10), let $\underline{g}_j$ and $\overline{g}_j$ be given such that $\underline{g}_j \leq g_j(\tau_k) \leq \overline{g}_j$ for all $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$ and $j \in \mathbb{Z}_{[0,\nu-1]}$. Then, for all $j \in \mathbb{Z}_{[0,\nu-1]}$ there exists $\mu_j \in \mathbb{R}_{[0,1]}$ such that

$$\Delta(\tau_k) = \sum_{j=0}^{\nu-1} \left(\frac{\mu_j}{\nu}\nu\underline{g}_j + \frac{1-\mu_j}{\nu}\nu\overline{g}_j\right) A_c^j B_c = \sum_{j=0}^{\nu-1} \delta_j \nu \underline{g}_j A_c^j B_c + \delta_{j+\nu}\nu\overline{g}_j A_c^j B_c,$$

where $\delta_j = \frac{\mu_j}{\nu}$ and $\delta_{j+\nu} = \frac{1-\mu_j}{\nu}$ for all $j \in \mathbb{Z}_{[0,\nu-1]}$. Noticing that $\sum_{j=0}^{2\nu-1} \delta_j = 1$ concludes the proof. $\square$

Thus, we have obtained two different expressions for the generators of the convex set defined in (5). The method corresponding to (9) will be referred to as CH1 and the method corresponding to (10) as CH2, to be consistent with the method JNF1 versus JNF2. As the function $g_j(\tau_k)$ is a summation of an infinite number of terms, the question now rises how to compute the bounds $\underline{g}_j$ and $\overline{g}_j$. We will present two solutions to this problem.

3.1. A truncation-based approach

Firstly, the sum can be truncated after some p terms, $p \in \mathbb{Z}_{>0}$. The truncated function, i.e.

$$g_j^p(\tau_k) := \int_0^{\tau_k} \sum_{i=0}^{p-1} \frac{c_{i,j}}{i!}(T_s-\theta)^i \mathrm{d}\theta, \tag{11}$$

is a polynomial. Therefore, $\underline{g}_j^p$ and $\overline{g}_j^p$ can be obtained using the derivative of (11). Let Δ^p denote the polytope obtained via Theorem 3 using these bounds. Next, we deal with the error due to truncation of the sum in (7).

Theorem 4. *Let $\rho := \frac{3\|A_c\|T_s}{p}$ and suppose[1] $\rho < 1$. Then $\left\|\int_0^{\tau_k} \sum_{i=p}^{\infty} \frac{A_c^i (T_s-\theta)^i}{i!} \mathrm{d}\theta B_c\right\| \leq \frac{\rho^p}{1-\rho}\overline{\tau}\|B_c\|$.*

Proof.

$$\left\|\int_0^{\tau_k} \sum_{i=p}^{\infty} \frac{A_c^i (T_s-\theta)^i}{i!} \mathrm{d}\theta B_c\right\| \leq \sum_{i=p}^{\infty} \left\|\int_0^{\tau_k} \frac{A_c^i (T_s-\theta)^i}{i!} \mathrm{d}\theta B_c\right\|$$

$$\leq \sum_{i=p}^{\infty} \left\|\frac{A_c^i T_s^i}{(\frac{i}{3})^i}\overline{\tau}B_c\right\| \leq \sum_{i=p}^{\infty} \left(\frac{3T_s}{i}\right)^i \|A_c^i\|\overline{\tau}\|B_c\| \tag{12a}$$

$$\leq \sum_{i=p}^{\infty} \left(\frac{3\|A_c\|T_s}{p}\right)^i \overline{\tau}\|B_c\| = \frac{\rho^p}{1-\rho}\overline{\tau}\|B_c\|, \tag{12b}$$

where the triangle norm inequality and the Cauchy–Schwarz inequality were used. The lower bound for the factorial, i.e. $i! \geq \left(\frac{i}{3}\right)^i$, that was used in the denominator of (12a) was established by induction. $\square$

Using Theorem 4 one can choose the degree of approximation p for the coefficients in (7) in order to control the overall approximation error. Moreover, one can correct the polytope Δ^p accordingly to guarantee that all values $\Delta(\tau_k)$ can take for $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$ are included in the pth order over-approximation.

Lemma 5. *Let $\mathcal{B} \subset \mathbb{R}^{n\times m}$ denote the closed unit ball in $\mathbb{R}^{n\times m}$, $\mathcal{B}^\Delta := \mathcal{B} \cap Span(A_c^0 B_c, \ldots, A_c^{\nu-1}B_c)$ and $\varepsilon := \frac{\rho^p}{1-\rho}\overline{\tau}\|B_c\|$. Then*

$$\Delta(\tau_k) \in \Delta^p \oplus \varepsilon\mathcal{B}^\Delta, \quad \forall \tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}. \tag{13}$$

Proof. Eq. (3) yields that

$$\Delta(\tau_k) = \int_0^{\tau_k} \sum_{i=0}^{p-1} \frac{A_c^i (T_s-\theta)^i}{i!} \mathrm{d}\theta B_c + \int_0^{\tau_k} \sum_{i=p}^{\infty} \frac{A_c^i (T_s-\theta)^i}{i!} \mathrm{d}\theta B_c. \tag{14}$$

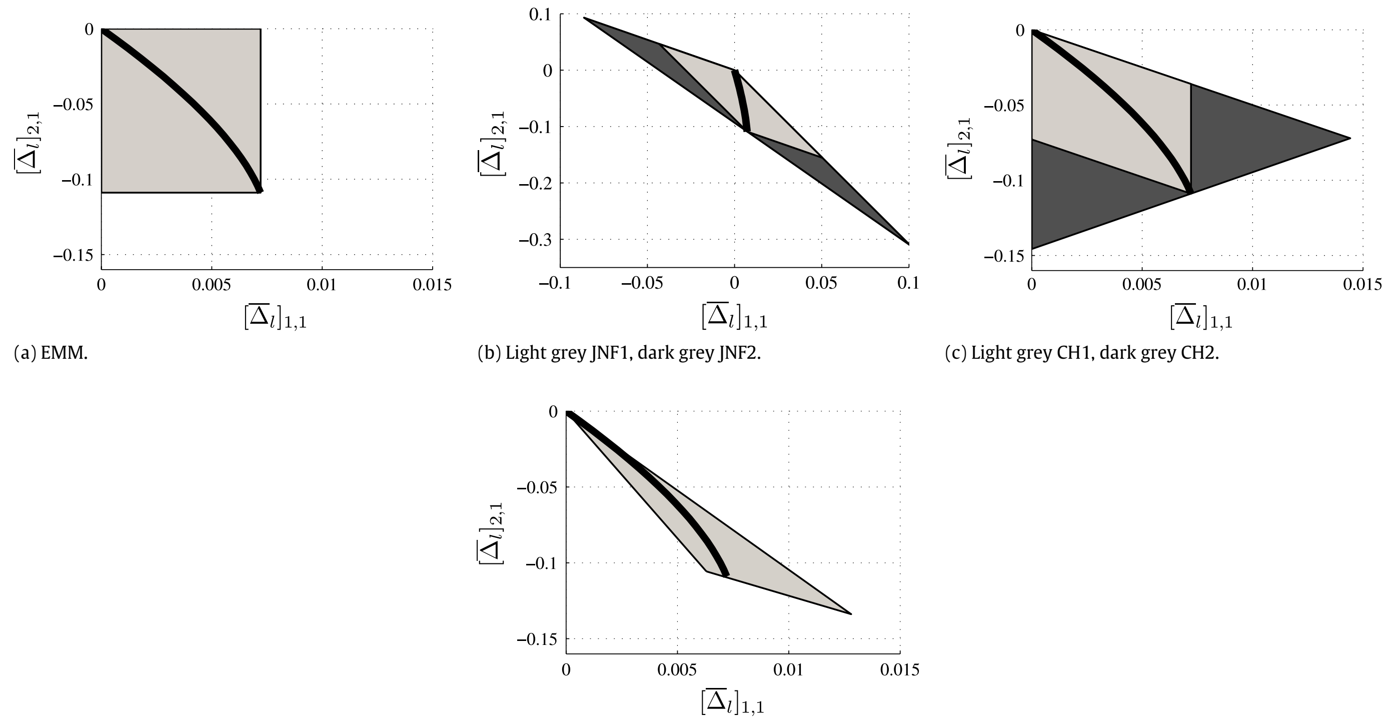
Using (11) instead of $g_j(\tau_k)$ one can, analogous to Lemma 2, rewrite the first term of (14) into (8). Analogous to Theorem 3, with the bounds $\underline{g}_j^p \in \mathbb{R}$ and $\overline{g}_j^p \in \mathbb{R}$ such that $\underline{g}_j^p \leq g_j^p(\tau_k) \leq \overline{g}_j^p$ for all $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$ one can obtain the polytopic over-approximation Δ^p for the first term of (14). Then, by Theorem 4 and the definition of ε, from (14) it follows that (13) holds. $\square$

Notice that the polytope $\Delta^p \oplus \varepsilon\mathcal{B}^\Delta$ has in general more generators than the original polytope. Next, we propose a way to avoid this increase in the number of generators. The matrix space $\mathbb{R}^{n\times m}$ is isomorph to the vector space $\mathbb{R}^{nm}$, hence we define the correspondences $\mathcal{B}^\Delta \leftrightarrow \mathcal{B}_v^\Delta \in \mathbb{R}^{nm}$ and $\Delta^p \leftrightarrow \Delta_v^p \in \mathbb{R}^{nm}$. Let $\Delta_v^p := \{x \in R^{nm} | H_{\Delta^p} x \leq h_{\Delta^p}\}$, let $\Omega(\mathcal{B}_v^\Delta)$ be the set of generators of $\mathcal{B}_v^\Delta$, let $w_i \in \Omega(\mathcal{B}_v^\Delta)$ for $i \in \mathbb{Z}_{[0,N-1]}$ denote the generators. An optimum translation $\delta_v^* \in \mathbb{R}^{nm}$ and scaling factor $\alpha^* \in \mathbb{R}_{>0}$ can be obtained by solving the optimization problem

$$\min_{\alpha,\delta_v} \alpha \tag{15}$$

subject to: $\alpha h_{\Delta^p} + H_{\Delta^p}\delta_v \geq H_{\Delta^p} w_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}$.

[1] Note that p can always be chosen such that $\rho < 1$.



(a) EMM.

(b) Light grey JNF1, dark grey JNF2.

(c) Light grey CH1, dark grey CH2.

(d) TA.

Fig. 1. Different polytopic approximations: Along the axes are the values of $[\overline{\Delta}_l]_{1,1}$ and $[\overline{\Delta}_l]_{2,1}$ for $l = 1, \dots, L$, in black all the possible realizations of $\Delta(\tau_k)$.

Notice that there exists a unique δ_v corresponding to the optimum α^* in (15), which is denoted by δ_v^*. Letting $\delta_v^* \leftrightarrow \delta^* \in \mathbb{R}^{n\times m}$ the set $\delta^* \oplus \alpha^* \Delta^p$ is such that $\varepsilon\mathcal{B}^{\Delta} \subseteq \delta^* \oplus \alpha^* \Delta^p$. Replacing $\varepsilon\mathcal{B}^{\Delta}$ with $\delta^* \oplus \alpha^* \Delta^p$ in (13) preserves the complexity of the over-approximation and yields $\Delta(\tau_k) \in \delta^* \oplus (1+\alpha^*)\Delta^p$, for all $\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}$.

Next, we present the second method to obtain the bounds $\underline{g}_j$ and $\overline{g}_j$.

3.2. An algebraic approach

The functions $g_j(\tau_k)$ can be derived exactly on the basis of the (real) Jordan form Usmani (1987). Using the real Jordan form of A_c it can be shown Cloosterman et al. (2009) that

$$\Delta(\tau_k) = \int_0^{\tau_k} e^{A_c(T_s-\theta)} d\theta B_c = \sum_{l=1}^{\nu} \alpha_l(\tau_k) S_l B_c, \tag{16}$$

where $S_l \in \mathbb{R}^{n\times n}$, $l \in \mathbb{Z}_{[1,\nu]}$. Furthermore, the functions $\alpha_l(\tau_k)$ correspond to the eigenvalues of A_c and are of the form $\int_0^{\tau_k} (T_s-\theta)^{\hat{l}-1} e^{\lambda_r(T_s-\theta)} d\theta$, $\hat{l} = 1, \dots, n_r$ in case $\lambda_r \in \mathbb{R}$ and $\int_0^{\tau_k} (T_s-\theta)^{\hat{l}-1} e^{a_r(T_s-\theta)} \cos(b_r(T_s-\theta)) d\theta$ and $\int_0^{\tau_k} (T_s-\theta)^{\hat{l}-1} e^{a_r(T_s-\theta)} \sin(b_r(T_s-\theta)) d\theta$, $\hat{l} = 1, \dots, n_r$ for complex pairs of eigenvalues $\lambda_r = a_r \pm ib_r \in \mathbb{C}$.

By minimality of the minimal polynomial the matrices $A_c^0, \dots, A_c^{\nu-1}$ are linearly independent in $\mathbb{R}^{n\times n}$, while by inspecting $e^{A_c(T_s-\theta)} = Q^{-1} e^{J(T_s-\theta)} Q$, where $A_c = Q^{-1} J Q$ with J the real Jordan form of A_c, it follows that $S_1, \dots, S_\nu$ are linearly independent as well and span the same linear space in $\mathbb{R}^{n\times n}$ as $A_c^0, \dots, A_c^{\nu-1}$. Hence, there is a unique invertible matrix $T \in \mathbb{R}^{\nu\times\nu}$ such that $S_l = \sum_{j=0}^{\nu-1} [T]_{j+1,l} A_c^j$, $l \in \mathbb{Z}_{[1,\nu]}$. Substituting this in (16) yields

$$\Delta(\tau_k) = \sum_{l=1}^{\nu} \alpha_l(\tau_k) \sum_{j=0}^{\nu-1} [T]_{j+1,l} A_c^j B_c.$$

Based on (8) it holds that $g_j(\tau_k) = \sum_{l=1}^{\nu} \alpha_l(\tau_k)[T]_{j+1,l}$, $j \in \mathbb{Z}_{[0,\nu-1]}$ and thus, we computed the functions $g_j(\tau_k)$ exactly. The lower and upper bounds $\underline{g}_j := \min_{\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}} g_j(\tau_k)$ and $\overline{g}_j := \max_{\tau_k \in \mathbb{R}_{[\underline{\tau},\overline{\tau}]}} g_j(\tau_k)$ can be computed directly from their explicit expressions.

Remark 6. All methods discussed in this paper are inherently related to calculating the matrix exponential. As such, different ways to calculate the matrix exponential (Moler & van Loan, 2003) might prove to be a fruitful starting point for further research. □

4. Illustrative example

We present an assessment of all modeling methods considered in this paper with a focus on the suitability for stabilizing controller synthesis. Consider the system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -1.2 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(t), \tag{17}$$

and the parameters $T_s = 0.1$, $\underline{\tau} = 0$ and $\overline{\tau} = 0.075$. The approximation order chosen for TA was $p = 10$. Recall that p defines the order of the Taylor approximation used and consequently also the number of generators L.

By applying all six methods to the system under study one obtains six polytopes, which are plotted in Fig. 1. Notice that the accuracy of the methods EMM, CH1, CH2 and TA is of the same order of magnitude. However, for JNF1 and JNF2 the polytope is much larger, note the different axes (the horizontal axis in particular). Table 1 shows the number of generators for each method.

Next, we consider the problem of controller synthesis. The model (4) in combination with the polytopic over-approximations JNF1, JNF2, TA, CH1 and CH2 is used to find a time-varying control law of the form $u_k = K_k \xi_k$. The controller minimizes a cost function of the form $J_k = \sum_{i=0}^{\infty} \xi_{k+i}^\top Q \xi_{k+i} + u_{k+i}^\top R u_{k+i}$, where Q and R are positive definite and symmetric matrices. Furthermore, a Lyapunov function of the form $V(k, \xi_k) = \xi_k^\top P_k \xi_k$ is employed

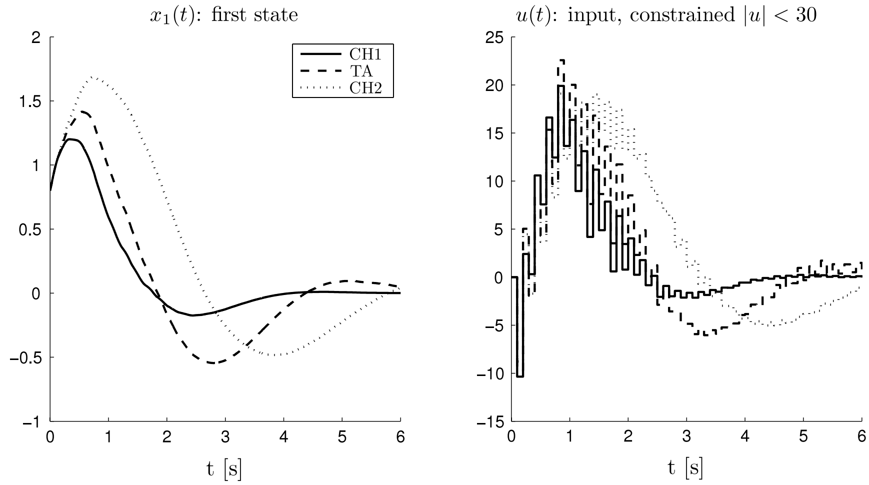


Fig. 2. Simulation of the same controller synthesis scheme for three different models.

Table 1
Complexity of the polytopic over-approximation per method.

Method:	EMM	JNF1	JNF2	TA	CH1	CH2
# gens.:	2^{nm}	2^{ν}	2ν	p	2^{ν}	2ν

to guarantee closed-loop asymptotic stability. The optimal state feedback and corresponding Lyapunov function are obtained online, for each measured state, as a solution to a semi-definite programming problem (Kothare et al., 1996).

In Fig. 2 we plot the results of a simulation for the system under observation. The method CH2 performs worse than its variant CH1, thus indicating the price paid for reducing computational complexity at the cost of over-approximating. In the simulations corresponding to JNF1 and JNF2 no robustly stabilizing controller and corresponding time-varying quadratic Lyapunov function could be obtained. This indicates that in general, over-approximating the nonlinearity can lead to infeasibility of controller synthesis. The latter observation stresses the need for tight over-approximation techniques and less conservative synthesis methods. The interested reader is referred to Gielen and Lazar (2009) for recent advances in stabilizing controller synthesis for systems with time-varying delays.

5. Conclusions

In this note we presented a novel method for modeling networked control systems (NCS) with time-varying input delays as a polytopic difference inclusion. The novelty consists of using the Cayley–Hamilton theorem to obtain the polytopic difference inclusion that contains all possible realizations of the nonlinear terms induced by delays. The developed method was compared with existing ones on the aspects of conservativeness, scalability and suitability for controller synthesis.

References

Cayley, A. (1857). A memoir on the theory of matrices. *Philosophical Transactions London*, *158*, 17–37.
Cloosterman, M. B. G., van de Wouw, N., Heemels, W. P. M. H., & Nijmeijer, H. (2006). Robust stability of networked control systems with time-varying network-induced delays. In *45th IEEE conference on decision and control*. (pp. 4980–4985). San Diego, CA.
Cloosterman, M. B. G., vande Wouw, N., Heemels, W. P. M. H., & Nijmeijer, H. (2007). Stability of networked control systems with large delays. In *46th IEEE conference on decision and control*. (pp. 5017–5022). New Orleans, LA.
Cloosterman, M. B. G., vande Wouw, N., Heemels, W. P. M. H., & Nijmeijer, H. (2009). Stability of networked control systems with uncertain time-varying delays. *IEEE Transactions on Automatic Control*, *54*(7), 1575–1580.
Gielen, R. H., & Lazar, M. (2009). Stabilization of networked control systems via non-monotone control Lyapunov functions. In *48th IEEE conference on decision and control*. (pp. 7942–7948). Shanghai, China.
Hespanha, J. P., Naghshtabrizi, P., & Xu, Y. (2007). A survey of recent results in networked control systems. *Proceedings of the IEEE*, *95*(1), 138–162.
Hetel, L., Daafouz, J., & Iung, C. (2006). Stabilization of arbitrary switched linear systems with unknown time varying delays. *IEEE Transactions on Automatic Control*, *51*(10), 1668–1674.
Kothare, M. V., Balakrishnan, V., & Morari, M. (1996). Robust constrained model predictive control using linear matrix inequalities. *Automatica*, *32*(10), 1361–1379.
Moler, C., & van Loan, C. (2003). Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Review*, *45*(1), 3–49.
Murray, R. M. (2002). *Control in an information rich world: Report of the panel on future directions in control, dynamics, and systems*. Philadelphia: SIAM.
Olaru, S., & Niculescu, S. -I. (2008). Predictive control for linear systems with delayed input subject to constraints In *Proceedings of the 17th IFAC world congres (CD-ROM)*. (pp. 11208–11213). Seoul, Korea.
Suh, Y. S. (2008). Stability and stabilization of nonuniform sampling systems. *Automatica*, *44*(12), 3222–3226.
Tipsuwan, Y., & Chow, M.-Y. (2003). Control methodologies in networked control systems. *Control Engineering Practice*, *11*, 1099–1111.
Usmani, R. A. (1987). Applied linear algebra. In *Pure and applied mathematics*. New York: Dekker, inc.