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# The uniform global output regulation problem

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*Abstract*— **In this paper, we present a new approach to the global output regulation problem. In this approach, a controller should be designed in such a way that the closed-loop system is uniformly convergent. This requirement allows to extend the theory developed for the local output regulation problem to what we will call the uniform global output regulation problem. Such extension is made using global invariant manifold theorems, which serve as global counterparts of the center manifold theorems. It is shown that within the proposed approach solvability of the (extended) regulator equations is a basic necessary condition for the solvability of the uniform global output regulation problem. As an illustration, we present a solution to the uniform global output regulation problem for a class of nonlinear systems.**

## I. INTRODUCTION

In this paper, we address the output regulation problem, which includes the problems of tracking reference signals and rejecting disturbances generated by an external autonomous system (exosystem). For linear systems, this problem was thoroughly investigated in the 1970-s [1], [2]. For nonlinear systems, intensive research started with the papers [3] and [4], which provided solutions to the local output regulation problem for general nonlinear systems. These papers were followed by a number of results dealing with different aspects of the output regulation problem for nonlinear systems, see [5], [6] and references therein.

Necessary and sufficient conditions for the solvability of the *local* output regulation problem were presented in [3]. This solution is based on the center manifold theorem [7]. Under the neutral stability assumption on the exosystem, this theorem allows to obtain the so-called regulator equations and to show that solvability of these equations is an important necessary condition for the solvability of the *local* output regulation problem. Extensions of this result to the global case have been obtained in [8], [9]. The results in these papers are based on certain quantitative conditions on the closed-loop system and the exosystem dynamics. A qualitative analysis of the global output regulation problem under the Poisson stability assumption on the exosystem has been done in [10].

In this paper, we present a new problem setting for the global output regulation problem – the so-called uniform global output regulation problem (see [10], [11] for alternative formulations of the global output regulation problem). In this new problem setting, a controller should be designed in such a way that the closed-loop system is uniformly convergent and the regulated output tends to zero. Roughly speaking, a system is called uniformly convergent if, being excited by a bounded signal, it has a unique uniformly bounded (in some sense) solution, which is uniformly globally asymptotically stable. The uniform convergence property originates from the convergence property introduced in [12] (see also [13]). This property is a natural extension of stability properties of asymptotically stable linear systems to nonlinear systems. For systems having the uniform convergence property, we present invariant manifold theorems. These theorems, which serve as non-local counterparts of center manifold theorems, allow to extend the analysis of the local output regulation problem to the global case while avoiding restrictive assumptions (either quantitative or qualitative) on the exosystem. For a controller that makes the closed-loop system uniformly convergent, we give necessary and sufficient conditions, under which this controller solves the uniform global output regulation problem. As an illustration, we present a solution to this problem for a class of nonlinear systems.

The paper is organized as follows. In Section II, we give definitions of convergent systems and review some related results. In Section III, the uniform global output regulation problem is formulated and discussed. Invariant manifold theorems are presented in Section IV. Results on the solvability of the uniform global output regulation problem are stated in Section V and Section VI presents conclusions. All proofs are provided in the Appendix.

## II. CONVERGENT SYSTEMS

In this section, we give definitions and review some results on convergent systems. Consider the system

$$\dot{z} = F(z, w), \tag{1}$$

where $z \in \mathbb{R}^d$, $w \in \mathbb{R}^m$ and $F(z, w)$ is locally Lipschitz in $z$ and continuous in $w$.

*Definition 1 ([12], [13]):* System (1) with a given continuous input $w(\cdot) : \mathbb{R} \to \mathbb{R}^m$ is said to be *convergent* if

i. all solutions $z(t)$ are defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}$, $z(t_0) \in \mathbb{R}^d$,
ii. there exists a unique solution $\bar{z}_w(t)$ bounded for all $t \in \mathbb{R}$,
iii. the solution $\bar{z}_w(t)$ is globally asymptotically stable.

System (1) is said to be convergent (for all inputs) if it is convergent for every continuous bounded $w(\cdot) : \mathbb{R} \to \mathbb{R}^m$.

We will refer to $\bar{z}_w(t)$ as the limit solution. The limit solution of a convergent system has certain natural properties. As shown in [12], from the definition of convergent systems one can easily obtain that for a constant input the corresponding limit solution is also constant. Similarly, if

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the input is periodic, the corresponding limit solution is also periodic with the same period. If an input $w(t)$ is bounded on $[t_0, +\infty)$ then every solution of the convergent system is also bounded on $[t_0, +\infty)$. For our purposes, we will need a stronger convergence property, which has an additional requirement of *uniform* asymptotic stability and *uniform* boundedness of the limit solutions.

*Definition 2:* System (1) is said to be *uniformly convergent* if it is convergent, for every $w(\cdot)$ the limit solution $\bar{z}_w(t)$ is *uniformly* globally asymptotically stable and for any $\rho > 0$ there exists $\mathcal{R} > 0$ such that

$$|w(t)| \leq \rho \quad \forall\, t \in \mathbb{R} \quad \Rightarrow \quad |\bar{z}_w(t)| \leq \mathcal{R} \quad \forall\, t \in \mathbb{R}. \tag{2}$$

The uniform convergence property is an extension of stability properties of asymptotically stable linear time-invariant systems. Recall, that for any continuous input $w(t)$, which is defined and bounded on $t \in \mathbb{R}$, the system $\dot{z} = Az + Bw(t)$ with a Hurwitz matrix $A$ has a unique solution $\bar{z}_w(t)$ which is defined and bounded on $t \in (-\infty, +\infty)$. This solution is given by the formula $\bar{z}_w(t) := \int_{-\infty}^{t} exp(A(t-s))Bw(s)ds$; it is globally exponentially stable with the rate of convergence depending only on the matrix $A$ and its upper bound is given by

$$|\bar{z}_w(t)| \leq \int_{-\infty}^{0} \|exp(-As)B\| ds \sup_{\tau \in \mathbb{R}} |w(\tau)|.$$

Thus, a linear time-invariant asymptotically stable system is uniformly convergent.

A simple sufficient condition for the uniform convergence property was proposed in [14] (see also [13]). Here we give a slightly different formulation of the result from [14] adapted for systems with inputs.

*Theorem 1:* Consider system (1) with the function $F(z, w)$ being $C^1$ with respect to $z$ and continuous with respect to $w$. Suppose, there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that

$$P\frac{\partial F}{\partial z}(z, w) + \frac{\partial F}{\partial z}^T(z, w)P \leq -Q, \quad \forall z \in \mathbb{R}^d, \quad w \in \mathbb{R}^m. \tag{3}$$

Then, system (1) is uniformly convergent.

*Remark.* Condition (3) is satisfied, for example, if there exist matrices $A_1, \ldots A_s$ such that

$$\frac{\partial F}{\partial z}(z, w) \in \mathrm{co}\{A_1, \ldots A_s\}, \quad \forall \;\; z \in \mathbb{R}^d,\; w \in \mathbb{R}^m,$$

and the following linear matrix inequalities

$$PA_i + A_i^T P < 0, \quad i = 1 \ldots s,$$

admit a common positive definite solution $P = P^T > 0$. Taking into account existence of powerful LMI solvers, this is a useful tool for checking the uniform convergence property. In certain cases, analytical results based on the Kalman-Yakubovich lemma can be used to check feasibility of these LMIs, see e.g. [15], [16].

## III. THE UNIFORM GLOBAL OUTPUT REGULATION PROBLEM

Consider systems modelled by equations of the form

$$\begin{aligned} \dot{x} &= f(x, u, w) && (4)\\ e &= h_r(x, w), \quad y = h_m(x, w), && (5) \end{aligned}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, regulated output $e \in \mathbb{R}^l$ and measured output $y \in \mathbb{R}^k$. The exogenous signal $w(t) \in \mathbb{R}^m$, which can be viewed as a disturbance in equation (4) or as a reference signal in (5), is generated by the exosystem

$$\dot{w} = s(w). \tag{6}$$

We are interested in solutions of the exosystem (6) starting in a bounded positively invariant set $\mathcal{W}_+ \subset \mathbb{R}^m$. The functions $f(x, u, w)$, $h_r(x, w)$, $h_m(x, w)$ and $s(w)$ are assumed to be continuous and, where necessary, locally Lipschitz in order to guarantee existence and uniqueness of solutions of the corresponding differential equations.

The uniform global output regulation problem is formulated in the following way: find, if possible, a feedback of the form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y), \quad \xi \in \mathbb{R}^q && (7)\\ u &= \theta(\xi, y), \end{aligned}$$

with continuous functions $\eta(\xi, y)$ and $\theta(\xi, y)$ and some $q$ such that

**a)** the right-hand side of the closed-loop system

$$\begin{aligned} \dot{x} &= f(x, \theta(\xi, h_m(x, w)), w)\\ \dot{\xi} &= \eta(\xi, h_m(x, w)) \end{aligned} \tag{8}$$

is locally Lipschitz with respect to $(x, \xi)$ and continuous with respect to $w$;

**b)** system (8) is uniformly convergent;

**c)** all solutions of the closed-loop system (8) and the exosystem (6) starting in $(x(0), \xi(0)) \in \mathbb{R}^{n+q}$, $w(0) \in \mathcal{W}_+$ satisfy $e(t) = h_r(x(t), w(t)) \to 0$ as $t \to \infty$.

Notice, that the requirement of uniform convergence implies that if $f(0, 0, 0) = 0$, $h_m(0, 0) = 0$, $\eta(0, 0) = 0$ and $\theta(0, 0) = 0$, then for $w(t) \equiv 0$ the closed-loop system (8) has a globally asymptotically stable (GAS) equilibrium at the origin $(x, \xi) = (0, 0)$. Another consequence of convergence is that for any bounded input $w(\cdot) : [0, +\infty) \to \mathbb{R}^m$ (and this is the case if w(t) is a solution of (6) starting in $\mathcal{W}_+$) every solution of (8) is bounded on $[0, +\infty)$.

The formulation of the output regulation problem given above is somewhat different from conventional ones. Usually, instead of condition **b)** some other internal stability property of the closed-loop system is required. For example, in the linear case, it is required that for $w(t) \equiv 0$ system (8) is asymptotically stable. But as we have seen in Section II, for linear systems this requirement implies uniform convergence. In the local output regulation problem for nonlinear systems, it is required that for $w(t) \equiv 0$ the closed-loop system (8) is locally exponentially stable at the origin. As it was shown in [17], this condition implies the local (in a certain sense) uniform convergence property of the closed-loop system. These observations allow to say that uniform convergence is a natural extension of the conventional internal stability requirements to the case of global output regulation for nonlinear systems. It will be shown in the next sections that the requirement of uniform convergence allows to treat the uniform *global* output regulation problem in a similar way as the *local* output regulation problem.

The analysis of the local output regulation problem is based on the center manifold theorem [7]. If the linearization of the closed-loop system (8) at the origin is asymptotically stable and the linearization of the exosystem at the origin is critically stable, then this theorem guarantees the existence of a locally defined and locally attractive invariant manifold of the form $(x, \xi) = (\pi(w), \sigma(w))$. In order to proceed with the analysis of the global output regulation problem, we need to find counterparts of this result for the global case. These results are presented in the next section.

## IV. Invariant manifold theorems

In this section, we study coupled systems of the form

$$\begin{aligned} \dot{z} &= F(z, w) && (9) \\ \dot{w} &= s(w), && (10) \end{aligned}$$

where $z \in \mathbb{R}^d$, $w \in \mathbb{R}^m$. The function $F(z, w)$ is locally Lipschitz in $z$ and continuous in $w$; $s(w)$ is locally Lipschitz. First, we consider the case of system (10) satisfying the following assumption:

**A1** All solutions of system (10) are defined for all $t \in (-\infty, +\infty)$ and for every $r > 0$ there exists $\rho > 0$ such that

$$|w_0| < r \quad \Rightarrow \quad |w(t, w_0)| < \rho \;\; \forall\, t \in \mathbb{R}. \qquad (11)$$

A simple example of a system satisfying **A1** is a linear harmonic oscillator. The next theorem gives sufficient conditions for the existence of a continuous globally asymptotically stable invariant manifold of the form $z = \alpha(w)$.

*Theorem 2:* Consider system (9) and system (10) satisfying assumption **A1**. Suppose, system (9) is uniformly convergent. Then, there exists a unique continuous mapping $\alpha(\cdot) : \mathbb{R}^m \to \mathbb{R}^d$ such that the graph $z = \alpha(w)$ is invariant with respect to systems (9) and (10) and for every $w(t)$, system (10) has a uniformly globally asymptotically stable solution $z(t) = \alpha(w(t))$.

In the output regulation problem we may deal with exosystems that do not satisfy **A1**. For example, it can be an exosystem with a limit cycle or any other attractor with an unbounded domain of attraction. Therefore, we need to relax the conditions of Theorem 2 in order to include exosystems with complex dynamics. This is done in the next theorem.

*Theorem 3:* Consider systems (9) and (10). Suppose, system (9) is uniformly convergent. Let $\mathcal{W}_+$ be a bounded positively invariant set of system (10) and $\mathcal{W}_\pm \subset \mathcal{W}_+$ be an invariant subset of $\mathcal{W}_+$. Then, there exists a continuous mapping $\alpha(\cdot) : \mathbb{R}^m \to \mathbb{R}^d$ such that the set

$$\mathcal{M}(\mathcal{W}_+) := \{z, w : \; z = \alpha(w), w \in \mathcal{W}_+\}$$

is positively invariant with respect to (9), (10) and for any $w(0) \in \mathcal{W}_+$ the solution $\bar{z}_w(t) = \alpha(w(t))$ is uniformly globally asymptotically stable. In general, the mapping $\alpha(w)$ is not unique. But for any two such mappings $\alpha_1(w)$ and $\alpha_2(w)$ and any $w(t)$ starting in $w(0) \in \mathcal{W}_+$ it holds that

$$|\alpha_1(w(t)) - \alpha_2(w(t))| \to 0 \quad \text{as} \;\; t \to +\infty \qquad (12)$$

and $\alpha_1(w) = \alpha_2(w)$ for all $w \in \mathcal{W}_\pm$.

## V. Conditions for uniform global output regulation

In this section, we apply the invariant manifold theorems to study the solvability of the uniform global output regulation problem.

Denote by $\Omega(w_0) \subset \mathbb{R}^m$ the set of all $\omega$-limit points of the trajectory $w(t, w_0)$. Recall that $w^* \in \Omega(w_0)$ if there exists a sequence $\{t_k\}$ such that $t_k \to +\infty$ and $w(t_k, w_0) \to w^*$ as $k \to +\infty$. Denote

$$\Omega(\mathcal{W}_+) := \bigcup_{w_0 \in \mathcal{W}_+} \Omega(w_0).$$

Since $\mathcal{W}_+$ is bounded, then $\Omega(\mathcal{W}_+)$ is nonempty and it attracts all solutions of the exosystem (6) starting in $\mathcal{W}_+$ [18], i.e. $\mathrm{dist}(w(t, w_0), \Omega(\mathcal{W}_+)) \xrightarrow[t \to +\infty]{} 0$ for all $w_0 \in \mathcal{W}_+$. The next theorem, which is based on Theorem 3, establishes necessary and sufficient conditions for a controller (7) to solve the uniform global output regulation problem.

*Theorem 4:* Suppose controller (7) is such that the closed-loop system (8) is uniformly convergent. Then this controller solves the uniform global output regulation problem if and only if there exist continuous mappings $\pi(w)$ and $\sigma(w)$ satisfying

$$\begin{aligned} \frac{d}{dt}\pi(w(t)) &= f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w)), w), \\ \frac{d}{dt}\sigma(w(t)) &= \eta(\sigma(w), h_m(\pi(w), w)), && (13) \\ & \qquad \forall \;\; w(t) = w(t, w_0) \in \mathcal{W}_+, \\ 0 &= h_r(\pi(w), w) \quad \forall \;\; w \in \Omega(\mathcal{W}_+). && (14) \end{aligned}$$

By denoting $c(w) := \theta(\sigma(w), h_m(\pi(w), w))$, we obtain the following necessary condition for the solvability of the problem:

*Lemma 1:* The uniform global output regulation problem is solvable only if there exist continuous mappings $\pi(w)$ and $c(w)$ satisfying the equations

$$\begin{aligned} \frac{d}{dt}\pi(w(t)) &= f(\pi(w(t)), c(w(t)), w(t)), && (15) \\ & \qquad \forall \; w(t) = w(t, w_0) \in \mathcal{W}_+, \\ 0 &= h_r(\pi(w), w) \quad \forall \;\; w \in \Omega(\mathcal{W}_+). && (16) \end{aligned}$$

Equations (15) and (16) are extensions of the so-called regulator equations. Originally, the regulator equations were obtained as a necessary condition for the solvability of the *local* output regulation problem under the assumption that the exosystem (6) is neutrally stable. Lemma 1 shows that solvability of the regulator equations (15) and (16) is also necessary for the solvability of the uniform *global* output regulation problem without restrictive assumptions on the exosystem such as neutral stability or Poisson stability.

As follows from Theorem 4, controller (7) solves the uniform global output regulation problem if and only if it makes the closed-loop system uniformly convergent and equations (13) and (14) are satisfied for some continuous $\pi(w)$ and $\sigma(w)$. In order to illustrate how a controller satisfying these two conditions can be found, we present a solution to the uniform global output regulation problem for a class of nonlinear systems.

Consider system (4)-(5) with the measured output $y = (x, w)$, i.e. we are dealing with the state feedback case. The function $f(x, u, w)$ is assumed to be continuously differentiable. Denote $\zeta := (x, u, w) \in \mathbb{R}^{n+p} \times \mathcal{W}_+$,

$$\mathcal{A}(\zeta) := \frac{\partial f}{\partial x}(x, u, w), \ \mathcal{B}(\zeta) := \frac{\partial f}{\partial u}(x, u, w).$$

The class of systems that we consider is limited by the following assumption:

**A2** There exist matrices $\mathcal{A}_1, \dots \mathcal{A}_s$ and $\mathcal{B}_1, \dots \mathcal{B}_s$ such that

$$[\mathcal{A}(\zeta), \mathcal{B}(\zeta)] \in \mathrm{co}\{[\mathcal{A}_1, \mathcal{B}_1], \dots [\mathcal{A}_s, \mathcal{B}_s]\},$$

for all $\zeta \in \mathbb{R}^{n+p} \times \mathcal{W}_+$.

The following theorem gives sufficient conditions for solvability of the uniform global output regulation problem.

*Theorem 5:* Consider system (4)-(5) with $y = (x, w)$ and satisfying assumption **A2**. Suppose there exist continuous mappings $\pi(w)$ and $c(w)$ satisfying the regulator equations (15), (16). If the linear matrix inequalities

$$\mathcal{A}_i \mathcal{P} + \mathcal{P} \mathcal{A}_i^T + \mathcal{B}_i \mathcal{Y} + \mathcal{Y}^T \mathcal{B}_i^T \quad < \quad 0, \quad i = 1, \dots s \quad (17)$$

admit a common solution $\mathcal{P} = \mathcal{P}^T > 0$, then the uniform global output regulation problem is solved by a controller of the form

$$u = c(w) + K(x - \pi(w)), \quad (18)$$

with the matrix $K = \mathcal{Y}\mathcal{P}^{-1}$, where $\mathcal{Y}$ and $\mathcal{P}$ satisfy (17).

## VI. Conclusions

We have presented and studied the uniform global output regulation problem. This is a new problem setting for the global output regulation problem. In this problem setting, a controller should be designed in such a way that the closed-loop system is uniformly convergent. This requirement is a natural extension of internal stability requirements from the linear and local nonlinear output regulation problems to the global nonlinear case. It allows to extend the theory developed for the local output regulation problem to the uniform global output regulation problem and to avoid restrictive assumptions on exosystem. Such extension has been made with the help of the invariant manifold theorems, which, in this case, serve as global counterparts of the center manifold theorems. We have presented necessary and sufficient conditions for a controller to solve the uniform global output regulation problem. It has been shown that solvability of the (extended) regulator equations is a necessary condition for the solvability of this problem. As an illustration, we have presented a solution to the state feedback case of the uniform global output regulation problem for a class of nonlinear systems.

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## Appendix

*Proof of Theorem 2.* First, we show, that if a continuous mapping $\alpha(w)$ exists, such that the graph $z = \alpha(w)$ is invariant, then it is unique. Suppose, $\alpha(w)$ and $\tilde{\alpha}(w)$ are two such distinct mappings. Consider a solution $w(t)$ of system (10). Due to condition **A1**, this solution is bounded on $t \in$

$\mathbb{R}$. Since $\alpha(w)$ and $\tilde{\alpha}(w)$ are continuous, then $\tilde{z}_w(t) := \tilde{\alpha}(w(t))$ and $\bar{z}_w(t) := \alpha(w(t))$ are two distinct bounded solutions of system (9) corresponding to the same input $w(t)$. This contradicts the convergence property of system (9). Thus, such mapping $\alpha(w)$, if it exists, is unique.

We prove the existence of $\alpha(w)$ by constructing this mapping. Due to assumption **A1**, for every $w_0 \in \mathbb{R}$ the solution $w(t, w_0)$ is defined and bounded for all $t \in \mathbb{R}$. Since system (9) is convergent, for this solution $w(t, w_0)$ there exists a unique limit solution $\bar{z}(t, w_0)$, which is defined and bounded on $t \in \mathbb{R}$. Construct the mapping $\alpha(w)$ in the following way: for every $w_0 \in \mathbb{R}^m$ and every $t \in \mathbb{R}$ set $\alpha(w(t, w_0)) := \bar{z}(t, w_0)$ or, equivalently, $\alpha(w_0) = \bar{z}(0, w_0)$. By the definition of the mapping $\alpha(w)$, the graph $z = \alpha(w)$ is invariant with respect to (9) and (10). Uniform global asymptotic stability of the solution $z(t) = \alpha(w(t, w_0))$ immediately follows from uniform global asymptotic stability of $\bar{z}(t, w_0)$.

It remains to show that the mapping $z = \alpha(w)$, constructed above, is continuous, i.e. that for any $w_1 \in \mathbb{R}^m$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|w_1 - w_2| < \delta$ implies $|\alpha(w_1) - \alpha(w_2)| < \varepsilon$. For simplicity, we will prove continuity in the ball $|w| < r$. Since $r$ can be chosen arbitrarily, this will imply continuity in $\mathbb{R}^m$. In the sequel, we assume that $|w_1| < r$ and $\varepsilon > 0$ are fixed and the point $w_2$ varies in the ball $|w_2| < r$.

As a preliminary observation, notice that $|w_1| \leq r$ and $|w_2| \leq r$ imply, due to assumption **A1**, that $|w(t, w_i)| \leq \rho$ for $i = 1, 2$ and for all $t \in \mathbb{R}$. This, in turn, due to uniform convergence of system (9) (see (2)) and due to the construction of $\alpha(w)$, implies that $|\alpha(w(t, w_i))| \leq \mathcal{R}$ for $i = 1, 2$ and for all $t \in \mathbb{R}$.

In order to prove continuity of $\alpha(w)$, we introduce the function

$$\varphi_T(w_1, w_2) := \hat{z}(0, -T, \alpha(w(-T, w_2)), w_1),$$

where the number $T > 0$ will be specified later and $\hat{z}(t, t_0, z_0, w_*)$ is the solution of the time-varying system

$$\dot{\hat{z}} = F(\hat{z}, w(t, w_*)) \tag{19}$$

satisfying the initial conditions $\hat{z}(t_0, t_0, z_0, w_*) = z_0$.

The function $\varphi_T(w_1, w_2)$ has the following meaning. First, consider the limit solution $\alpha(w(t, w_2))$, which is a solution of system (19) with the input $w(t, w_2)$. At time $t = 0$, $\alpha(w(0, w_2)) = \alpha(w_2)$. Next, we shift along $\alpha(w(t, w_2))$ to time $t = -T$ and appear in $\alpha(w(-T, w_2))$. Then, we switch the input to $w(t, w_1)$, shift forward to the time instant $t = 0$ along the solution $\hat{z}(t)$ starting in $\hat{z}(-T) = \alpha(w(-T, w_2))$ and appear in $\hat{z}(0) = \varphi_T(w_1, w_2)$. Notice, that $\varphi_T(w_0, w_0) = \alpha(w_0)$ (there is no switch of inputs and we just shift back and forth along the same solution $\alpha(w(t, w_0))$). Thus,

$$\begin{aligned}
\alpha(w_1) - \alpha(w_2) &= \varphi_T(w_1, w_1) - \varphi_T(w_2, w_2) \\
&= \varphi_T(w_1, w_1) - \varphi_T(w_1, w_2) \\
&\quad + \varphi_T(w_1, w_2) - \varphi_T(w_2, w_2).
\end{aligned} \tag{20}$$

By the triangle inequality, this implies

$$\begin{aligned}
|\alpha(w_1) - \alpha(w_2)| &\leq |\varphi_T(w_1, w_1) - \varphi_T(w_1, w_2)| \\
&+ |\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)|.
\end{aligned} \tag{21}$$

In the next steps, we will first show that there exist $T > 0$ such that

$$|\varphi_T(w_1, w_1) - \varphi_T(w_1, w_2)| < \varepsilon/2 \quad \forall \ |w_2| < r. \tag{22}$$

Second, we will show that given a number $T > 0$ satisfying (22), there exists $\delta > 0$ such that

$$|\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)| < \varepsilon/2 \quad \forall \ w_2 : |w_1 - w_2| < \delta. \tag{23}$$

Unifying inequalities (22) and (23), we obtain $|\alpha(w_1) - \alpha(w_2)| < \varepsilon$ for all $w_2$ satisfying $|w_1 - w_2| < \delta$. Due to the arbitrary choice of $\varepsilon > 0$ and $|w_1| < r$, this proves continuity of $\alpha(w)$ in the ball $|w| < r$.

In order to show (22), notice that $\varphi_T(w_1, w_1) = \hat{z}_1(0)$ and $\varphi_T(w_1, w_2) = \hat{z}_2(0)$, where $\hat{z}_1(t)$ and $\hat{z}_2(t)$ are solutions of system (19) with the input $w(t, w_1)$ satisfying the initial conditions $\hat{z}_1(-T) = \alpha(w(-T, w_1))$ and $\hat{z}_2(-T) = \alpha(w(-T, w_2))$. By the conditions of the theorem, $\hat{z}_1(t) = \alpha(w(t, w_1))$ is a bounded uniformly globally asymptotically stable solution of (19). This implies, that it attracts all other solutions $\hat{z}(t)$ of system (19) uniformly over the initial conditions $t_0 \in \mathbb{R}$ and $\hat{z}(t_0)$ from any compact set. In particular, for the compact set $K(\mathcal{R}) := \{z : |z| \leq \mathcal{R}\}$ and for fixed $\varepsilon > 0$ there exists $\tilde{T}_\varepsilon(\mathcal{R})$ such that $\hat{z}(t_0) \in K(\mathcal{R})$ implies

$$|\hat{z}_1(t) - \hat{z}(t)| < \varepsilon/2, \quad \forall \ t \geq t_0 + \tilde{T}_\varepsilon(\mathcal{R}), \ t_0 \in \mathbb{R}. \tag{24}$$

Let us choose $T := \tilde{T}_\varepsilon(\mathcal{R})$. By the definition of $\hat{z}_2(t)$, $\hat{z}_2(-T) = \alpha(w(-T, w_2))$. Since $\alpha(w(t, w_2)) \in K(\mathcal{R})$ for all $t \in \mathbb{R}$ and all $|w_2| < r$ (see above), then $\hat{z}_2(-T) \in K(\mathcal{R})$. Thus, for $t_0 = -T$ and $t = 0$ formula (24) implies

$$|\hat{z}_1(0) - \hat{z}_2(0)| < \varepsilon/2, \tag{25}$$

which is equivalent to (22).

In order to show (23), notice that for a fixed $T > 0$, the function $\hat{z}(0, -T, z_0, w_0)$ is continuous with respect to $z_0$ and $w_0$. Thus, it is uniformly continuous over the compact set $G := \{(z_0, w_0) : \ |z_0| \leq \mathcal{R}, |w_0| \leq r\}$. Hence, there exists $\delta > 0$ such that if $|z_0| \leq \mathcal{R}$, $|w_1| \leq r$, $|w_2| \leq r$ and $|w_1 - w_2| < \delta$, then

$$|\hat{z}(0, -T, z_0, w_1)) - \hat{z}(0, -T, z_0, w_2)| \leq \varepsilon/2. \tag{26}$$

Recall, that by the definition of $\varphi_T(w_1, w_2)$

$$\begin{aligned}
&\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2) = \\
&\hat{z}(0, -T, z_0, w_1)) - \hat{z}(0, -T, z_0, w_2),
\end{aligned} \tag{27}$$

where $z_0 := \alpha(w(-T, w_2))$. Since $|w_1| \leq r$, $|w_2| \leq r$ and, due to uniform convergence property, $|\alpha(w(-T, w_2))| \leq \mathcal{R}$, then, as follows from (26) and (27),

$$|w_1 - w_2| < \delta \quad \Rightarrow \quad |\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)| < \varepsilon/2.$$

Thus, we have shown (23). This completes the proof of continuity of $\alpha(w)$. $\square$

*Proof of Theorem 3.* Since $\mathcal{W}_+$ is bounded, we can choose $R > 0$ such that $\mathcal{W}_+ \subset B(R) := \{w : \ |w| < R\}$. Consider system (9) and the following auxiliary system

$$\dot{\tilde{w}} = S(\tilde{w}), \tag{28}$$

where $S(\tilde{w})$ is a locally Lipschitz function such that $S(\tilde{w}) = s(\tilde{w})$ for all $|\tilde{w}| \leq R$ and $S(\tilde{w}) = 0$ for all $|\tilde{w}| \geq 2R$. For example, $S(\tilde{w})$ can be chosen equal to $S(\tilde{w}) := \psi(|\tilde{w}|)s(\tilde{w})$, where $\psi(v)$ is a smooth scalar function satisfying $\psi(v) = 1$ for $v \leq R$ and $\psi(v) = 0$ for $v \geq 2R$. An example of such function can be found in [16], p. 662.

Notice, that system (28) satisfies assumption **A1**. Indeed, for any trajectory $\tilde{w}(t, \tilde{w}_0)$ of (28) it holds that $|\tilde{w}(t, \tilde{w}_0)| \leq \max\{2R, |\tilde{w}_0|\}$ for all $t \in \mathbb{R}$. Thus, for any $r > 0$ we can set $\rho := \max\{2R, r\}$ such that

$$|\tilde{w}_0| < r \quad \Rightarrow \quad |\tilde{w}(t, \tilde{w}_0)| < \rho \ \ \forall \ t \in \mathbb{R}.$$

Hence, we can apply Theorem 2. By Theorem 2, there exists a continuous mapping $\alpha(\cdot) : \mathbb{R}^m \to \mathbb{R}^d$ such that the set $\mathcal{M} := \{(z, \tilde{w}) : z = \alpha(\tilde{w}), \ \tilde{w} \in \mathbb{R}^m\}$ is invariant with respect to systems (9) and (28) and every solution $z(t) = \alpha(\tilde{w}(t))$ on this manifold is uniformly globally asymptotically stable. Since the dynamics of systems (10) and (28) coincide inside the ball $B(R)$, and $\mathcal{W}_+ \subset B(R)$ then $\mathcal{M}(\mathcal{W}_+) = \{(z, w) : z = \alpha(w), \ w \in \mathcal{W}_+\}$ is a positively invariant set with respect to (9) and (10) and every solution $\bar{z}_w(t) = \alpha(w(t, w_0))$ on this set is uniformly globally asymptotically stable.

The mapping $\alpha(w)$ depends on the choice of the auxiliary system (28), which can be made in many ways. So in general, such mapping $\alpha(w)$ is not unique. If $\alpha_1(w)$ and $\alpha_2(w)$ are two such mappings, then for any solution of system (10) starting in $w(0) \in \mathcal{W}_+$, the functions $z_1(t) := \alpha_1(w(t))$ and $z_2(t) := \alpha_2(w(t))$ are two solutions of system (9). Since they are uniformly globally asymptotically stable, then relation (12) holds. If $w(0) \in \mathcal{W}_\pm$, then the solution $w(t)$ of system (10) is defined and bounded for all $t \in \mathbb{R}$. Hence, $\alpha_1(w(t))$ and $\alpha_2(w(t))$ are two solutions of system (9) defined and bounded for all $t \in \mathbb{R}$. But due to convergence of system (9), there exists only one such solution. Hence, $\alpha_1(w(t)) \equiv \alpha_2(w(t))$ for all $t \in \mathbb{R}$. Therefore, $\alpha_1(w) = \alpha_2(w)$ for all $w \in \mathcal{W}_\pm$.$\square$

*Proof of Theorem 4.* Since the closed-loop system is uniformly convergent, it satisfies the conditions of Theorem 3. By Theorem 3 there exist continuous mappings $\bar{\pi}(\cdot) : \mathbb{R}^m \to \mathbb{R}^n$, $\bar{\sigma}(\cdot) : \mathbb{R}^m \to \mathbb{R}^q$ such that the set

$$\mathcal{M}(\mathcal{W}_+) := \{x = \bar{\pi}(w), \xi = \bar{\sigma}(w) \ w \in \mathcal{W}_+\}$$

is positively invariant with respect to (8) and (6) (in this case, $z = (x, \xi)$ and $\alpha(w) := (\bar{\pi}(w), \bar{\sigma}(w))$). Also, for every solution $w(t)$ of system (6) starting in $w(0) \in \mathcal{W}_+$, $(\bar{x}_w(t), \bar{\xi}_w(t) := (\bar{\pi}(w(t)), \bar{\sigma}(w(t)))$ is a uniformly globally asymptotically stable solution of (8).

First, we prove the "only if" implication. Due to positive invariance of $\mathcal{M}(\mathcal{W}_+)$, the mappings $\pi(w) := \bar{\pi}(w)$ and $\sigma(w) := \bar{\sigma}(w)$ satisfy equations (13). In order to show (14), consider a point $w^* \in \Omega(\mathcal{W}_+)$. By the definition of the set $\Omega(\mathcal{W}_+)$, there exists a solution $w(t, w_0)$ of system (6) with $w_0 \in \mathcal{W}_+$ and a sequence $\{t_k\}$ such that $t_k \to +\infty$ and $w(t_k, w_0) \to w^*$ as $k \to +\infty$. By continuity of $\pi(w)$ and $h(x, w)$, the limit solution $x(t) = \pi(w(t))$ satisfies

$$e(t_k) = h(\pi(w(t_k)), w(t_k)) \xrightarrow[k \to +\infty]{} h(\pi(w^*), w^*).$$

At the same time, by condition **c)** from the formulation of the uniform global output regulation problem, $e(t_k) \to 0$ as $k \to +\infty$. Hence, $h(\pi(w^*), w^*) = 0$. Due to arbitrary choice of $w^* \in \Omega(\mathcal{W}_+)$, we obtain (14).

Next, we prove the "if" implication. Equations (13) and (14) guarantee that the graph $(x, \xi) = (\pi(w), \sigma(w))$ is invariant with respect to systems (8) and (6) with $w(0) \in \mathcal{W}_+$. By formula (12) it holds that $\pi(w(t)) - \bar{\pi}(w(t)) \to 0$ and $\sigma(w(t)) - \bar{\sigma}(w(t)) \to 0$ as $t \to +\infty$. Together with global asymptotic stability of the solution $(\bar{x}_w(t), \bar{\xi}_w(t)) = (\bar{\pi}(w(t)), \bar{\sigma}(w(t)))$, this implies that for any $(x(0), \xi(0), w(0)) \in \mathbb{R}^{n+q} \times \mathcal{W}_+$

$$x(t) \to \pi(w(t)) \quad \text{as} \quad t \to +\infty.$$

By continuity of $h(x, w)$ and boundedness of $w(t)$ and $\pi(w(t))$, this implies

$$e(t) = h(x(t), w(t)) \xrightarrow[t \to +\infty]{} h(\pi(w(t)), w(t)).$$

Notice that for any solution $w(t) \in \mathcal{W}_+$ it holds that $\operatorname{dist}(w(t), \Omega(\mathcal{W}_+)) \to 0$ as $t \to +\infty$ (see, for example [18]). As follows from (14), on the set $\Omega(\mathcal{W}_+)$ the regulated output equals to zero. Hence,

$$e(t) = h(x(t), w(t)) \xrightarrow[t \to +\infty]{} h(\pi(w(t)), w(t)) \xrightarrow[t \to +\infty]{} 0.$$

$\square$

*Proof of Theorem 5.* Since we have a static controller, then $q = 0$ and $\theta(x, w) = c(w) + K(x - \pi(w))$. Denote the right-hand side of the closed-loop system $F(x, w) := f(x, c(w) + K(x - \pi(w)), w)$. The Jacobian of $F(x, w)$ equals to $\frac{\partial F}{\partial x}(x, w) = \mathcal{A}(\zeta) + \mathcal{B}(\zeta)K$. As follows from a standard result on LMI from [19], assumption **A2** and the choice of $K$ imply that

$$\mathcal{P}^{-1}(\mathcal{A}(\zeta) + \mathcal{B}(\zeta)K) + (\mathcal{A}(\zeta) + \mathcal{B}(\zeta)K)^T\mathcal{P}^{-1} \leq -Q$$

for some $Q = Q^T > 0$ and all $\zeta \in \mathbb{R}^n \times \mathcal{W}_+$. Thus, by Theorem 1 the closed-loop system $\dot{x} = F(x, w)$ is uniformly convergent. Hence, it satisfies conditions **a)** and **b)** from the formulation of the global output regulation problem. Notice, that since $\theta(\pi(w), w) = c(w)$, the continuous mapping $x = \pi(w)$ satisfies equations (13) and (14). Thus, by Theorem 4, controller (18) solves the uniform global output regulation problem. $\square$