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Global robust output regulation for Lur'e systems

Alexei Pavlov, Nathan van de Wouw and Henk Nijmeijer

Abstract— **In this paper, we consider the global output regulation problem for systems with an output depending nonlinearity. If the system model is known and the corresponding regulator equations are solvable, an observer-based controller solving the global output regulation problem is presented. Sufficient conditions for the existence of such a controller are formulated in terms of an LMI. If the system parameters are not exactly known and the nonlinearity can vary over a given class, we show that there exists a robust controller provided that the robust output regulation problem for the system without the nonlinearity is solvable and that there exists a suboptimal solution to a certain H_∞ optimization problem.**

I. Introduction

In this paper we address the output regulation problem, which includes the problems of tracking reference signals and rejecting disturbances generated by an external autonomous system (exosystem). For linear systems, this problem was thoroughly investigated in 1970-s [1], [2]. For nonlinear systems, intensive research started with the papers [3] and [4], which provided solutions to the local output regulation problem for general nonlinear systems. These papers were followed by a number of results dealing with different aspects of the output regulation problem for nonlinear systems: approximate, robust and adaptive output regulation, see [5] and references therein.

At the moment, most of the existing results on the output regulation problem for nonlinear systems are either local or semiglobal. This can be explained by the fact that in the local case one can still use linear techniques to make desired trajectories locally asymptotically stable; semiglobal solutions are usually based on high-gain controllers or observers, which allow to cope with nonlinearities in a bounded region of the state space, see e.g. [6]. The global output regulation problem requires a better "understanding" of the system dynamics on the whole state space, which makes it difficult to tackle. Only a few global results exist and are mostly limited to systems which are linear in the unmeasured variables [7], [8], see also [9], [10] for recent results on the global robust output regulation problem. A new approach to solving the global output regulation problem was proposed in [11]. The approach is based on the so-called incremental stability property [12], [13], [14], [15] (in [16] the incremental stability is used in the context of the local output regulation problem). Within this approach, a controller is designed in such a way that the closed-loop system has the following properties: **a)** there exists a solution of the closed-loop system on which the regulated output equals to zero and **b)** *every* solution of the closed-loop system is globally asymptotically stable. These two properties imply that the desired solution is globally asymptotically stable. This approach can be beneficial if the desired trajectory is known to exist, but is not known in advance, which is the case in robust output regulation.

Here, we apply the incremental stability approach to systems with an output depending nonlinearity, also known as Lur'e systems. The structure of such systems allows us to formulate straightforward LMI-based sufficient conditions for solvability of the global output regulation problem both for state- and output-feedback cases. If the system contains uncertainties both in the system matrices and in the nonlinearity, we show that the problem of finding a robust controller can be reduced to solving the linear robust output regulation problem for the system without the nonlinearity and finding a suboptimal solution to a certain H_∞ optimization problem. The robust output regulation problem with gain-bounded uncertainties has been previously considered in [17] for the case of linear systems with linear (dynamical) uncertainties and for nonlinear systems in [18] for the case of approximate local output regulation. We provide a global solution to the robust output regulation problem for linear systems with nonlinear static uncertainties.

The paper is organized as follows. In Section II, we formulate the global output regulation problem and introduce some preliminary notions and results. The main results are given in Section III – on an observer-based controller design and in Section IV – on a robust controller design. Section V contains examples and Section VI presents conclusions.

II. Problem formulation and preliminaries

Consider systems modelled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + D\phi(z) + Pw && (1)\\ z &= C_z x + Q_z w && (2)\\ e &= C_r x + Q_r w && (3)\\ y &= Cx + Qw, && (4) \end{aligned}$$

with state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^m$, auxiliary output $z \in \mathbb{R}$, regulated output $e \in \mathbb{R}^l$ and measured output $y \in \mathbb{R}^p$. The nonlinearity $\phi(z)$ is scalar and continuously differentiable. The exogenous signal $w(t) \in \mathbb{R}^k$, which can be viewed as a disturbance in equation (1) or as a reference signal in (3), is generated by the exosystem

$$\dot{w} = Sw, \qquad (5)$$

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The authors are with the Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. A.Pavlov@tue.nl, N.v.d.Wouw@tue.nl, H.Nijmeijer@tue.nl

where S is such that all its eigenvalues are simple and lie on the imaginary axis. Such exosystem generates constant signals and harmonic signals at a fixed set of frequencies.

The nonlinearity $\phi(z)$ is assumed to belong to a class $\mathcal{F}(\gamma)$ defined as

$$\mathcal{F}(\gamma) := \left\{ \phi(\cdot) \in C^1 : \ \phi(0) = 0, \ \left| \frac{\partial \phi}{\partial z}(z) \right| \leq \gamma \right\}. \quad (6)$$

The global output regulation problem is formulated in the following way: find, if possible, a feedback of the form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y) \\ u &= \theta(\xi, y) \end{aligned} \quad (7)$$

such that **a)** all solutions of the system

$$\dot{x} = Ax + B\theta(\xi, y) + D\phi(C_z x + Q_z w) + Pw \quad (8)$$
$$\dot{\xi} = \eta(\xi, y) \quad (9)$$
$$y = Cx + Qw \quad (10)$$
$$\dot{w} = Sw \quad (11)$$

are bounded and satisfy

$$e(t) = C_r x(t) + Q_r w(t) \to 0 \quad \text{as} \quad t \to +\infty;$$

b) for $w(t) \equiv 0$ the origin $(x, \xi) = (0, 0)$ is a globally asymptotically stable equilibrium of (8)–(10).

If system (1)-(4) depends on some vector of parameters $\mathcal{H} \in \mathbb{R}^h$, with a nominal value $\mathcal{H}^\circ$, we say that controller (7) is structurally stable with respect to $\mathcal{H}$ if properties **a)** and **b)** hold for all $\mathcal{H}$ from some neighborhood of $\mathcal{H}^\circ$. If **a)** and **b)** hold for all $\mathcal{H} \in \mathbb{R}^h$, we call the controller (7) robust with respect to $\mathcal{H}$. We call controller (7) robust with respect to $\phi(\cdot) \in \mathcal{F}(\gamma)$, if properties **a)** and **b)** hold for all $\phi(\cdot) \in \mathcal{F}(\gamma)$.

Quadratic stability

Prior to solving the problem, we introduce and discuss the following notion.

Definition 1. A matrix function $\mathcal{A}(\chi) \in \mathbb{R}^{n \times n}$, $\chi \in \mathbb{R}^k$, is called quadratically stable if for some $\mathcal{P} = \mathcal{P}^T > 0$ and $\mathcal{Q} = \mathcal{Q}^T > 0$

$$\mathcal{P}\mathcal{A}(\chi) + \mathcal{A}(\chi)^T \mathcal{P} \leq -\mathcal{Q}, \quad \forall \chi \in \mathbb{R}^k. \quad (12)$$

The purpose of this notion becomes clear from the following lemma, see [12], [13].

Lemma 1: Consider the system

$$\dot{\zeta} = F(\zeta, v(t)), \quad (13)$$

where $\zeta \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^m$ is a continuous input defined on $t \in [0, +\infty)$, and F is C^1 with respect to ζ and continuous with respect to v. Suppose $\frac{\partial F}{\partial \zeta}(\zeta, v)$ is quadratically stable. Then, for every continuous bounded $v(t)$ every solution of system (13) is globally exponentially stable (GES) with a rate of convergence independent of $v(t)$.

Notice, that for Lur'e systems

$$\begin{aligned} \dot{\zeta} &= F(\zeta, v) := \mathcal{A}\zeta + \mathcal{D}\phi(z) + f_1(v) \\ z &= \mathcal{C}_z \zeta + f_2(v) \end{aligned} \quad (14)$$

with a scalar nonlinearity $\phi(\cdot) \in \mathcal{F}(\gamma)$, it holds that

$$\frac{\partial F}{\partial \zeta}(\zeta, v) = \mathcal{A} + \frac{\partial \phi}{\partial z}(z)\mathcal{D}\mathcal{C}_z \in \text{co}\{\mathcal{A}^-, \mathcal{A}^+\}, \quad (15)$$

where $\mathcal{A}^- := \mathcal{A} - \gamma\mathcal{D}\mathcal{C}_z$, $\mathcal{A}^+ := \mathcal{A} + \gamma\mathcal{D}\mathcal{C}_z$. Therefore, quadratic stability of $\frac{\partial F}{\partial \zeta}(\zeta, v)$ can be established by checking feasibility of the following LMI:

$$\begin{aligned} \mathcal{P}\mathcal{A}^- + (\mathcal{A}^-)^T\mathcal{P} &< 0, \\ \mathcal{P}\mathcal{A}^+ + (\mathcal{A}^+)^T\mathcal{P} &< 0 \\ \mathcal{P} = \mathcal{P}^T &> 0 \end{aligned} \quad (16)$$

Indeed, if (16) is feasible, then any matrix function $\tilde{\mathcal{A}}(\zeta, v)$ satisfying $\tilde{\mathcal{A}}(\zeta, v) \in \text{co}\{\mathcal{A}^-, \mathcal{A}^+\}$ for all $(\zeta, v) \in \mathbb{R}^{n+m}$ is quadratically stable with a matrix $\mathcal{P}$ satisfying (16). In particular, due to (15), $\frac{\partial F}{\partial \zeta}(\zeta, v)$ is also quadratically stable. Solvability of the LMI (16), in turn, can be established by the circle criterion. This is stated in the following lemma (see e.g. [19], [20]).

Lemma 2: Suppose, the matrix $\mathcal{A}$ is Hurwitz. Then, the LMI (16) is feasible iff

$$\left| \mathcal{C}_z (i\omega I - \mathcal{A})^{-1} \mathcal{D} \right| < 1/\gamma, \quad \forall\, \omega \in \mathbb{R}.$$

Next, consider system (14) with control u and measured output y given by

$$\begin{aligned} \dot{\zeta} &= \mathcal{A}\zeta + \mathcal{D}\phi(z) + f_1(v) + \mathcal{B}u \\ z &= \mathcal{C}_z \zeta + f_2(v) \\ y &= \mathcal{C}\zeta + f_3(v). \end{aligned} \quad (17)$$

For the sake of designing controllers and observers, it is of interest to find out under what conditions there exist matrices K and L such that $\frac{\partial F}{\partial \zeta}(\zeta, v) + \mathcal{B}K$ and $\frac{\partial F}{\partial \zeta}(\zeta, v) + L\mathcal{C}$ are quadratically stable. These conditions can be expressed in terms of an LMI, as follows from the next lemma (see e.g. [21]).

Lemma 3: Consider system (17).

i) Suppose, the LMI

$$\begin{aligned} \mathcal{A}^-\mathcal{P} + \mathcal{P}(\mathcal{A}^-)^T + \mathcal{B}\mathcal{Y} + \mathcal{Y}^T\mathcal{B}^T &< 0, \\ \mathcal{A}^+\mathcal{P} + \mathcal{P}(\mathcal{A}^+)^T + \mathcal{B}\mathcal{Y} + \mathcal{Y}^T\mathcal{B}^T &< 0, \\ \mathcal{P} = \mathcal{P}^T &> 0. \end{aligned} \quad (18)$$

is feasible. Then, $\frac{\partial F}{\partial \zeta}(\zeta, v) + \mathcal{B}K$ is quadratically stable for $K = \mathcal{Y}\mathcal{P}^{-1}$, where $\mathcal{Y}$ and $\mathcal{P}$ satisfy (18).

ii) Suppose, the LMI

$$\begin{aligned} \mathcal{P}\mathcal{A}^- + (\mathcal{A}^-)^T\mathcal{P} + \mathcal{X}\mathcal{C} + \mathcal{C}^T\mathcal{X}^T &< 0, \\ \mathcal{P}\mathcal{A}^+ + (\mathcal{A}^+)^T\mathcal{P} + \mathcal{X}\mathcal{C} + \mathcal{C}^T\mathcal{X}^T &< 0, \\ \mathcal{P} = \mathcal{P}^T &> 0. \end{aligned} \quad (19)$$

is feasible. Then, $\frac{\partial F}{\partial \zeta}(\zeta, v) + L\mathcal{C}$ is quadratically stable for $L = \mathcal{P}^{-1}\mathcal{X}$, where $\mathcal{X}$ and $\mathcal{P}$ satisfy (19).

III. Observer-based controller

The results in this section are based on the assumption of solvability of the so-called regulator equations:

A1 There exist C^1 mappings $\pi(w)$ and $c(w)$ defined on $\mathbb{R}^m$ and satisfying the relations $\pi(0) = 0$, $c(0) = 0$ and

$$\frac{\partial \pi}{\partial w} Sw = A\pi(w) + Bc(w) + D\phi(C_z\pi(w) + Q_z w) + Pw,$$
$$C_r\pi(w) + Q_r w = 0. \tag{20}$$

In the sequel, the following notations will be used:

$$A_c^-(\gamma) := A - \gamma DC_z, \quad A_c^+(\gamma) := A + \gamma DC_z,$$

where $\gamma > 0$ is such that $\phi(\cdot) \in \mathcal{F}(\gamma)$. In order to simplify the subsequent formulas, we will write A_c^- instead of $A_c^-(\gamma)$ and A_c^+ instead of $A_c^+(\gamma)$. Let us first consider the static state feedback case when the states x and w are available for measurements, i.e. $y = (x, w)$.

Theorem 1: Consider system (1)-(4) with $y = (x, w)$ and exosystem (5). Suppose, assumption **A1** holds, $\phi(\cdot) \in \mathcal{F}(\gamma)$ and the LMI

$$\begin{array}{rcl} A_c^+\mathcal{P}_c + \mathcal{P}_c(A_c^+)^T + B\mathcal{Y} + \mathcal{Y}^T B^T & < & 0, \\ A_c^+\mathcal{P}_c + \mathcal{P}_c(A_c^+)^T + B\mathcal{Y} + \mathcal{Y}^T B^T & < & 0, \\ \mathcal{P}_c = \mathcal{P}_c^T & > & 0. \end{array} \tag{21}$$

is feasible. Then, the global output regulation problem is solved by a controller of the form

$$u = c(w) + K(x - \pi(w)), \quad K := \mathcal{Y}\mathcal{P}_c^{-1}, \tag{22}$$

where $\mathcal{P}_c$ and $\mathcal{Y}$ satisfy (21) and $\pi(w)$ and $c(w)$ satisfy (20).

Proof: Consider the closed-loop system

$$\dot{x} = F(x, w) + B(c(w) + K(x - \pi(w))), \tag{23}$$

where $F(x, w) := Ax + D\phi(C_z x + Q_z w) + Pw$. By the choice of K, the Jacobian of the right-hand side of (23), which equals $\frac{\partial F}{\partial x}(x, w) + BK$, is quadratically stable (see (21) and Lemma 3). Thus, by Lemma 1, every solution of the closed-loop system is GES. Due to assumption **A1**, for any solution $w(t)$ of the exosystem (5), system (23) has a solution $\bar{x}_w(t) := \pi(w(t))$ along which $e(t) \equiv 0$. Hence, $\bar{x}_w(t)$ is GES and for any other solution $x(t)$ it holds that

$$e(t) = C_r x(t) + Q_r w(t) \xrightarrow[t\to+\infty]{} C_r\pi(w(t)) + Q_r w(t) \equiv 0.$$

Since $w(t)$ is bounded, $\pi(w(t))$ is also bounded. This implies boundedness of all solutions of the closed-loop system. Since $w(t) \equiv 0$ is a solution of the exosystem (5) and $\pi(0) = 0$, then for $w(t) \equiv 0$ the origin $\bar{x}_0(t) \equiv \pi(0) = 0$ is GES. □

Next, we consider the case when only the output y is available for feedback. At this point, we will need the following notations: $\mathcal{C} := [C, Q]$,

$$A_o^-(\gamma) := \begin{bmatrix} A - \gamma DC_z & P - \gamma DQ_z \\ 0 & S \end{bmatrix},$$

$$A_o^+(\gamma) := \begin{bmatrix} A + \gamma DC_z & P + \gamma DQ_z \\ 0 & S \end{bmatrix}.$$

The following theorem provides conditions for the solvability of the global output regulation problem in the case of output feedback.

Theorem 2: Consider system (1)-(4) and exosystem (5). Suppose, the following conditions are satisfied: assumption **A1** holds, $\phi(\cdot) \in \mathcal{F}(\gamma)$ and the LMIs (21) and

$$\begin{array}{rcl} \mathcal{P}_o A_o^+ + (A_o^+)^T\mathcal{P}_o + \mathcal{X}\mathcal{C} + \mathcal{C}^T\mathcal{X}^T & < & 0, \\ \mathcal{P}_o A_o^- + (A_o^-)^T\mathcal{P}_o + \mathcal{X}\mathcal{C} + \mathcal{C}^T\mathcal{X}^T & < & 0, \\ \mathcal{P}_o = \mathcal{P}_o^T & > & 0. \end{array} \tag{24}$$

are feasible. Then, the global output regulation problem is solved by a controller of the form

$$u = c(\hat{w}) + K(\hat{x} - \pi(\hat{w})) \tag{25}$$
$$\dot{\hat{x}} = A\hat{x} + Bu + D\phi(\hat{z}) + P\hat{w} + L_1(\hat{y} - y) \tag{26}$$
$$\dot{\hat{w}} = S\hat{w} + L_2(\hat{y} - y) \tag{27}$$
$$\hat{z} = C_z\hat{x} + Q_z\hat{w}, \quad \hat{y} = C\hat{x} + Q\hat{w} \tag{28}$$

with $K = \mathcal{Y}\mathcal{P}_c^{-1}$, where $\mathcal{P}_c$ and $\mathcal{Y}$ satisfy (21), and $L = [L_1^T, L_2^T]^T = \mathcal{P}_o^{-1}\mathcal{X}$, where $\mathcal{P}_o$ and $\mathcal{X}$ satisfy (24).

Proof: Here, we provide a sketch of the proof. A detailed proof (for a more general class of systems) can be found in [11]. Consider system (1)-(4) in closed-loop with (25)-(28). Denote $\Delta x = \hat{x} - x$, $\Delta w = \hat{w} - w$,

$$\xi := \begin{bmatrix} x \\ w \end{bmatrix}, \quad \Delta\xi = \begin{bmatrix} \Delta x \\ \Delta w \end{bmatrix}.$$

The closed-loop system can be written in the form

$$\dot{x} = F(x, w) + BU(x, w) + B\rho(\Delta x, \Delta w) \tag{29}$$
$$\Delta\dot{\xi} = G(\xi + \Delta\xi) - G(\xi) \tag{30}$$
$$u = U(\xi + \Delta\xi), \tag{31}$$

where

$$U(x, w) := c(w) + K(x - \pi(w)),$$
$$F(x, w) := Ax + D\phi(C_z x + Q_z w) + Pw,$$
$$\rho(\Delta x, \Delta w) := U(x + \Delta x, w + \Delta w) - U(x, w),$$
$$G(\xi) := \begin{bmatrix} Ax + D\phi(C_z x + Q_z w) + Pw \\ Sw \end{bmatrix} + L(Cx + Qw).$$

Notice, that $\rho(0, 0) \equiv 0$. Thus, as follows from the proof of Theorem 1, for $\Delta x = 0$ and $\Delta w = 0$ system (29) has a GES solution $\bar{x}_w(t) = \pi(w(t))$. Moreover, it can be shown (see [11]), that along this solution system (29) is input-to-state stable (ISS) with respect to the inputs Δx and Δw. Consider the estimation error dynamics (30). The Jacobian of the right-hand side of (30) with respect to $\Delta\xi$ equals to

$$\begin{bmatrix} A + \frac{\partial\phi}{\partial z}DC_z & P + \frac{\partial\phi}{\partial z}DQ_z \\ 0 & S \end{bmatrix} + L[C\ Q].$$

By the choice of the matrix L (see (24) and Lemma 3), it is quadratically stable. Hence, by Lemma 1, $\Delta\xi = 0$ is a GES solution of (30). Thus, system (29)-(31), treated

as a cascade, has a uniformly globally asymptotically stable (UGAS) solution $(x, \Delta\xi) = (\pi(w(t)), 0)$ (see Lemma 5.6 in [19]). For the closed-loop system in the original coordinates $(x, \hat{x}, \hat{w})$ this implies that the solution $(x(t), \hat{x}(t), \hat{w}(t)) = (\pi(w(t)), \pi(w(t)), w(t))$ is UGAS. Hence, since $w(t)$ and $\pi(w(t))$ are bounded, every solution of the closed-loop system is also bounded and

$$e(t) = C_r x(t) + Qw(t) \to C_r \pi(w(t)) + Qw(t) \equiv 0,$$

as $t \to +\infty$. Global asymptotic stability of the origin $(0, 0, 0)$ for $w(t) \equiv 0$ is proved in the same way as in the proof of Theorem 1. □

Remark. If z is measured, then the condition on feasibility of the LMI (24) can be relaxed by demanding that the pair of matrices

$$\begin{pmatrix} A & P \\ 0 & S \end{pmatrix}, \quad (C\ Q)$$

is detectable. Under this condition the observer (26)–(28) can be replaced by the observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + D\phi(z) + P\hat{w} + L_1(\hat{y} - y) && (32) \\ \dot{\hat{w}} &= S\hat{w} + L_2(\hat{y} - y) && (33) \\ \hat{y} &= C\hat{x} + Q\hat{w}, && (34) \end{aligned}$$

where $L := [L_1^T\ L_2^T]^T$ is taken such that

$$\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + L[C\ Q]$$

is Hurwitz. Observer (32)-(34) has linear exponentially stable estimation error dynamics. Therefore, the proof of Theorem 2 can be repeated for controller (25) with the observer (32)-(34).

As can be seen from Theorems 1 and 2, the proposed controllers require accurate knowledge of the system model and the mappings $\pi(w)$ and $c(w)$. In practice, both the system model and the mappings $\pi(w)$ and $c(w)$ may be not known exactly or they may change if certain system parameters are varied. This rises the problem of the design of a robust controller that would cope with these uncertainties. This problem is addressed in the next section.

IV. Robust regulator

In this section, we aim at designing a controller that would solve the global output regulation problem not only for the nominal system parameters, but also for the parameters from some neighborhood of the nominal ones and for all nonlinearities $\phi(\cdot) \in \mathcal{F}(\gamma)$. In order to design a robust controller, we assume the following:

A2 There exist matrices $\alpha \in \mathbb{R}^{l \times p}$ and $\beta \in \mathbb{R}^{1 \times p}$ such that $e = \alpha y$ and $z = \beta y$.

For simplicity, we also assume that

A3 both y and u are of the same dimension.

At this point, instead of solving the robust output regulation problem for system (1)-(4), we will solve it for the system

$$\begin{aligned} \dot{x} &= Ax + Bu + D\phi(\beta y) + Pw && (35) \\ \bar{e} &= y = Cx + Qw \end{aligned}$$

with the new regulated output $\bar{e}$. Obviously, since the original regulated output e is a linear function of $\bar{e}$, then by solving the problem for system (35) we also solve it for the original system. The nominal parameters of system (35) are denoted by A°, B°, C°, D°. The proposed design of a robust controller is based on the following lemma.

Lemma 4: Suppose, there exists a linear controller

$$\begin{aligned} \dot{\xi} &= F\xi + Gy, && (36) \\ u &= H_1\xi + H_2 y \end{aligned}$$

such that

i) it solves the robust output regulation problem with internal stability for the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw, && (37) \\ \bar{e} &= y = Cx + Qw, \end{aligned}$$

and the exosystem (5);

ii) for $w \equiv 0$, the transfer function $\mathcal{W}^\circ_{z\phi}(s)$ of the closed-loop system with the nominal parameters

$$\begin{aligned} \dot{x} &= A^\circ x + B^\circ(H_1\xi + H_2 C^\circ x) + D^\circ\phi && (38) \\ \xi &= F\xi + GC^\circ x \\ z &= \beta C^\circ x \end{aligned}$$

from input ϕ to output z satisfies $\|\mathcal{W}^\circ_{z\phi}\|_\infty < 1/\gamma$.

Then the controller (36) solves the global output regulation problem for system (35); it is robust with respect to P, Q and $\phi(\cdot) \in \mathcal{F}(\gamma)$ and structurally stable with respect to A, B, C and D.

Proof: System (35) in closed loop with (36) is a Lur'e system of the form

$$\begin{aligned} \dot{\zeta} &= F(\zeta, w) := \mathcal{A}\zeta + \mathcal{D}\phi(z) + f_1(w) && (39) \\ z &= \mathcal{C}_z\zeta + f_2(w) \end{aligned}$$

with $f_1(w) := ((Pw)^T, 0)^T$, $f_2(w) = \beta Qw$,

$$\mathcal{A} := \begin{bmatrix} A + BH_2C & BH_1 \\ GC & F \end{bmatrix}, \quad \mathcal{D} := \begin{bmatrix} D \\ 0 \end{bmatrix} \qquad (40)$$

and $\mathcal{C}_z = \beta[C\ 0]$. Notice, that $\mathcal{W}^\circ_{z\phi}(s) = \mathcal{C}^\circ_z(sI - \mathcal{A}^\circ)^{-1}\mathcal{D}^\circ$, where $\mathcal{A}^\circ$, $\mathcal{C}^\circ$ and $\mathcal{D}^\circ$ equal to $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$ defined for the nominal system parameters. Since $\sup_{\omega\in\mathbb{R}} |\mathcal{C}^\circ_z(i\omega I - \mathcal{A}^\circ)^{-1}\mathcal{D}^\circ| = \|\mathcal{W}^\circ_{z\phi}\|_\infty < 1/\gamma$ then, by continuity,

$$\sup_{\omega\in\mathbb{R}} |\mathcal{C}_z(i\omega I - \mathcal{A})^{-1}\mathcal{D}| < 1/\gamma$$

for all $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{C}_z$ from some neighborhood of the nominal ones. Thus, by Lemma 2 the Jacobian of the right-hand side of (39) is quadratically stable for any $\phi(\cdot) \in \mathcal{F}(\gamma)$ and for all A, B, C and D from some neighborhood of the corresponding nominal matrices. By Lemma 1, every solution of the closed-loop system (39) is GES. Notice, that since the controller (36) also solves the robust output regulation problem for the linear system (37), for all A, B, C and D close enough to the nominal ones and for all P and Q, system (37) in closed loop with (36) has a bounded

solution $(\bar{x}_w(t), \bar{\xi}_w(t))$ along which $\bar{e}(t) = y(t) \equiv 0$ (see, e.g. [5]). Since $z = \beta y$ and $\phi(0) = 0$, then $(\bar{x}_w(t), \bar{\xi}_w(t))$ is also a solution of the closed-loop system (39). Since all solutions of (39) are GES, then $(\bar{x}_w(t), \bar{\xi}_w(t))$ is also GES. This implies that, along any solution of the closed-loop system (39) and exosystem (5), the regulated output satisfies

$$\bar{e}(t) = Cx(t) + Qw(t) \xrightarrow[t\to+\infty]{} C\bar{x}_w(t) + Qw(t) \equiv 0$$

and this property holds for all A, B, C and D close enough to the corresponding nominal matrices A°, B°, C° and D°, for all P and Q and for all $\phi(\cdot) \in \mathcal{F}(\gamma)$.□

Remark. The problem of finding a controller that satisfies conditions **i)** and **ii)** has been solved in [17]. Yet, careful examination shows that the conditions, under which the problem was solved in [17], are not satisfied in our case. So, we proceed with our own controller design.

Necessary and sufficient conditions for solvability of the robust output regulation problem for the linear system (37) are [5]:

A4 the pair (A°, B°) is stabilizable, the pair (A°, C°) is detectable and for every λ being an eigenvalue of the matrix S the matrix

$$\begin{pmatrix} A^\circ - \lambda I & B^\circ \\ C^\circ & 0 \end{pmatrix}$$

has full row rank.

We assume that condition **A4** is satisfied and proceed with a design of a robust regulator. The design closely follows the design of a robust controller for the linear robust output regulation problem (see e.g. [5]). Let S_{min} be a $q \times q$ matrix whose characteristic polynomial coincides with the minimal polynomial of S. Construct a block-diagonal $mq \times mq$ matrix Φ which has m blocks S_{min} on its diagonal, where m is the number of inputs (see Assumption **A3**). Choose an $mq \times m$ matrix N and an $m \times mq$ matrix Γ such that (Φ, Γ) is controllable and (Φ, N) is observable. Consider the augmented system

$$\dot{x} = A^\circ x + B^\circ \Gamma \xi_1 + B^\circ v + D^\circ \phi \quad (41)$$
$$\dot{\xi}_1 = \Phi \xi_1 + NC^\circ x \quad (42)$$
$$z = \beta C^\circ x \quad (43)$$

Suppose, there exists a controller

$$\dot{\xi}_2 = K\xi_2 + LC^0 x \quad (44)$$
$$v = M\xi_2 + RC^\circ x$$

such that system (41) in closed-loop with this controller is asymptotically stable for $\phi = 0$ and the transfer function $\mathcal{W}^\circ_{z\phi}(s)$ from input ϕ to output z satisfies $\|\mathcal{W}^\circ_{z\phi}\|_\infty < 1/\gamma$. As follows from the linear regulator theory [5], the controller

$$\dot{\xi}_1 = \Phi \xi_1 + Ny \quad (45)$$
$$\dot{\xi}_2 = K\xi_2 + Ly \quad (46)$$
$$u = \Gamma \xi_1 + M\xi_2 + Ry \quad (47)$$

solves the robust output regulation problem for system (37). At the same time, the transfer function $\mathcal{W}^\circ_{z\phi}(s)$ satisfies $\|\mathcal{W}^\circ_{z\phi}\|_\infty < 1/\gamma$. Thus, the statement of Lemma 4 holds for controller (45). The problem of finding a controller (44) that guarantees $\|\mathcal{W}^\circ_{z\phi}\|_\infty < 1/\gamma$ is a standard problem in H_∞ optimization, for which efficient solvers are available, for example, in MATLAB.

V. Examples

A. Observer-based controller

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 - x_2 + \sin(x_2) \\ \dot{x}_3 &= u \\ e &= y = x_1 - w_1 \end{aligned} \quad (48)$$

and the exosystem

$$\dot{w}_1 = w_2, \quad \dot{w}_2 = -w_1. \quad (49)$$

The control goal is to find an output feedback controller such that all solutions of the closed-loop system are bounded and $e(t) \to 0$, as $t \to +\infty$.

The regulator equations admit the solution $\pi_1(w) = w_1$, $\pi_2(w) = w_2$, $\pi_3(w) = w_2 - w_1 - \sin(w_2)$, $c(w) = -w_1 - w_2 + w_1 \cos(w_2)$. The mappings $\pi(w)$ and $c(w)$ are globally defined and continuously differentiable. Hence, assumption **A1** holds. Let us apply Theorem 2. In our case,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$B = [0\ 0\ 1]^T$, $P \equiv 0$, $C_r = C = [1\ 0\ 0]$, $Q_r = Q = [-1\ 0]$, $z = x_2$, $C_z = [0, 1, 0]$, $Q_z = 0$, $\phi(z) = sin(z) \in \mathcal{F}(1)$.

Denote

$$A_c^- := \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_c^+ := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_o^- := \begin{bmatrix} A_c^- & P \\ 0 & S \end{bmatrix}, \quad \mathcal{A}_o^+ := \begin{bmatrix} A_c^+ & P \\ 0 & S \end{bmatrix},$$

$\mathcal{C} := [C\ \ Q]$. Numeric computations show that both LMIs (21) and (24) are feasible and, for example, the matrices $K = [-6 - 11,\ -6]^T$ and $L = [-153,\ -78,\ -13,\ -132,\ 52]$ can be used as parameters in the controller (25)-(28).

Thus, all conditions of Theorem 2 are satisfied. By this theorem, controller (25)-(28) with the system matrices, mappings $\pi(w)$, $c(w)$ and controller parameters K, L specified above solves the global output regulation problem globally. In Fig. 1 simulation results of the closed-loop dynamics are presented.

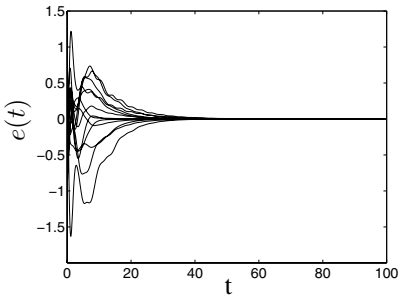

Fig. 1. Simulations results for observer-based controller: $e(t)$ for different initial conditions of the closed-loop system and the exosystem.

B. Robust controller

To demonstrate the design of a robust controller, we consider system (1) with the nominal values

$$A^\circ = \begin{bmatrix} 1 & -2 & 0 \\ 40 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}, \quad B^\circ = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad D^\circ = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$C^\circ = [1, 0, 0]$. The exosignal w is generated by the exosystem (49). The outputs of the system are equal: $z = e = y = Cx + Qw$. The matrices Q and P are arbitrary. The value γ for the class of nonlinearities $\mathcal{F}(\gamma)$ is chosen $\gamma = 0.1$. Notice, that with such choice of system matrices assumptions **A2** and **A3** hold. Following the design procedure given in Section IV, we set

$$\Phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Gamma = [1\ 0].$$

Next, we search for a controller (44) that would satisfy the inequality $\|\mathcal{W}^\circ_{z\phi}\| < 1/\gamma$. Such controller is found using the MATLAB routine `hinflmi`. Results of the simulations of the closed-loop system are given in Fig. 2. In the simulations, we choose $\phi(z) = \gamma \sin(z)$, system matrices equal to their nominal values and P and Q – random matrices of the corresponding dimensions.

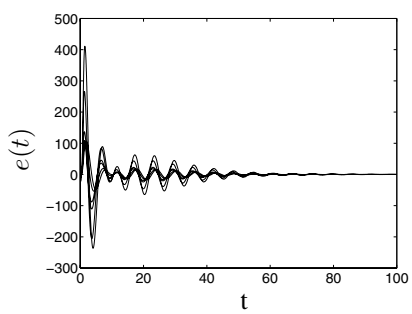

Fig. 2. Simulations results for robust controller: $e(t)$ for different initial conditions and for different P and Q.

VI. Conclusions

In this paper, we have considered the global output regulation problem for systems with an output dependent nonlinearity. If the system model is known and the corresponding regulator equations are solvable, we presented an observer-based controller solving the global output regulation problem. Sufficient conditions for the existence of such a controller are formulated in terms of LMIs. For the case that the system parameters are not known exactly and the nonlinearity can be arbitrary from a given class of functions with a bounded derivative, we have demonstrated that there exists a robust controller provided that the robust output regulation problem for the system without the nonlinearity is solvable and that there exists a suboptimal solution to a certain H_∞ optimization problem.

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