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Brief paper

Extremum-seeking control for nonlinear systems with periodic steady-state outputs☆

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ABSTRACT

Extremum-seeking control is a powerful adaptive technique to optimize system performance. To this date, extremum-seeking control has mainly been used to optimize plants with constant steady-state outputs, whereas the non-equilibrium case, in which the steady-state outputs are time varying, has received relatively little attention compared to the equilibrium case. In this paper, we propose an extremum-seeking scheme for the optimization of nonlinear plants with periodic steady-state outputs. Extremum-seeking control in this non-equilibrium setting is relevant in, for example, the scope of tracking and disturbance rejection problems. Using the concept of semi-global practical asymptotic stability, we show that under certain assumptions the proposed extremum-seeking controller design guarantees that for an arbitrarily large set of initial conditions the steady-state performance of the plant converges arbitrarily close to its optimal value.

1. Introduction

Extremum-seeking control is an adaptive control approach that optimizes a performance measure in terms of the steady-state output of a stable or stabilized plant in real time by automated tuning of the system parameters. In many applications of extremum-seeking control, only limited knowledge of the plant dynamics is available. Hence, the steady-state output of the plant (as a function of the system parameters) is not analytically known to the designer, and the output can only be measured. So, the purpose of an extremum-seeking controller is to drive the system parameters to their optimizing values, using merely output measurements of the plant. Since only output measurements are used, a model of the plant is not required. Therefore, extremum-seeking control can be applied to many different engineering domains; see, e.g., Ariyur and Krstić (2003) and Tan, Moase, Manzie, Nešić, and Mareels (2010), and the references therein.

In the majority of the works on extremum seeking, the steady-state output of the plant is assumed to be *constant*; see, e.g., Tan et al. (2010). To deal with general time-varying outputs for Wiener–Hammerstein-type plants, Krstić (2000) included a dynamic compensator in the extremum-seeking algorithm; see also Ariyur and Krstić (2003). In many cases, the performance of engineering systems is related to time-varying repetitive behavior (think for example of tracking or disturbance rejection problems). Examples are repetitive motion tasks in high-tech motion systems such as wafer scanners (Heertjes & van Engelen, 2011), and the control of sawtooth instabilities in fusion tokamak plasmas (Bolder et al., 2012).

Wang and Krstić (2000) designed an extremum-seeking controller to minimize the amplitude of a sinusoidal steady-state output using a detector. Although the steady-state output of the plant is time varying, the amplitude of the sinusoidal steady-state output is constant. Therefore, the same extremum-seeking method as for the optimization of plants with constant steady-state outputs can be applied to minimize the detected amplitude. The results in Wang and Krstić (2000) are tailored to sinusoidal outputs, whereas in the current paper we develop a more general framework for performance optimization of arbitrary periodic outputs.

Guay, Dochain, Perrier, and Hudson (2007) developed an extremum-seeking control scheme for the steady-state output optimization of a class of differentially flat periodic nonlinear

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plants. Flatness is exploited to compute one period of the steady-state output of the plant. Extremum-seeking control is used to optimize the computed output in real time. This method utilizes explicit knowledge of the relation between the parameters and the steady-state output of the plant, i.e., an accurate model of the system is required. A similar approach is used in Höffner, Hudon, and Guay (2007) for the steady-state output optimization of a class of periodic Hamiltonian systems. We stress that the wide application of extremum-seeking control in engineering is a result of the distinguishing feature that extremum-seeking control is model free, which is not the case in Guay et al. (2007) and Höffner et al. (2007).

The contribution of this paper can be summarized as follows. First, we propose an extremum-seeking control method for steady-state performance optimization of general nonlinear plants with arbitrary periodic steady-state outputs without requiring explicit knowledge of the relation between the parameters and the steady-state output of the plant. Second, we present a novel extremum-seeking controller with moving-average filter, which leads to an improved performance. Third, we present a stability analysis showing the semi-global practical asymptotic stability of the performance-optimal solution. Due to the nature of the proposed extremum-seeking scheme, the closed-loop dynamics of the plant and extremum-seeking controller are described by functional differential equations instead of ordinary differential equations, which requires an essentially different analysis compared to, for example, Krstić and Wang (2000) and Tan, Nešić, and Mareels (2006).

The paper is organized as follows. In Section 2, we present preliminaries, followed by the problem formulation in Section 3. In Section 4, we introduce an extremum-seeking controller to optimize the steady-state output of the plant. The stability analysis of the extremum-seeking scheme is presented in Section 5. An example with simulations is given in Section 6 followed by the conclusions in Section 7. The proofs of the results are found in the Appendix.

2. Preliminaries

The sets of real numbers and natural numbers (nonnegative integers) are denoted as $\mathbb{R}$ and $\mathbb{N}$, respectively. The sets of real numbers larger than zero and larger than or equal to zero are given by $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$, respectively. The following notation is adopted from Hale (1977) and Teel (1998). Let t_d be a nonnegative real number. Given a function $q : \mathbb{R} \to \mathbb{R}^n$ and $t \in \mathbb{R}$, we define $q_d(t)(\cdot)$ such that $q_d(t)(\tau) := q(t+\tau)$ for all $\tau \in [-t_d, 0]$. We say that $q_d(t) \in \mathcal{C}([-t_d, 0]; \mathbb{R}^n)$, where $\mathcal{C}$ is the Banach space of continuous functions mapping the interval $[-t_d, 0]$ to $\mathbb{R}^n$. We define (when it makes sense) $|q_d(t)| := \max_{s\in[t-t_d,t]} |q(s)|$, where $|\cdot|$ denotes the Euclidean norm.

Given two functions $f, g : \mathbb{R} \to \mathbb{R}$, by $f \circ g(\cdot)$ we denote $f(g(\cdot))$. We define the following function classes. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to the class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to the class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded (that is, $\alpha(r) \to \infty$ as $r \to \infty$). A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if $\beta(r, s)$ is of class $\mathcal{K}$ in its first argument for each fixed $s \geq 0$ and monotonically decreasing to zero in its second argument, i.e., for each fixed $r > 0$ the function $\beta(r, s) \to 0$ as $s \to \infty$.

We consider a parameterized family of $N \in \mathbb{N}$ interconnected systems:

$$\dot{x}_i = f_i(t, x_{1d}, x_{2d}, \ldots, x_{nd}, \epsilon), \tag{1}$$

with states $x_i \in \mathbb{R}^{v_i}$, $v_i \in \mathbb{N}$, for all $i \in \{1, 2, \ldots, N\}$ and parameter vector $\epsilon \in \mathbb{R}^k_{>0}$. Given the vector $x(t) := [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T$, we denote $x^+(t) := [|x_1(t)|, |x_2(t)|, \ldots, |x_N(t)|]^T$. This notation is adopted from Polushin, Marquez, Tayebi, and Liu (2009).

Definition 1. The interconnected system in (1) with parameter vector $\epsilon := [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]^T$ is said to be *semi-globally practically asymptotically stable* (SGPAS) if for any $\rho^0, \nu \in \mathbb{R}^N_{>0}$ the following holds. There exists an $\varepsilon_1^* \in \mathbb{R}_{>0}$ such that for any $\varepsilon_1 \in (0, \varepsilon_1^*)$ there exists an $\varepsilon_2^* = \varepsilon_2^*(\varepsilon_1) \in \mathbb{R}_{>0}$, such that for any $\varepsilon_2 \in (0, \varepsilon_2^*)$ there exists an $\varepsilon_3^* = \varepsilon_3^*(\varepsilon_1, \varepsilon_2) \in \mathbb{R}_{>0}$, such that ..., such that for any $\varepsilon_{k-1} \in (0, \varepsilon_{k-1}^*)$ there exists an $\varepsilon_k^* = \varepsilon_k^*(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{k-1}) \in \mathbb{R}_{>0}$, such that for any $\varepsilon_k \in (0, \varepsilon_k^*)$ and for all $x_d^+(0) \leq \rho^0$, the solutions $x_i(t)$, $\forall i \in \{1, 2, \ldots, N\}$, of (1) are well defined for all $t \geq 0$ and satisfy the following properties:

(1) uniform boundedness: $\sup_{t\geq 0} x^+(t) \leq C$;
(2) ultimate boundedness: $\limsup_{t\to\infty} x^+(t) \leq \nu$,

where the inequalities hold in an elementwise sense and where $C = C(\rho^0, \epsilon) \in \mathbb{R}^N_{>0}$ is a constant vector.

Note that Definition 1 defines the order in which the parameters should be tuned, namely, ε_1 should be tuned first, followed by ε_2, ε_3, etc.

3. Extremum-seeking problem for periodic steady states

Consider a nonlinear plant of the following form:

$$\begin{aligned} \dot{x} &= f(x, u, \theta, w(t)), \\ y &= h(x, w(t)), \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}$ are respectively the state, the control input, and the output, where $w(t) \in \mathbb{R}^l$ are input disturbances, and where $\theta \in \mathbb{R}$ is a scalar parameter. The function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^n$ is twice continuously differentiable in x, u, and θ, and continuous in $w(t)$. The function $h : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ is twice continuously differentiable in x and continuous in $w(t)$. The disturbances $w(t)$ correspond to the solution of an *exosystem* of the following form:

$$\dot{w} = \varphi(w), \tag{3}$$

where $\varphi : \mathbb{R}^l \to \mathbb{R}^l$ is such that the exosystem (3) exhibits the existence and uniqueness of solutions and the continuous dependence of solutions on initial conditions (in backward and forward time). Moreover, we assume that the following assumption on the exosystem holds.

Assumption 2. For any initial condition $w(0) \in \mathbb{R}^l$, the solution of system (3) is uniformly bounded (in backward and forward time) and periodic with a known constant period $T_w \in \mathbb{R}_{>0}$, yielding $w(t + T_w) \equiv w(t)$ for all $t \in \mathbb{R}$.

Note that Assumption 2 is (also) satisfied for constant solutions of (3), because constant solutions are periodic with any period $T_w \in \mathbb{R}_{>0}$. Moreover, T_w may depend on the initial condition $w(0)$.

Consider a state-feedback controller of the following form:

$$u = \alpha(x, \theta), \tag{4}$$

where the function $\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is twice continuously differentiable in x and θ. We assume that we can find a stabilizing controller (4) such that the following assumption holds.

Assumption 3. For all fixed $\theta \in \mathbb{R}$, there exists a unique, bounded for all $t \in \mathbb{R}$, uniformly globally asymptotically stable (UGAS) steady-state solution $\bar{x}_{\theta,w}(t)$ of the stabilized plant in (2), (4). Moreover, there exists a map $M : \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^n$, twice continuously differentiable in θ and continuous in $w(t)$, such that

$$\bar{x}_{\theta,w}(t) = M(\theta, w(t)), \tag{5}$$

for fixed values of $\theta \in \mathbb{R}$ and all $t \in \mathbb{R}$. In addition, there exist functions $\alpha_{x1}, \alpha_{x2} \in \mathcal{K}_\infty, \alpha_f \in \mathcal{K}$ and a (smooth) Lyapunov function $V_x(\tilde{x})$ such that

$$\alpha_{x1}(|\tilde{x}|) \le V_x(\tilde{x}) \le \alpha_{x2}(|\tilde{x}|), \tag{6}$$

and

$$\frac{dV_x}{d\tilde{x}}\tilde{f}\left(\tilde{x}, M(\theta, w(t)), \theta, w(t)\right) \le -\alpha_f(|\tilde{x}|), \tag{7}$$

for all fixed $\theta \in \mathbb{R}$ and all $t \ge 0$, with $\tilde{x} := x - M(\theta, w(t))$ and $\tilde{f}\left(\tilde{x}, M, \theta, w(t)\right) := f(\tilde{x} + M, \alpha(\tilde{x} + M, \theta), \theta, w(t)) - f(M, \alpha(M, \theta), \theta, w(t))$.

Remark 4. For uniformly convergent plants, it was shown in Pavlov, van de Wouw, and Nijmeijer (2007, Theorem 2) that $\bar{x}_{\theta,w}(t)$ is UGAS and that there exists a map M as in (5) for each fixed $\theta \in \mathbb{R}$. See Pavlov, van de Wouw, and Nijmeijer (2005) and Pavlov et al. (2007) for a definition of uniform convergence. It was shown in Pavlov et al. (2005) how for some classes of systems (2) it is possible to design a controller of the form (4) such that the stabilized plant in (2), (4) is uniformly convergent. Note that finding a controller (4) such that Assumption 3 holds may require explicit knowledge of the plant in (2).

Moreover, the existence of a Lyapunov function satisfying (6), (7) actually guarantees the UGAS property for $\bar{x}_{\theta,w}(t)$. There exist converse theorems for UGAS parameterized families of systems such that similar inequalities as in (6) and (7) hold; see, e.g., Lin, Sontag, and Wang (1995).

We aim to find the fixed value of $\theta \in \mathbb{R}$ that optimizes the steady-state performance of the stabilized plant in (2), (4). In order to do so, we design a cost function that links the output of the stabilized plant in (2), (4) to its performance. As a stepping stone, we introduce various performance measures of the following form:

$$\begin{aligned} L_p(y_d(t)) &:= \left(\frac{1}{T_w}\int_{t-T_w}^{t} |y(\tau)|^p d\tau\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{T_w}\int_{-T_w}^{0} |y_d(t)(\tau)|^p d\tau\right)^{\frac{1}{p}}, \\ L_\infty(y_d(t)) &:= \max_{\tau\in[t-T_w,t]} |y(\tau)| = \max_{\tau\in[-T_w,0]} |y_d(t)(\tau)|, \end{aligned} \tag{8}$$

with $p \in [1, \infty)$. Here, we emphasize that the period time T_w is known to the designer; see Assumption 2. The argument of the performance measures in (8) is defined by $y_d(t)(\tau) := y(t + \tau)$ for all $\tau \in [-t_d, 0]$, where $t_d > T_w$ is the maximal delay in the extremum-seeking scheme, which will be defined in Section 4. We use one of the performance measures in (8) in the design of the cost function, which is given by

$$Q_i(y_d(t)) := g \circ L_i(y_d(t)), \quad i \in [1, \infty], \tag{9}$$

where $g : \mathbb{R}_{\ge 0} \to \mathbb{R}$ is a twice continuously differentiable function chosen by the designer. We say that the steady-state performance of the stabilized plant in (2), (4) is optimized if the steady-state output of the cost function Q_i in (9) is maximized. The output of the cost function in (9) will be referred to as the performance of the stabilized plant in (2), (4), and it is denoted by $q \in \mathbb{R}$, i.e., $q(t) = Q_i(y_d(t))$ with $i \in [1, \infty]$.

The stabilized plant in (2), (4) and the cost function in (9) are considered as one lumped plant with θ and $w(t)$ as input and q as output, as shown in Fig. 1. Combining (2), (4) and (9), the lumped plant is given by

$$\begin{aligned} \dot{x} &= f(x, \alpha(x, \theta), \theta, w(t)), \\ q &= J_i(x_d, w_d(t)), \end{aligned} \tag{10}$$

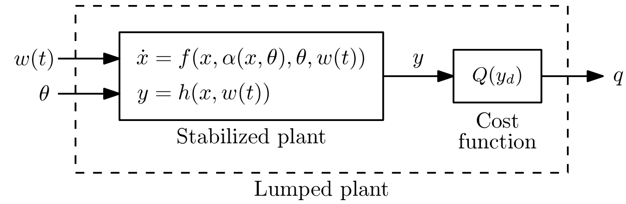

Fig. 1. Lumped plant in (10).

with

$$J_i(x_d, w_d(t)) := Q_i \circ h(x_d, w_d(t)) = g \circ L_i \circ h(x_d, w_d(t)), \tag{11}$$

with $i \in [1, \infty]$. Herein, we have adopted the notation $y_d = h(x_d, w_d(t))$ for the sake of simplicity. We will refer to J_i in (11) as the performance function.

Next, we introduce a useful property on the periodicity of the steady-state solution $\bar{x}_{\theta,w}(t)$.

Property 5. *Suppose that Assumptions 2 and 3 hold and $\theta \in \mathbb{R}$ is fixed. Then, the corresponding steady-state solution $\bar{x}_{\theta,w}(t)$ is periodic, with period $T_w \in \mathbb{R}_{>0}$.*

Proof. Using the uniqueness and UGAS properties of the steady-state solution in Assumption 3, the proof of the property follows similar steps as the proof of Pavlov et al. (2005, Property 2.23). □

Assuming that Assumptions 2 and 3 hold, note that, from (2) and Property 5, it follows that the steady-state output $\bar{y}_{\theta,w}(t) = h(\bar{x}_{\theta,w}(t), w(t))$ is T_w-periodic if $\theta \in \mathbb{R}$ is fixed. Subsequently, the steady-state performance $\bar{q}_{\theta,w}(t) = Q_i(\bar{y}_{\theta,w,d}(t)) = g \circ L_i(\bar{y}_{\theta,w,d}(t))$ is constant for each fixed $\theta \in \mathbb{R}$, because $\bar{y}_{\theta,w}(t)$ is T_w-periodic and the output of L_i in (8) is constant for T_w-periodic inputs. We obtain that the relation between fixed values of the parameter θ and the steady-state performance $\bar{q}_{\theta,w}$ is given by the following static map:

$$\begin{aligned} J_{sta,p}(\theta) &:= g \circ \left(\frac{1}{T_w}\int_0^{T_w} |h(M(\theta, w(\tau)), w(\tau))|^p d\tau\right)^{\frac{1}{p}} \\ J_{sta,\infty}(\theta) &:= g \circ \left(\max_{\tau\in[0,T_w]} |h(M(\theta, w(\tau)), w(\tau))|\right), \end{aligned} \tag{12}$$

with $p \in [1, \infty)$, where we used (5), the definitions of L_i in (8) and J_i in (11), and the periodicity of $w(t)$ to obtain (12).

Consider some $i \in [1, \infty]$. We assume that the output function h in (2), and/or the map M in (5) and/or the input $w(t)$ are unknown[2] to the designer. Note that this implies that the static map $J_{sta,i}$ in (12) is also unknown. Nonetheless, we adopt the following assumption on the existence of a unique maximum of $J_{sta,i}$.

Assumption 6. Consider some $i \in [1, \infty]$. It is assumed that the static map $J_{sta,i}$ in (12) and its first two derivatives with respect to θ are continuous and bounded on compact sets of θ. Moreover, it is assumed that there exists a function $\alpha_J \in \mathcal{K}$ and a constant $\theta^* \in \mathbb{R}$, corresponding to the value of θ that optimizes the steady-state performance of the stabilized plant in (2), (4), such that

$$\frac{dJ_{sta,i}}{d\theta}(\theta)[\theta - \theta^*] \le -\alpha_J(|\theta - \theta^*|), \tag{13}$$

for all $\theta \in \mathbb{R}$. In other words, for $\theta = \theta^*$ the map $J_{sta,i}$ achieves a unique maximum in $\mathbb{R}$.

[2] Note that the period T_w of the unknown input $w(t)$ is assumed to be known, since $w(t)$ satisfies Assumption 2.

By finding the maximum of the map $J_{sta,i}$ at $\theta = \theta^*$, we find the value of $\theta \in \mathbb{R}$ that optimizes the steady-state performance $\bar{q}_{\theta,w}$ of the stabilized plant. Hence, we can rephrase the objective of finding the value of $\theta \in \mathbb{R}$ that optimizes $\bar{q}_{\theta,w}$ by finding the value of $\theta \in \mathbb{R}$ that corresponds to the maximum of the static map $J_{sta,i}$, i.e., by finding $\theta = \theta^*$.

4. Extremum-seeking controller design

Consider the extremum-seeking scheme in Fig. 2. The extremum-seeking scheme consists of the lumped plant in (10) and an extremum-seeking controller consisting of a perturbation-based gradient estimator and an optimizer. The optimizer is given by

$$\dot{\hat{\theta}} = Ke, \tag{14}$$

where e is the estimate of the gradient $\frac{dJ_{sta,i}}{d\theta}(\hat{\theta})$. Here, we propose a novel gradient estimator based on a moving-average filter, which we will call a *mean-over-perturbation-period (MOPP) filter*, given by

$$e = \frac{\omega}{a\pi}\int_{t-\frac{2\pi}{\omega}}^{t} q(\tau)\sin(\omega[\tau-\phi])d\tau. \tag{15}$$

Using $\theta = \hat{\theta} + a\sin(\omega t)$, (10), (14) and (15), the closed-loop dynamics are given by

$$\begin{aligned}\dot{x} &= f(x, \alpha(x, \hat{\theta} + a\sin(\omega t)), \hat{\theta} + a\sin(\omega t), w(t)),\\ \dot{\hat{\theta}} &= \frac{\omega K}{a\pi}\int_{t-\frac{2\pi}{\omega}}^{t} J_i(x_d(\tau), w_d(\tau))\sin(\omega[\tau-\phi])d\tau,\end{aligned} \tag{16}$$

with $i \in [1, \infty]$ and $\hat{\theta} \in \mathbb{R}$, where $a, \omega, K \in \mathbb{R}_{>0}$ are controller parameters and $\phi \in \mathbb{R}_{\geq 0}$ is a constant. Note that the maximal delay of the extremum-seeking scheme and, therefore, the maximal delay for delayed signals with subscript d is $t_d = T_w + \frac{2\pi}{\omega}$; the delay T_w is introduced by the performance measure L_i in (8), while the delay $\frac{2\pi}{\omega}$ is introduced by the MOPP filter.

Remark 7. To motivate the use of the MOPP filter, assume that the performance q is equal to the steady-state performance $\bar{q}_{\theta,w}$ and that $\hat{\theta} \in \mathbb{R}$ is fixed. Then, we can write the performance q as

$$\begin{aligned}q &= J_{sta,i}(\theta) = J_{sta,i}(\hat{\theta} + a\sin(\omega t))\\ &= J_{sta,i}(\hat{\theta}) + a\frac{dJ_{sta,i}}{d\theta}(\hat{\theta})\sin(\omega t) + \mathcal{O}\left(a^2\right).\end{aligned} \tag{17}$$

The corresponding gradient estimate e in (15) is given by

$$\begin{aligned}e &= \frac{\omega}{a\pi}J_{sta,i}(\hat{\theta})\int_{t-\frac{2\pi}{\omega}}^{t}\sin(\omega[\tau-\phi])d\tau\\ &\quad + \frac{\omega}{\pi}\frac{dJ_{sta,i}}{d\theta}(\hat{\theta})\int_{t-\frac{2\pi}{\omega}}^{t}\sin(\omega\tau)\sin(\omega[\tau-\phi])d\tau + \mathcal{O}(a)\\ &= \frac{dJ_{sta,i}}{d\theta}(\hat{\theta})\cos(\omega\phi) + \mathcal{O}(a),\end{aligned} \tag{18}$$

where $\omega\phi$ and a are typically small. Note that the magnitude of the frequency response function of the MOPP filter $e(t) = \frac{\omega}{2\pi}\int_{t-\frac{2\pi}{\omega}}^{t} i(\tau)d\tau$ is given by $\left|\frac{E(j\hat{\omega})}{I(j\hat{\omega})}\right| = \left|\mathrm{sinc}\left(\frac{\hat{\omega}\pi}{\omega}\right)\right|$, where $\mathrm{sinc}(\cdot)$ is the unnormalized sinc function. Contrary to finite-order filters such as the low-pass and/or high-pass filters in Krstić and Wang (2000) and Tan et al. (2006), angular frequencies $\hat{\omega} = \omega$ and higher-order harmonics related to the perturbation signal $a\sin(\omega t)$ are filtered out completely as $\left|\frac{E(j\hat{\omega})}{I(j\hat{\omega})}\right| = 0$ for all $\hat{\omega} = k\omega$ with $k \in \mathbb{N}_{>0}$, which results in a more accurate gradient estimate in (18). Similar to Krstić (2000), the phase shift $\omega\phi$ between $a\sin(\omega t)$ and $\sin(\omega[t-\phi])$ is introduced to compensate for the delays introduced by the plant dynamics and L_i in (8), which is part of the performance function J_i in (11). A good choice for the constant $\phi \in \mathbb{R}_{\geq 0}$ is an estimate of the sum of the time-varying delay of the plant dynamics and the performance measure L_i.

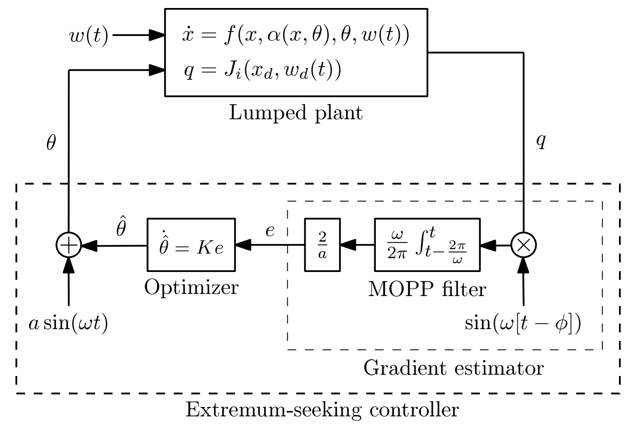

Fig. 2. Extremum-seeking scheme in (16).

For analysis purposes, we select $K = a^2\omega\delta$, where $\delta \in \mathbb{R}_{>0}$ is a constant. In addition, the following change of variables is introduced:

$$\tilde{x} := x - M(\theta, w(t)) \quad \text{and} \quad \tilde{\theta} := \hat{\theta} - \theta^*, \tag{19}$$

where M and θ^* are defined in (5) and Assumption 6, respectively. Using $K = a^2\omega\delta$ and the change of variables in (19), the system equations in (16) are transformed to

$$\begin{aligned}\frac{d\tilde{x}}{dt} &= \tilde{f}\left(\tilde{x}, M, \theta, w(t)\right) - \frac{\partial M}{\partial\theta}\left[\frac{d\tilde{\theta}}{dt} + a\omega\cos(\omega t)\right],\\ \frac{d\tilde{\theta}}{dt} &= \frac{a\omega^2\delta}{\pi}\int_{t-\frac{2\pi}{\omega}}^{t} J_i(\tilde{x}_d(\tau) + M_d(\tau), w_d(\tau))s(\tau)d\tau,\end{aligned} \tag{20}$$

with $i \in [1, \infty]$, $\tilde{f}\left(\tilde{x}, M, \theta, w(t)\right) := f(\tilde{x} + M, \alpha(\tilde{x} + M, \theta), \theta, w(t)) - f(M, \alpha(M, \theta), \theta, w(t))$ as in Assumption 3, $M_d(t) := M(\theta_d(t), w_d(t))$, and $s(t) := \sin(\omega[t-\phi])$. To prevent lengthy expressions, we have not substituted $\theta = \tilde{\theta} + \theta^* + a\sin(\omega t)$, and we have written M instead of $M(\theta, w(t))$.

5. Stability analysis

5.1. Main result

We first present our main result, which states the conditions under which the system in (20) is SGPAS as defined in Definition 1.

Theorem 8. *Suppose that Assumptions 2, 3 and 6 hold. Then, the closed-loop dynamics of the extremum-seeking scheme in (20) is SGPAS, where the parameter vector is given by $\epsilon = [a, \omega, \delta]^T$.*

Proof. See Appendix A.1. □

Let us make explicit the implications of this result. Under the conditions of Theorem 8, the state x of the stabilized plant in (2), (4) converges to an arbitrarily small neighborhood of the steady-state solution given by the map M in (5) for sufficiently small $a, \omega, \delta \in \mathbb{R}_{>0}$, since $x = \tilde{x} + M(\theta, w(t))$ and $\tilde{x}$ converges to an arbitrarily small neighborhood of the origin; see the definition of an SGPAS system above (Definition 1). Then, from the continuity of J_i in (11), it follows that the performance q of the plant converges to an arbitrarily small neighborhood of the steady-state performance for sufficiently small $a, \omega, \delta \in \mathbb{R}_{>0}$. Note that, from Theorem 8 and $\theta = \tilde{\theta} + \theta^* + a\sin(\omega t)$, it also follows that the value of the parameter θ converges to an arbitrarily small neighborhood of the performance-optimizing value θ^* for sufficiently small $a, \omega, \delta \in \mathbb{R}_{>0}$, since $\tilde{\theta}$ converges to an arbitrarily small neighborhood of the origin and $|a\sin(\omega t)| \leq a$. Hence, it follows that the performance q of the stabilized plant in (2), (4) converges arbitrarily close to the optimal steady-state performance for sufficiently small $a, \omega, \delta \in \mathbb{R}_{>0}$ (or equivalently $a, \omega, K \in \mathbb{R}_{>0}$).

5.2. Supporting technical results

To prove Theorem 8, the extremum-seeking scheme in (20) is regarded as a feedback interconnection between its $\tilde{x}$-dynamics and its $\tilde{\theta}$-dynamics. First, bounds on the solutions $\tilde{\theta}(t)$ and $\tilde{x}(t)$ of (20) are derived. Let $t^* \geq t_d$ be a constant. For $t \in [0, t^*]$, we use the following bound on the solution $\tilde{\theta}(t)$, which follows directly from the boundedness of the right-hand side of the $\tilde{\theta}$-dynamics in (20) for compact sets of $\tilde{\theta}$ and $\tilde{x}$, and is obtained by integrating the $\tilde{\theta}$-dynamics in (20).

Proposition 9. *Suppose that Assumptions 2, 3 and 6 hold. Then, for any $\rho^\theta, \rho^x \in \mathbb{R}_{>0}$ there exists a constant $c_\theta \in \mathbb{R}_{>0}$ such that for all $\sup_{t\geq 0} |\tilde{\theta}_d(t)| \leq \rho^\theta$, $\sup_{t\geq 0} |\tilde{x}_d(t)| \leq \rho^x$ and bounded values of $a, \omega, \delta \in \mathbb{R}_{>0}$ the following bound holds for all $t \geq 0$:*

$$|\tilde{\theta}(t)| \leq |\tilde{\theta}(0)| + a\omega\delta c_\theta t. \tag{21}$$

The following proposition provides a bound on the solution $\tilde{\theta}(t)$ for $t \geq t^*$.

Proposition 10. *Suppose that Assumptions 2, 3 and 6 hold. Then, for any $\rho^\theta, \rho^x \in \mathbb{R}_{>0}$ there exist functions $\beta_\theta \in \mathcal{KL}, \gamma_{\theta x} \in \mathcal{K}$ and a constant $C_\theta \in \mathbb{R}_{>0}$ such that for all $\sup_{t\geq 0} |\tilde{\theta}_d(t)| \leq \rho^\theta$, $\sup_{t\geq 0} |\tilde{x}_d(t)| \leq \rho^x$ and bounded values of $a, \omega, \delta \in \mathbb{R}_{>0}$ the following bound holds for any $t^* \geq t_d$ and all $t \geq t^*$:*

$$|\tilde{\theta}(t)| \leq \max\left\{\beta_\theta\left(|\theta(t^*)|, a^2\omega\delta(t-t^*)\right), \right.$$
$$\left. \gamma_{\theta x}\left(\sup_{t\geq t^*} |\tilde{x}_d(t)|\right), C_\theta\right\}. \tag{22}$$

Moreover, there exist some functions $\tilde{\gamma}_{\theta x1}, \tilde{\gamma}_{\theta x2}, \tilde{\gamma}_{\theta C} \in \mathcal{K}$ and some constant $\tilde{C}_\theta \in \mathbb{R}_{>0}$ such that $\gamma_{\theta x}(\cdot) = \tilde{\gamma}_{\theta x1}(\frac{1}{a}\tilde{\gamma}_{\theta x2}(\cdot))$ and $C_\theta = \tilde{\gamma}_{\theta C}([a + \omega T_w + \delta]\tilde{C}_\theta)$.

Proof. See Appendix A.2. □

Similar to Tan et al. (2006), note that the convergence rate of the $\tilde{\theta}$-dynamics in (20) depends on the controller parameters $a, \omega, \delta \in \mathbb{R}_{>0}$ (see the second argument of the function β_θ in Proposition 10). Note that small $a, \omega, \delta \in \mathbb{R}_{>0}$ imply a slow convergence of $\tilde{\theta}$. The next proposition provides a bound on the solution $\tilde{x}(t)$ of the $\tilde{x}$-dynamics in (20).

Proposition 11. *Suppose that Assumptions 2, 3 and 6 hold. Then, for any $\rho^\theta, \rho^x \in \mathbb{R}_{>0}$ there exist a function $\beta_x \in \mathcal{KL}$ and a constant $C_x \in \mathbb{R}_{>0}$ such that for all $\sup_{t\geq 0} |\tilde{\theta}_d(t)| \leq \rho^\theta$, $\sup_{t\geq 0} |\tilde{x}_d(t)| \leq \rho^x$ and bounded values of $a, \omega, \delta \in \mathbb{R}_{>0}$ the following bound holds for all $t \geq 0$:*

$$|\tilde{x}(t)| \leq \max\left\{\beta_x(|\tilde{x}(0)|, t), C_x\right\}. \tag{23}$$

Moreover, there exist some function $\tilde{\gamma}_{xC} \in \mathcal{K}$ and some constant $\tilde{C}_x \in \mathbb{R}_{>0}$ such that $C_x = \tilde{\gamma}_{xC}(a\omega\tilde{C}_x)$.

Proof. See Appendix A.3. □

To complete the proof of Theorem 8, the bounds obtained in Propositions 9–11 are exploited using a small-gain argument; see Appendix A.1.

6. Illustrative example

Consider a plant with the following dynamics and output:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -25x_1 - b(\theta)x_2 + w_1(t) \\ y &= x_1, \end{aligned} \tag{24}$$

where $b(\theta) \in \mathbb{R}_{>0}$ is a nonlinear characteristic that depends on the system parameter $\theta \in \mathbb{R}$ and is given by

$$b(\theta) = 10 + 5(\theta - 10)^2. \tag{25}$$

The input disturbance $w_1(t) = 20\sin(80t)$ is part of the solution of the exosystem $\dot{w}_1 = vw_2$, $\dot{w}_2 = -vw_1$, with $v = 80$ and initial conditions $w_1(0) = 0$ and $w_2(0) = 20$. To find the value of $\theta \in \mathbb{R}$ that maximizes the amplitude of the steady-state output of the plant in (24)–(25), we introduce the following cost function:

$$Q_\infty(y_d) = L_\infty(y_d), \tag{26}$$

where L_∞ is defined in (8). Using (24)–(26), the relation between fixed values of θ and the steady-state plant performance $\bar{q}_{\theta,w}$ is given by $\bar{q}_{\theta,w} = J_{sta,\infty}(\theta)$, with

$$J_{sta,\infty}(\theta) = \frac{20}{\sqrt{6375^2 + 80^2 b^2(\theta)}}. \tag{27}$$

Note that the extremum of the map is located at $\theta = \theta^* = 10$; see Fig. 3.

The extremum-seeking controller in Fig. 2 is used to optimize the steady-state performance of the plant in (24)–(25). Simulation results in Fig. 3 (in black) show that θ converges to a small neighborhood of the performance-optimizing value $\theta^* = 10$. As θ converges to θ^*, the amplitude of the output y increases and the performance q of the plant converges to a small neighborhood of the optimal steady-state performance indicated by the maximum of the static map $J_{sta,\infty}$.

To emphasize the benefit of the novel MOPP filter for gradient estimation, Fig. 3 also shows simulation results (in gray) for a similar extremum-seeking scheme, where the MOPP filter is replaced by a first-order low-pass filter with (properly tuned) angular cutoff frequency $\omega_l = 1.1$; see Krstić and Wang (2000) and Tan et al. (2006). Fig. 3 clearly shows that the use of the MOPP filter results in a better estimate e of the gradient $\frac{dJ_{sta,i}}{d\theta}(\hat{\theta})$. As mentioned in Remark 7 and illustrated in Fig. 3, the main reason for this fact is that (for fixed $\hat{\theta}$) the MOPP filter filters out all oscillations with angular frequency ω (and higher-order harmonics), while the low-pass filter does not. Moreover, by using the MOPP filter we obtain a smaller estimation delay compared to when the low-pass filter is used. Because the MOPP filter provides a better gradient estimate of $\frac{dJ_{sta,i}}{d\theta}(\hat{\theta})$, the system parameter θ converges faster to the optimal value θ^* when the MOPP filter is used; see Fig. 3.

7. Conclusions

In this paper, we have presented an extremum-seeking control method for steady-state performance optimization of general nonlinear plants with periodic steady-state outputs. This methodology allows us to consider arbitrary periodic steady-state outputs without requiring explicit knowledge of the relation between the system parameter and the steady-state output of the plant. Furthermore, we have presented a novel extremum-seeking controller with moving-average filter, which we call a mean-over-perturbation-period (MOPP) filter. Simulations indicate that our design leads to an improved performance with respect to a comparable extremum-seeking controller with a low-pass filter. Moreover, conditions have been presented under which semi-global practical asymptotic stability of the closed-loop system is guaranteed, which implies the achievement of performance optimization using extremum seeking.

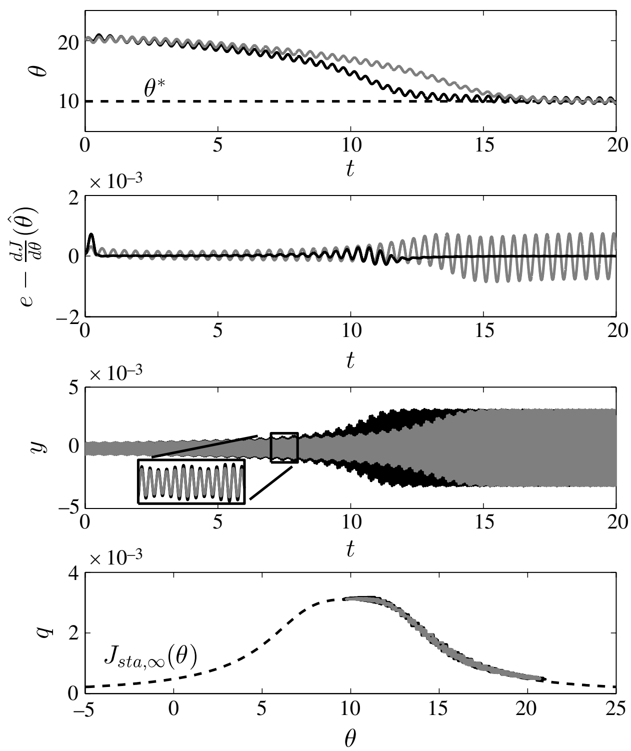

Fig. 3. Simulation results of the extremum-seeking scheme with a MOPP filter (black) and a low-pass filter (gray), for $a = 0.5$, $K = 4 \times 10^3$, $\omega = 15$ and $\phi = 0.02$, with initial conditions $\hat{\theta}(0) = 20$ and $x_{1d}(0)(\tau) = -5.4 \times 10^{-4}$, $x_{2d}(0)(\tau) = -6.8 \times 10^{-3}$ for all $\tau \in [-t_d, 0]$, with $t_d = T_w + \frac{2\pi}{\omega} = \frac{\pi}{40} + \frac{2\pi}{15} = \frac{19\pi}{120}$.

Appendix. Proofs

A.1. Proof of Theorem 8

Define $z^+ := [|\tilde{\theta}|, |\tilde{x}|]^T$. To prove the validity of the statement in the theorem, we will show that for any $\rho^0 := [\rho_\theta^0, \rho_x^0]^T \in \mathbb{R}^2_{>0}$ and any $\nu := [\nu_\theta, \nu_x]^T \in \mathbb{R}^2_{>0}$ there exist some constant vector $C \in \mathbb{R}^2_{>0}$ and some real numbers $a^* \in \mathbb{R}_{>0}$, $\omega^* = \omega^*(a) \in \mathbb{R}_{>0}$, $\delta^* = \delta^*(a) \in \mathbb{R}_{>0}$ such that for all $a \in (0, a^*)$, $\omega \in (0, \omega^*)$, $\delta \in (0, \delta^*)$ and all $z_d^+(0) \leq \rho^0$ the solutions of the closed-loop extremum-seeking dynamics in (20) are well defined for all $t \geq 0$ and satisfy the following inequalities:

$$\sup_{t\geq 0} z^+(t) \leq C, \tag{A.1}$$

$$\limsup_{t\to\infty} z^+(t) \leq \nu. \tag{A.2}$$

First, choose $\rho^\theta, \rho^x \in \mathbb{R}_{>0}$ in Propositions 9–11 sufficiently large such that

$$\beta_\theta(\rho_\theta^0, 0) < \rho^\theta \quad \text{and} \quad \beta_x(\rho_x^0, 0) \leq \rho^x, \tag{A.3}$$

where $\beta_\theta \in \mathcal{KL}$ is defined in Proposition 10 and $\beta_x \in \mathcal{KL}$ is defined in Proposition 11. Next, fix $a \in \mathbb{R}_{>0}$ at a sufficiently small value such that

$$C_\theta = \tilde{\gamma}_{\theta C}([a + \omega T_w + \delta]\tilde{C}_\theta) \leq \min\{\rho^\theta, \nu_\theta\}, \tag{A.4}$$

for sufficiently small $\omega, \delta \in \mathbb{R}_{>0}$, where $\tilde{\gamma}_{\theta C} \in \mathcal{K}$ and $\tilde{C}_\theta \in \mathbb{R}_{>0}$ are defined in Proposition 10. Let $c_{t^*} \in \mathbb{R}_{>0}$ be a constant such that

$$\gamma_{\theta x}(c_{t^*}) = \tilde{\gamma}_{\theta x1}\left(\frac{1}{a}\tilde{\gamma}_{\theta x2}(c_{t^*})\right) \leq \min\{\rho^\theta, \nu_\theta\}, \tag{A.5}$$

where $\tilde{\gamma}_{\theta x1}, \tilde{\gamma}_{\theta x2} \in \mathcal{K}$ are defined in Proposition 10. We choose $t^* \geq t_d$ sufficiently large such that

$$\beta_x(\rho_x^0, t^* - t_d) \leq c_{t^*}, \tag{A.6}$$

where $\beta_x \in \mathcal{KL}$ is defined in Proposition 11. From $|\tilde{\theta}_d(0)| \leq \rho_\theta^0$ and (21) in Proposition 9, it follows that

$$\sup_{t\in[0,t^*]} |\tilde{\theta}_d(t)| \leq \rho_\theta^0 + a\omega\delta c_\theta t^*. \tag{A.7}$$

Using that $\beta_\theta \in \mathcal{KL}$ in Proposition 10 is continuous and, without loss of generality, $\beta_\theta(s, 0) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$, from (A.3) and (A.7), it follows that

$$\sup_{t\in[0,t^*]} |\tilde{\theta}_d(t)| \leq \beta_\theta(\rho_\theta^0 + a\omega\delta c_\theta t^*, 0) \leq \rho^\theta, \tag{A.8}$$

for sufficiently small $\omega, \delta \in \mathbb{R}_{>0}$. Note that, for sufficiently small $\omega \in \mathbb{R}_{>0}$, we have that

$$C_x = \tilde{\gamma}_{xC}(a\omega\tilde{C}_x) \leq \min\{\rho^x, c_{t^*}, \nu_x\}, \tag{A.9}$$

where $\tilde{\gamma}_{xC} \in \mathcal{K}$ and $\tilde{C}_x \in \mathbb{R}_{>0}$ are defined in Proposition 11. From (A.3), (A.9), $|\tilde{x}_d(0)| \leq \rho_x^0$, and (23) in Proposition 11, it follows that

$$\sup_{t\geq 0} |\tilde{x}_d(t)| \leq \max\left\{\beta_x(\rho_x^0, 0), C_x\right\} \leq \rho^x, \tag{A.10}$$

for sufficiently small $\omega \in \mathbb{R}_{>0}$. Note that $|\tilde{x}_d(t)| := \max_{s\in[t-t_d,t]} |\tilde{x}(s)|$ implies that $\sup_{t\geq t^*} |\tilde{x}_d(t)| = \sup_{t\geq t^*-t_d} |\tilde{x}(t)|$. Then, from (A.6), (A.9) and (23) in Proposition 11, we have

$$\sup_{t\geq t^*} |\tilde{x}_d(t)| \leq \max\left\{\beta_x(\rho_x^0, t^* - t_d), C_x\right\} \leq c_{t^*}, \tag{A.11}$$

for sufficiently small $\omega \in \mathbb{R}_{>0}$. From (A.8), (A.11) and (22) in Proposition 10, it follows that

$$|\tilde{\theta}(t)| \leq \max\left\{\beta_\theta\left(\rho_\theta^0 + a\omega\delta c_\theta t^*, a^2\omega\delta(t - t^*)\right), \gamma_{\theta x}(c_{t^*}), C_\theta\right\}, \tag{A.12}$$

for all $t \geq t^*$. From (A.4), (A.5) and (A.12), we immediately obtain that

$$\limsup_{t\to\infty} |\tilde{\theta}(t)| \leq \nu_\theta, \tag{A.13}$$

for sufficiently small $\omega, \delta \in \mathbb{R}_{>0}$. Moreover, from (A.4), (A.5), (A.8) and (A.12), it follows that

$$\sup_{t\geq t^*} |\tilde{\theta}(t)| \leq \max\left\{\beta_\theta\left(\rho_\theta^0 + a\omega\delta c_\theta t^*, 0\right), \gamma_{\theta x}(c_{t^*}), C_\theta\right\} \leq \rho^\theta, \tag{A.14}$$

for sufficiently small $\omega, \delta \in \mathbb{R}_{>0}$. From (A.9) and (23) in Proposition 11, we obtain

$$\limsup_{t\to\infty} |\tilde{x}(t)| \leq \nu_x, \tag{A.15}$$

for sufficiently small $\omega \in \mathbb{R}_{>0}$. From (A.8), (A.10) and (A.14), it follows that

$$\sup_{t\geq 0} |\tilde{\theta}_d(t)| \leq \rho^\theta \quad \text{and} \quad \sup_{t\geq 0} |\tilde{x}_d(t)| \leq \rho^\theta, \tag{A.16}$$

for sufficiently small $\omega, \delta \in \mathbb{R}_{>0}$. This implies that the bounds in Propositions 9–11 are valid for sufficiently small $a, \omega, \delta \in \mathbb{R}_{>0}$. From (A.8), (A.10) and (A.14), it follows that (A.1) is satisfied with

$$C = \begin{bmatrix} \max\left\{\beta_\theta\left(\rho_\theta^0 + a\omega\delta c_\theta t^*, 0\right), \gamma_{\theta x}(c_{t^*}), C_\theta\right\} \\ \max\left\{\beta_x(\rho_x^0, 0), C_x\right\} \end{bmatrix}. \tag{A.17}$$

Moreover, (A.2) directly follows from (A.13) and (A.15). Recall that we first fixed $a \in \mathbb{R}_{>0}$ before we selected sufficiently small values of $\omega, \delta \in \mathbb{R}_{>0}$. Hence, there exist some (sufficiently small) $a^* \in \mathbb{R}_{>0}, \omega^* = \omega^*(a) \in \mathbb{R}_{>0}$ and $\delta^* = \delta^*(a) \in \mathbb{R}_{>0}$ such that (A.1) and (A.2) hold for all $a \in (0, a^*)$, $\omega \in (0, \omega^*)$ and all $\delta \in (0, \delta^*)$. This completes the proof of the theorem.

A.2. Proof of Proposition 10

Throughout the proof, we assume that $a, \omega, \delta \in \mathbb{R}_{>0}$ are bounded. Consider some $i \in [1, \infty]$. To prove the proposition, we first write the $\tilde{\theta}$-dynamics in (20) as

$$\frac{d\tilde{\theta}}{dt} = \frac{a\omega^2\delta}{\pi}\int_{t-\frac{2\pi}{\omega}}^{t} J_{sta,i}(\tilde{\theta}(t)+\theta^*+as(\tau))s(\tau)d\tau + a\omega\delta\Delta_1 + a\omega\delta\Delta_2 + a\omega\delta\Delta_3, \tag{A.18}$$

where

$$\Delta_1 := \frac{\omega}{\pi}\int_{t-\frac{2\pi}{\omega}}^{t}\Big\{J_{sta,i}(\tilde{\theta}(\tau)+\theta^*+as(\tau)) - J_{sta,i}(\tilde{\theta}(t)+\theta^*+as(\tau))\Big\}s(\tau)d\tau, \tag{A.19}$$

$$\Delta_2 := \frac{\omega}{\pi}\int_{t-\frac{2\pi}{\omega}}^{t}\Big\{J_i(M_d(\tau), w_d(\tau)) - J_{sta,i}(\tilde{\theta}(\tau)+\theta^*+as(\tau))\Big\}s(\tau)d\tau, \tag{A.20}$$

$$\Delta_3 := \frac{\omega}{\pi}\int_{t-\frac{2\pi}{\omega}}^{t}\Big\{J_i(\tilde{x}_d(\tau)+M_d(\tau), w_d(\tau)) - J_i(M_d(\tau), w_d(\tau))\Big\}s(\tau)d\tau. \tag{A.21}$$

Applying the Taylor series expansion, we write

$$J_{sta,i}(\tilde{\theta}(t)+\theta^*+as(\tau)) = J_{sta,i}(\tilde{\theta}(t)+\theta^*) + as(\tau)\frac{dJ_{sta,i}}{d\theta}(\tilde{\theta}(t)+\theta^*) + a^2R(\tilde{\theta}(t),\tau,a), \tag{A.22}$$

with

$$R(\tilde{\theta}(t),\tau,a) := s^2(\tau)\int_0^1(1-r)\cdot\frac{d^2J_{sta,i}}{d\theta^2}(\tilde{\theta}(t)+\theta^*+as(\tau)r)dr. \tag{A.23}$$

Using (A.22), the $\tilde{\theta}$-dynamics in (A.18) are rewritten as

$$\frac{d\tilde{\theta}}{dt} = a^2\omega\delta\frac{dJ_{sta,i}}{d\theta}(\tilde{\theta}+\theta^*) + a\omega\delta\Delta_1 + a\omega\delta\Delta_2 + a\omega\delta\Delta_3 + a^3\omega\delta\Delta_4, \tag{A.24}$$

with

$$\Delta_4 := \frac{\omega}{\pi}\int_{t-\frac{2\pi}{\omega}}^{t} R(\tilde{\theta}(t),\tau,a)s(\tau)d\tau. \tag{A.25}$$

Using the Lyapunov–Razumikhin function candidate $V_\theta(\tilde{\theta}) := \frac{\tilde{\theta}^2}{2}$, see for example Teel (1998), we obtain

$$\frac{dV_\theta}{dt} = a^2\omega\delta\frac{dJ_{sta,i}}{d\theta}(\tilde{\theta}+\theta^*)\tilde{\theta} + a\omega\delta\Delta_1\tilde{\theta} + a\omega\delta\Delta_2\tilde{\theta} + a\omega\delta\Delta_3\tilde{\theta} + a^3\omega\delta\Delta_4\tilde{\theta}. \tag{A.26}$$

From (A.26) and (13) in Assumption 6, it follows that

$$\frac{dV_\theta}{dt} \le -a^2\omega\delta\alpha_J(|\tilde{\theta}|) + a\omega\delta|\Delta_1||\tilde{\theta}| + a\omega\delta|\Delta_2||\tilde{\theta}| + a\omega\delta|\Delta_3||\tilde{\theta}| + a^3\omega\delta|\Delta_4|\,|\tilde{\theta}|. \tag{A.27}$$

Let us now derive bounds for $|\Delta_i|$, $i = 1, 2, 3, 4$, to be used in the inequality in (A.27), starting with $|\Delta_1|$. Note that, using (20), we obtain that

$$\begin{aligned} &J_{sta,i}(\tilde{\theta}(\tau)+\theta^*+as(\tau)) - J_{sta,i}(\tilde{\theta}(t)+\theta^*+as(\tau)) \\ &= \int_t^\tau \frac{dJ_{sta,i}}{d\theta}(\tilde{\theta}(r)+\theta^*+as(\tau))\frac{d\tilde{\theta}}{dt}(r)dr \\ &= 2a\omega\delta\int_t^\tau \frac{dJ_{sta,i}}{d\theta}(\tilde{\theta}(r)+\theta^*+as(\tau)) \\ &\quad\cdot\frac{\omega}{2\pi}\int_{r-\frac{2\pi}{\omega}}^{r} J_i(\tilde{x}_d(\rho)+M_d(\rho), w_d(\rho))s(\rho)d\rho dr, \end{aligned} \tag{A.28}$$

for all $\tau \in [t-\frac{2\pi}{\omega}, t]$ and all $t \ge \frac{2\pi}{\omega}$. From $\sup_{t\ge0}|\tilde{\theta}_d(t)| \le \rho^\theta$ and $\sup_{t\ge0}|\tilde{x}_d(t)| \le \rho^x$, we conclude that $|\tilde{\theta}| \le \rho^\theta$ and $|\tilde{x}| \le \rho^x$ for all $t \ge -t_d$. Combining (A.19) and (A.28), we obtain that there exists some constant $C_{\Delta_1} \in \mathbb{R}_{>0}$ such that

$$|\Delta_1| \le a\delta C_{\Delta_1}, \tag{A.29}$$

for all $|\tilde{\theta}| \le \rho^\theta$, $|\tilde{x}| \le \rho^x$ and all $t \ge \frac{2\pi}{\omega}$.

Let us now upper bound $|\Delta_2|$. Suppose that the performance function J_i in (11) is subjected to steady-state inputs $M_{1d}(t) = M(\theta_{1d}(t), w_d(t))$ and $M_{2d}(t) = M(\theta_{2d}(t), w_d(t))$ with $(\theta_{1d}, w_d(t)), (\theta_{2d}, w_d(t)) \in \mathcal{C}([-t_d, 0]; \mathcal{Q}) \times \mathcal{C}([-t_d, 0]; \mathbb{R}^l)$, where $\mathcal{Q} \subset \mathbb{R}$ is a compact set. Note that, from the definition of the performance measures L_i in (8), it follows that $L_i(y_d(t)) \le L_\infty(y_d(t)) = \max_{\tau\in[-T_w,0]}|y(t+\tau)|$ for all $i \in [1, \infty]$. Then, from (11) and the continuity of the functions g, h and M, it follows that

$$\begin{aligned} &|J_i(M_{1d}(t), w_d(t)) - J_i(M_{2d}(t), w_d(t))| \\ &= |g\circ L_i\circ h(M(\theta_{1d}(t), w_d(t)), w_d(t)) \\ &\quad - g\circ L_i\circ h(M(\theta_{2d}(t), w_d(t)), w_d(t))| \\ &\le C_M \max_{\tau\in[-T_w,0]}|\theta_1(t+\tau)-\theta_2(t+\tau)|, \end{aligned} \tag{A.30}$$

for some constant $C_M \in \mathbb{R}_{>0}$. Let the function $\theta_1(t) = \theta_1$ be constant for all $t \in \mathbb{R}$. Then, from the definitions of J_i in (11) and $J_{sta,i}$ in (12), we have that

$$J_i(M(\theta_{1d}(t), w_d(t)), w_d(t)) = J_{sta,i}(\theta_1), \tag{A.31}$$

for all $t \in \mathbb{R}$. Hence, for constant $\theta_1(t) = \theta_1$, from (A.30), (A.31), and $M_{1d}(t) = M(\theta_{1d}(t), w_d(t))$, we obtain that

$$\begin{aligned} &|J_{sta,i}(\theta_1) - J_i(M_{2d}(t), w_d(t))| \\ &\le C_M \max_{\tau\in[-T_w,0]}|\theta_1-\theta_2(t+\tau)|. \end{aligned} \tag{A.32}$$

Now, let $\theta_3(t) \in \mathcal{Q}$, for all $t \in \mathbb{R}$, be an arbitrary (time-varying) function. Note that, for any arbitrary fixed value of $t \in \mathbb{R}$, we can define θ_1 such that $\theta_1 = \theta_3(t)$. Since the fixed value of $t \in \mathbb{R}$ can be chosen arbitrarily, from $\theta_1 = \theta_3(t)$ and (A.32), it follows that

$$\begin{aligned} &|J_{sta,i}(\theta_3(t)) - J_i(M_{2d}(t), w_d(t))| \\ &\le C_M \max_{\tau\in[-T_w,0]}|\theta_3(t)-\theta_2(t+\tau)|, \end{aligned} \tag{A.33}$$

for all $t \in \mathbb{R}$. Hence, (A.33) holds for any arbitrary (time-varying) function $\theta_3(t) \in \mathcal{Q}$ and all $t \in \mathbb{R}$. Next, for all $t \ge -t_d$, let $\theta_3(t) = \tilde{\theta}(t)+\theta^*+as(t)$ and $\theta_2(t) = \theta(t) = \tilde{\theta}(t)+\theta^*+a\sin(\omega t)$, which implies that $M_{2d}(t) = M_d(t)$ for all $t \ge 0$. Then, from (A.20), (A.33), and $s(t) := \sin(\omega[t-\phi])$, it follows that

$$|\Delta_2| \le 2C_M \max_{\tau\in[t-\frac{2\pi}{\omega},t], r\in[-T_w,0]}\Bigg[\Big|\tilde{\theta}(\tau)-\tilde{\theta}(\tau+r)\Big| + a\,|\sin(\omega[\tau-\phi])-\sin(\omega[\tau+r])|\Bigg], \tag{A.34}$$

for all $t \geq 0$. Using the $\tilde{\theta}$-dynamics in (20), we have that

$$\max_{\tau \in [t-\frac{2\pi}{\omega},t], r \in [-T_w,0]} \left|\tilde{\theta}(\tau) - \tilde{\theta}(\tau + r)\right|$$
$$\leq \max_{\tau \in [t-\frac{2\pi}{\omega},t]} \int_{\tau - T_w}^{\tau} \left|\dot{\tilde{\theta}}(r)\right| dr \leq T_w \max_{\tau \in [t-\frac{2\pi}{\omega}-T_w,t]} \left|\dot{\tilde{\theta}}(\tau)\right|$$
$$\leq 2a\omega\delta T_w \max_{\tau \in [t-\frac{2\pi}{\omega}-T_w,t]} \frac{\omega}{2\pi}$$
$$\cdot \int_{\tau - \frac{2\pi}{\omega}}^{\tau} \left|J_i(\tilde{x}_d(r) + M_d(r), w_d(r))\right| dr, \tag{A.35}$$

for all $t \geq \frac{2\pi}{\omega} + T_w = t_d$. Furthermore, we have that

$$\max_{\tau \in [t-\frac{2\pi}{\omega},t], r \in [-T_w,0]} |\sin(\omega[\tau - \phi]) - \sin(\omega[\tau + r])|$$
$$\leq \omega \max_{\tau \in [t-\frac{2\pi}{\omega},t], r \in [-T_w,0]} \int_{\tau + r}^{\tau - \phi} |\cos(\omega\rho)| \, d\rho$$
$$\leq \omega T_w \left[\left|\frac{1}{2} - \frac{\phi}{T_w}\right| + \frac{1}{2}\right]. \tag{A.36}$$

Combining (A.34)–(A.36) yields that there exists some constant $C_{\Delta_2} \in \mathbb{R}_{>0}$ such that

$$|\Delta_2| \leq a\omega T_w C_{\Delta_2}, \tag{A.37}$$

for all $|\tilde{\theta}| \leq \rho^\theta$, $|\tilde{x}| \leq \rho^x$ and all $t \geq t_d$. Note that (A.30), and therefore (A.37), is valid, because for $\theta_3(t) = \tilde{\theta}(t) + \theta^* + as(t)$ and $\theta_2(t) = \tilde{\theta}(t) + \theta^* + a\sin(\omega t)$ there always exists a compact set $\mathcal{Q} \subset \mathbb{R}$ such that $\theta_3(t), \theta_2(t) \in \mathcal{Q}$ for all $|\tilde{\theta}| \leq \rho^\theta$.

Next, let us upper bound $|\Delta_3|$. Similar to (A.30), using the definition of J_i in (11), it follows that there exists some constant $C_{J_i} \in \mathbb{R}_{>0}$ such that

$$|J_i(\tilde{x}_d(\tau) + M_d(\tau), w_d(\tau)) - J_i(M_d(\tau), w_d(\tau))|$$
$$= |g \circ L_i \circ h(\tilde{x}_d(\tau) + M_d(\tau), w_d(\tau))$$
$$- g \circ L_i \circ h(M_d(\tau), w_d(\tau))|$$
$$\leq C_{J_i} \max_{r \in [-T_w,0]} |\tilde{x}(\tau + r)|, \tag{A.38}$$

for all $|\tilde{\theta}| \leq \rho^\theta$, $|\tilde{x}| \leq \rho^x$. Combining (A.21) and (A.38), we obtain that there exists a constant $C_{\Delta_3} \in \mathbb{R}_{>0}$ such that

$$|\Delta_3| \leq C_{\Delta_3} |\tilde{x}_d|, \tag{A.39}$$

for all $t \geq 0$, with $|\tilde{x}_d(t)| := \max_{s \in [t-t_d,t]} |\tilde{x}(s)|$. Hence, from (A.27), (A.29), (A.37), (A.39) and $|\tilde{\theta}| \leq \rho^\theta$, it follows that

$$\frac{dV_\theta}{dt} \leq -a^2\omega\delta\alpha_J(|\tilde{\theta}|) + a\omega\delta C_{\Delta_3}\rho^\theta |\tilde{x}_d|$$
$$+ a^2\omega\delta[a + \omega T_w + \delta] \max\{C_{\Delta_1}, C_{\Delta_2}, C_{\Delta_4}\}\rho^\theta, \tag{A.40}$$

for all $|\tilde{\theta}| \leq \rho^\theta$, $|\tilde{x}| \leq \rho^x$ and all $t \geq t_d$, where the constant $C_{\Delta_4} \in \mathbb{R}_{>0}$ upper bounds $|\Delta_4|$, with Δ_4 in (A.25). Note that we used that the function R in (A.23) is bounded for all $|\tilde{\theta}| \leq \rho^\theta$ and all $t \in \mathbb{R}$ to conclude that $|\Delta_4| \leq C_{\Delta_4}$ for some constant $C_{\Delta_4} \in \mathbb{R}_{>0}$.

From (A.40), it follows that, if $|\tilde{\theta}| \geq \max\{\alpha_J^{-1}(\frac{4}{a}C_{\Delta_3}\rho^\theta \, |\tilde{x}_d|), \alpha_J^{-1}(4[a + \omega T_w + \delta] \max\{C_{\Delta_1}, C_{\Delta_2}, C_{\Delta_4}\}\rho^\theta)\}$, then

$$\frac{dV_\theta}{dt} \leq -\frac{a^2\omega\delta}{2}\alpha_J(|\tilde{\theta}|), \tag{A.41}$$

for all $|\tilde{\theta}| \leq \rho^\theta$, $|\tilde{x}| \leq \rho^x$ and all $t \geq t_d$. Using a similar approach as in for example Sontag (1989), from (A.41), it can be shown that (22) is satisfied with $\tilde{\gamma}_{\theta x1}(r) := \tilde{\gamma}_{\theta C}(r) := \alpha_J^{-1}(r)$, $\tilde{\gamma}_{\theta x2}(r) := 4C_{\Delta_3}\rho^\theta r$ for all $r \in \mathbb{R}_{\geq 0}$ and $\tilde{C}_\theta = 4\max\{C_{\Delta_1}, C_{\Delta_2}, C_{\Delta_4}\}\rho^\theta$. This completes the proof of the proposition.

A.3. Proof of Proposition 11

Throughout the proof, we assume that $a, \omega, \delta \in \mathbb{R}_{>0}$ are bounded. Using $V_x(\tilde{x})$ in Assumption 3 as a Lyapunov function candidate for the $\tilde{x}$-dynamics in (20) yields

$$\frac{dV_x}{dt} = \frac{dV_x}{d\tilde{x}}\tilde{f}\left(\tilde{x}, M(\theta, w(t)), \theta, w(t)\right)$$
$$- \frac{dV_x}{d\tilde{x}}\frac{\partial M}{\partial\theta}\left[\frac{d\tilde{\theta}}{dt} + a\omega\cos(\omega t)\right]. \tag{A.42}$$

Substituting (7) and the $\tilde{\theta}$-dynamics in (20) in (A.42), we obtain

$$\frac{dV_x}{dt} \leq -\alpha_f(|\tilde{x}|) - a\omega\Delta_5, \tag{A.43}$$

with

$$\Delta_5 := \frac{dV_x}{d\tilde{x}}\frac{\partial M}{\partial\theta}\left[\frac{\omega\delta}{\pi}\int_{t-\frac{2\pi}{\omega}}^{t} J_i(\tilde{x}_d(\tau) + M_d(\tau), w_d(\tau))\right.$$
$$\left.\cdot s(\tau)d\tau + \cos(\omega t)\right]. \tag{A.44}$$

As in the proof of Proposition 10, $\sup_{t\geq 0}|\tilde{\theta}_d(t)| \leq \rho^\theta$ and $\sup_{t\geq 0}|\tilde{x}_d(t)| \leq \rho^x$ imply that $|\tilde{\theta}| \leq \rho^\theta$ and $|\tilde{x}| \leq \rho^x$ for all $t \geq -t_d$. For all $|\tilde{\theta}| \leq \rho^\theta$, $|\tilde{x}| \leq \rho^x$ and all $t \geq 0$, there exists a constant $C_{\Delta_5} \in \mathbb{R}_{>0}$ such that

$$|\Delta_5| \leq C_{\Delta_5}, \tag{A.45}$$

where Δ_5 is defined in (A.44). Combining (A.43) and (A.45) yields

$$\frac{dV_x}{dt} \leq -\alpha_f(|\tilde{x}|) + a\omega C_{\Delta_5}, \tag{A.46}$$

for all $|\tilde{\theta}| \leq \rho^\theta$, $|\tilde{x}| \leq \rho^x$ and all $t \geq 0$. From (A.46), it follows that, if $|\tilde{x}| \geq \alpha_f^{-1}(2a\omega C_{\Delta_5})$, then

$$\frac{dV_x}{dt} \leq -\frac{1}{2}\alpha_f(|\tilde{x}|), \tag{A.47}$$

for all $|\tilde{\theta}| \leq \rho^\theta$, $|\tilde{x}| \leq \rho^x$ and all $t \geq 0$. Using a similar approach as in for example Sontag (1989), from (A.47), it can be shown that (23) is satisfied with $\tilde{\gamma}_{xC}(r) := \alpha_{x1}^{-1} \circ \alpha_{x2} \circ \alpha_f^{-1}(r)$ for all $r \in \mathbb{R}_{\geq 0}$ and $\tilde{C}_x := 2C_{\Delta_5}$, where $\alpha_{x1}, \alpha_{x2} \in \mathcal{K}_\infty$ are defined in (6). This completes the proof of the proposition.

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