# Frequency response functions and Bode plots for nonlinear convergent systems 

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#### Abstract

Convergent systems constitute a practically important class of nonlinear systems that extends the class of asymptotically stable LTI systems. In this paper we extend frequency response functions defined for linear systems to nonlinear convergent systems. Such nonlinear frequency response functions for convergent systems give rise to nonlinear Bode plots, which serve as a graphical tool for performance analysis of nonlinear convergent systems in the frequency domain. The results are illustrated with an example.


## I. Introduction

A common way to analyze the behavior of a (closed-loop) dynamical system is to investigate its responses to harmonic excitations at different frequencies. For linear systems, the information on responses to harmonic excitations, which is contained in frequency response functions, allows one to identify the system and analyze its properties such as performance and robustness. There exists a vast literature on frequency domain identification, analysis, and controller design methods for linear systems, see, e.g., [13], [22]. Most (high-performance) industrial controllers, especially for motion systems, are designed and tuned based on these methods, since these methods allow one to analyze the performance of the closed-loop system. The lack of such methods for nonlinear systems is one of the reasons why nonlinear systems and controllers are not popular in industry. Even if a (nonlinear) controller achieves a certain control goal (e.g., tracking), which can be proved, for example, using Lyapunov stability methods, it is very difficult to conclude how the closed-loop system would respond to external signals at various frequencies, such as, for example, high-frequency measurement noise or low-frequency disturbances. Such performance characteristics are critical in many industrial applications. So, there is a need to extend the linear frequency domain performance analysis tools, which are based on the analysis of frequency response functions, to nonlinear systems. Such an extension for the class of nonlinear convergent systems is the subject of this paper.

Convergent systems are systems that, although may be nonlinear, have relatively simple dynamics. In particular, for any bounded input such a system has a unique bounded globally asymptotically stable solution, which is called a

[^0]steady-state solution [3], [18], [17]. This property makes convergent systems convenient to deal with. Nonlinear systems with similar properties have been considered in [1], [4], [14]. In [7], [8] nonlinear controllers for a controlled Optical Pickup Unit (OPU) of DVD storage drives have been proposed to overcome linear controller design limitations. These controllers, in fact, make the corresponding closedloop system convergent. The latter fact facilitates frequencydomain performance analysis of such nonlinear though convergent closed-loop systems. In [6] even experimental frequency-domain performance analysis based on measuring steady-state responses of the closed-loop OPU to harmonic excitations has been done.

Input-output characterizations of smooth nonlinear systems have been pursued in the form of Volterra series descriptions both in time and frequency domain, see, e.g., [21], [20], [23]. The practical application of such descriptions is hampered by, firstly, the fact that the Volterra kernels in the Volterra series are, in general, difficult to compute; and, secondly, the accuracy of the truncated Volterra seriesand truncation is necessary for practical applications-is, in general, an open problem.

In this paper we show that for convergent systems all steady-state solutions corresponding to harmonic excitations at various amplitudes and frequencies can be characterized by one function. This function, which we call a nonlinear frequency response function (FRF), extends the conventional frequency response functions defined for linear systems. Contrary to the describing functions method (see, e.g., [12]), which provides only approximations of periodic steadystate responses of nonlinear systems to harmonic excitations, the nonlinear FRF provides exact steady-state responses to harmonic excitations at various amplitudes and frequencies. Similar to the linear case, the nonlinear FRF gives rise to nonlinear Bode plots, which provide information on how a convergent system amplifies harmonic inputs of various frequencies and amplitudes. This information is essential for performance analysis of convergent closed-loop systems since it allows one to quantify the influence of the highfrequency measurement noise on the steady-state response of the system, or how close the output of a closed-loop system will track certain low-frequency reference signals. Such frequency-domain performance information is extremely important in control applications.

The results in this paper are based on the idea of considering harmonic excitations as outputs of a linear harmonic oscillator or, more generally, of an exosystem. This idea has proved to be beneficial in the steady-state analysis of
nonlinear systems. In the scope of the local output regulation problem it has been used in [10], [2]. Developments in non-local steady-state analysis of nonlinear systems and its applications can be found in [11], [18]. In [9] the idea of using an exosystem has been used for quantitative analysis of steady-state as well as transient dynamics of systems excited by harmonic inputs.

The paper is organized as follows. In Section II we present definitions and basic facts on convergent systems. In Section III we review frequency response functions for linear systems. The main result on frequency response functions for nonlinear convergent systems is presented in Section IV. Nonlinear Bode plots are presented in Section V. In Section VI we present an example. Section VII contains conclusions.

## II. Convergent systems

Consider systems of the form

$$
\begin{equation*}
\dot{x}=F(x, w) \tag{1}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}$ and $w \in \mathbb{R}^{m}$. The inputs $w(t)$ are assumed to belong to the class $\overline{\mathbb{P C}}_{m}$ of piecewise-continuous functions defined and bounded on $\mathbb{R}$. The function $F(x, w)$ is assumed to be continuous with respect to $w$ and locally Lipschitz with respect to $x$. Below we give a definition of convergent systems.

Definition 1 ([3], [18]): System (1) with a given input $w(\cdot) \in \overline{\mathbb{P C}}_{m}$ is said to be (uniformly, exponentially) convergent if
i. all solutions $x_{w}(t)=x\left(t, t_{0}, x_{0}, w(\cdot)\right)$ are defined for all $t \in\left[t_{0},+\infty\right)$ and all initial conditions $t_{0} \in \mathbb{R}$, $x\left(t_{0}\right) \in \mathbb{R}^{n}$,
ii. there is a solution $\bar{x}_{w}(t)$ defined and bounded on $\mathbb{R}$,
iii. the solution $\bar{x}_{w}(t)$ is (uniformly, exponentially) globally asymptotically stable.
System (1) is said to be (uniformly, exponentially) convergent for all inputs if it is (uniformly, exponentially) convergent for every input $w(\cdot) \in \overline{\mathbb{P C}}_{m}$.

We will refer to $\bar{x}_{w}(t)$ as the steady-state solution. It is known, see, e.g., [18], that for uniformly convergent systems the steady-state solution is unique in the sense that for any input $w(\cdot) \in \overline{\mathbb{P}}_{m}$ there exists only one solution of system (1) that is bounded on $\mathbb{R}$. For our purposes we will need the following definition.

Definition 2 ([18]): A convergent system (1) is said to have the Uniformly Bounded Steady-State (UBSS) property if for any $\rho>0$ there exists $\mathcal{R}>0$ such that for any input $w(\cdot) \in \overline{\mathbb{P C}}_{m}$ the following implication holds:

$$
\begin{equation*}
|w(t)| \leq \rho \forall t \in \mathbb{R} \quad \Rightarrow \quad\left|\bar{x}_{w}(t)\right| \leq \mathcal{R} \forall t \in \mathbb{R} \tag{2}
\end{equation*}
$$

Systems that are uniformly convergent with the UBSS property extend the class of asymptotically stable linear timeinvariant (LTI) systems. One can easily verify that a linear system of the form $\dot{x}=A x+B w(t)$ with a Hurwitz matrix $A$ is uniformly and exponentially convergent with the UBSS property for all inputs.

A simple sufficient condition for the exponential convergence property, presented in the next theorem, was proposed in [3] (see also [15], [18]).

Theorem 1: Consider system (1) with $F \in C^{1}$. Suppose, there exist symmetric matrices $P>0$ and $Q>0$ such that
$P \frac{\partial F}{\partial x}(x, w)+\frac{\partial F}{\partial x}^{T}(x, w) P \leq-Q, \quad \forall x \in \mathbb{R}^{n}, \quad w \in \mathbb{R}^{m}$.
Then, system (1) is exponentially convergent with the UBSS property for all inputs.

Remark 1. It is shown in [18] that a cascade of systems satisfying the conditions of Theorem 1 is a uniformly convergent system with the UBSS property for all inputs. Further (interconnection) properties of convergent systems can be found in [18], [17].

Conditions for exponential and, therefore, uniform convergence for systems in Lur'e form with a possibly discontinuous scalar nonlinearity are presented in [24], and for piecewise-affine systems in [16].

Below we formulate a fundamental property of uniformly convergent systems, which forms a foundation for the main results of the paper. This property corresponds to the uniformly convergent system (1) excited by the input $w(t)$ being a solution of the differential equation

$$
\begin{equation*}
\dot{w}=s(w), \quad w \in \mathbb{R}^{m} \tag{4}
\end{equation*}
$$

with a locally Lipschitz right-hand side. By $w\left(t, w_{0}\right)$ we denote the solution of system (4) with the initial condition $w\left(0, w_{0}\right)=w_{0}$.

Theorem 2: Consider system (1) coupled with system (4). Suppose system (1) is uniformly convergent with the UBSS property for all inputs and system (4) satisfies the assumption
BA all solutions of system (4) are defined for all $t \in$ $(-\infty,+\infty)$ and for every $r>0$ there exists $\rho>0$ such that

$$
\begin{equation*}
\left|w_{0}\right|<r \Rightarrow\left|w\left(t, w_{0}\right)\right|<\rho \quad \forall t \in \mathbb{R} \tag{5}
\end{equation*}
$$

Then there exists a unique continuous mapping $\alpha: \mathbb{R}^{m} \rightarrow$ $\mathbb{R}^{n}$ such that for any solution $w(t)=w\left(t, w_{0}\right)$ of system (4) the corresponding steady-state solution of system (1) equals $\bar{x}_{w}(t) \equiv \alpha\left(w\left(t, w_{0}\right)\right)$.

Proof: See the Appendix.

## III. LINEAR FREQUENCY RESPONSE FUNCTIONS

Prior to considering the case of nonlinear systems, let us have a look at LTI systems of the form

$$
\begin{equation*}
\dot{x}=A x+B u, \tag{6}
\end{equation*}
$$

with $x \in \mathbb{R}^{n}, u \in \mathbb{R}$ and a Hurwitz matrix $A$. System (6) can be equivalently represented in Laplace domain by its transfer function $G(s):=(s I-A)^{-1} B$. With this function one can immediately compute the steady-state solution corresponding to the complex harmonic excitation $a e^{i \omega t}$, which equals $G(i \omega) a e^{i \omega t}$. This, in turn, implies that the steadystate solution corresponding to the real harmonic excitation
$a \sin (\omega t)$ equals $\bar{x}_{a \omega}(t)=\operatorname{Im}\left(G(i \omega) a e^{i \omega t}\right)$. This method is not applicable to nonlinear systems since the transformation into Laplace domain is, in general, not applicable to nonlinear systems.

An alternative way of finding steady-state solutions of system (6) is based on the fact that a harmonic excitation can be considered as an output of the linear harmonic oscillator

$$
\begin{align*}
& \dot{w}=S(\omega) w, w:=\left[\begin{array}{l}
w_{1} \\
w_{2}
\end{array}\right], S(\omega):=\left[\begin{array}{cc}
0 & \omega \\
-\omega & 0
\end{array}\right]  \tag{7}\\
& u=\Gamma w, \quad \Gamma:=\left[\begin{array}{ll}
1 & 0
\end{array}\right] .
\end{align*}
$$

This system generates harmonic outputs of the form $u(t)=$ $a \sin (\omega t+\phi)$, where the phase $\phi$ and amplitude $a$ are determined by the initial conditions of (7). Therefore, to study responses to harmonic excitations, we can consider steady-state solutions of the system

$$
\begin{equation*}
\dot{x}=A x+B \Gamma w, \tag{8}
\end{equation*}
$$

with $w(t)$ being solutions of the harmonic oscillator (7). Since the eigenvalues of the matrices $A$ and $S(\omega)$ do not coincide, for any $\omega \geq 0$ there exists a unique matrix $\Pi(\omega) \in$ $\mathbb{R}^{n \times 2}$ satisfying the matrix equation (see, e.g., [5])

$$
\begin{equation*}
\Pi(\omega) S(\omega)=A \Pi(\omega)+B \Gamma \tag{9}
\end{equation*}
$$

By substitution one can easily verify that for any solution $w(t)$ of (7), the corresponding steady-state solution of (6) equals $\bar{x}_{w}(t)=\Pi(\omega) w(t)$. Moreover, it can be verified that $\Pi(\omega)=[\operatorname{Re}(G(i \omega)) \quad \operatorname{Im}(G(i \omega))]$. Therefore, the function $\alpha(w, \omega):=\Pi(\omega) w$ can be considered as a frequencyresponse function of system (6) since it contains information on all steady-state responses to harmonic excitations at different frequencies and amplitudes. Notice that due to the linearity of system (8) the function $\alpha(w, \omega)$ is linear in $w$ and all essential information is contained in $\Pi(\omega)$. For this reason, in linear systems theory only $\Pi(\omega)$ or, equivalently, $G(i \omega)$, is considered as a frequency response function. For nonlinear systems the linearity in $w$ will apparently be lost and we will have to consider frequency response functions as functions of both $\omega$ and $w$.

## IV. Nonlinear frequency response functions

In this section we consider uniformly convergent systems

$$
\begin{equation*}
\dot{x}=f(x, u), \quad y=h(x) \tag{10}
\end{equation*}
$$

with state $x \in \mathbb{R}^{n}$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$. Recall that according to Definition 1, for any bounded input $u(t)$ system (10) has a unique steady-state solution $\bar{x}_{u}(t)$, which is UGAS. We are interested in a characterization of all steady-state responses corresponding to harmonic excitations $u(t):=a \sin (\omega t)$ with various frequencies $\omega \geq 0$ and amplitudes $a \geq 0$. The main result of the paper is formulated in the following theorem.

Theorem 3: Suppose system (10) is uniformly convergent with the UBSS property for all inputs. Then there exists a continuous function $\alpha: \mathbb{R}^{3} \rightarrow \mathbb{R}^{n}$ such that for any harmonic
excitation of the form $u(t)=a \sin (\omega t)$, system (10) has a unique periodic solution

$$
\begin{equation*}
\bar{x}_{a \omega}(t):=\alpha(a \sin (\omega t), a \cos (\omega t), \omega) \tag{11}
\end{equation*}
$$

and this solution is UGAS.
Proof: The proof of this theorem follows from the fact that harmonic signals of the form $u(t)=a \sin (\omega t)$ for various amplitudes $a \geq 0$ and frequencies $\omega \geq 0$ are generated by the system

$$
\begin{align*}
\dot{w}_{1} & =w_{3} w_{2}, \quad \dot{w}_{2}=-w_{3} w_{1}, \quad \dot{w}_{3}=0,  \tag{12}\\
u & =\Gamma w, \quad \Gamma:=\left[\begin{array}{lll}
1 & 0 & 0
\end{array}\right],
\end{align*}
$$

with the initial conditions $w_{1}(0)=0, w_{2}(0)=a, w_{3}(0)=$ $\omega$. Consequently, we can treat system (10) excited by the input $u(t)=a \sin (\omega t)$ as the system

$$
\begin{equation*}
\dot{x}=f(x, \Gamma w) \tag{13}
\end{equation*}
$$

excited by a solution of the system (12). According to the conditions of the theorem, system (13) is uniformly convergent with the UBSS property for all inputs. One can easily check that system (12) satisfies the boundedness assumption BA of Theorem 2. Therefore, by Theorem 2 there exists a unique continuous function $\alpha: \mathbb{R}^{3} \rightarrow \mathbb{R}^{n}$ such that for any solution $w(t)$ of the system (12) the corresponding steady-state solution of system (13), which is UGAS due to the uniform convergence property, equals $\bar{x}_{w}(t)=\alpha\left(w_{1}(t), w_{2}(t), w_{3}(t)\right)$. In particular, for the solution of system (12) $w(t)=[a \sin (\omega t), a \cos (\omega t), \omega]^{T}$, which corresponds to the input $u(t)=a \sin (\omega t)$, the corresponding steady-state solution equals $\bar{x}_{a \omega}(t)$ given in (11).

As follows from Theorem 3, the function $\alpha\left(w_{1}, w_{2}, \omega\right)$ contains all information on the steady-state solutions of system (10) corresponding to harmonic excitations. For this reason, we give the following definition.

Definition 3: The function $\alpha\left(w_{1}, w_{2}, \omega\right)$ defined in Theorem 3 is called the state frequency response function. The function $h\left(\alpha\left(w_{1}, w_{2}, \omega\right)\right)$ is called the output frequency response function.
In the nonlinear case, the dependency of the frequency response functions on $w_{1}, w_{2}$ and $\omega$ is, in general, nonlinear. This implies, for example, that for nonlinear convergent systems we may observe a non-proportional change in the amplitude of the steady-state responses with respect to a change of the excitation amplitude. At the same time, the steady-state solution $\bar{x}_{a \omega}(t)$ given in (11) is a unique periodic solution with the same period time as the period of the harmonic excitation. This resembles properties of asymptotically stable linear systems. Notice that for general nonlinear systems, one can have multiple coexisting attractors, which excludes the possibility of the existence of the single-valued mapping $\alpha$ and makes analysis of the steady-state behavior corresponding to harmonic excitations much more involved.

In general, it is not easy to find such frequency response functions analytically. Yet, as will be illustrated with an example in Section VI, for some systems this can be done relatively easily. For general uniformly convergent systems
(10), the frequency response functions can always be found numerically by simulating system (10) with the input $u(t)=$ $a \sin (\omega t)$. All solutions of this system converge to the UGAS steady-state solution equal to $\alpha(a \sin (\omega t), a \cos (\omega t), \omega)$. By performing these simulations for various excitation amplitudes $a$ and frequencies $\omega$ we will find the state frequency response function in the range of interest of $a$ and $\omega$. To reduce computational costs for this numerical procedure, instead of finding exact values of the periodic steady-state solutions $\alpha(a \sin (\omega t), a \cos (\omega t), \omega)$, one can find their approximations using the describing function method, see, e.g., [12]. Notice that the describing function method provides only an approximation of the periodic solution $\alpha(a \sin (\omega t), a \cos (\omega t), \omega)$ based on its first harmonic (or the first $k$ harmonics). Also such an approximation requires an additional justification. Similar to this simulation-based numerical procedure, in practice, when one has a convergent system, its output frequency response function $h\left(\alpha\left(w_{1}, w_{2}, \omega\right)\right)$ can be obtained by exciting the system with harmonic signals at various amplitudes and frequencies and measuring the corresponding steady-state outputs.

## V. Nonlinear Bode plot

In practice it is very important to know how a system amplifies inputs at various frequencies. In performance analysis of control systems this information allows one to quantify the influence of high frequency measurement noise on the steady-state response of the system, or how close a closedloop system will track low-frequency reference signals. In the case of LTI systems, this essentially important information is usually represented in the Bode magnitude plots. The Bode magnitude plot is a graphical representation of the gain with which the system amplifies harmonic signals at various frequencies.

Similar to linear systems, for uniformly convergent systems we can define a counterpart of the Bode magnitude plot, which then can be used for the purpose of frequency domain performance analysis. Suppose the system is excited by the harmonic signal $a \sin (\omega t)$ with amplitude $a$. Denote the maximal absolute value of the output in steady-state by $B(\omega, a)$. We are interested in the ratio $\gamma_{a}(\omega):=B(\omega, a) / a$ at various amplitudes and frequencies. This ratio can be considered as an amplification gain of the convergent system. Notice that in the nonlinear case $\gamma_{a}(\omega)$ depends not only on the frequency, as in the linear case, but also on the amplitude of the excitation.

Formally, the amplification gain $\gamma_{a}(\omega)$ is defined as

$$
\begin{equation*}
\gamma_{a}(\omega):=\frac{1}{a}\left(\sup _{w_{1}^{2}+w_{2}^{2}=a^{2}}\left|h\left(\alpha\left(w_{1}, w_{2}, \omega\right)\right)\right|\right) . \tag{14}
\end{equation*}
$$

If we are interested in the maximum of the amplification gain for a given frequency $\omega$ and over a range of amplitudes $a \in(\underline{a}, \bar{a})$, then we can extend the definition of $\gamma_{a}(\omega)$ as follows: $\Upsilon_{(\underline{a}, \bar{a})}(\omega):=\sup _{a \in(\underline{a}, \bar{a})} \gamma_{a}(\omega)$.

For linear SISO systems of the form $\dot{x}=A x+B u$ with a Hurwitz matrix $A$ and output $y=C x$, the gain $\gamma_{a}(\omega)$ is independent of the amplitude $a$ and it equals $\gamma(\omega)=$
$\left|C(i \omega I-A)^{-1} B\right|$. Therefore, we see that for linear systems the graph of the amplification gain $\gamma_{a}(\omega)$ as in (14) versus the excitation frequency $\omega$ coincides with the Bode magnitude plot. Although for linear systems the Bode plot also contains phase information, at the moment it is not clear yet how to define a meaningful counterpart of the Bode phase plot for nonlinear convergent systems.

## VI. Example

For general convergent systems it is rather difficult to find the frequency response function $\alpha\left(w_{1}, w_{2}, \omega\right)$ analytically. Yet, for some systems this can be done rather easily, as illustrated by the following example. Consider the system

$$
\begin{align*}
& \dot{x}_{1}=-x_{1}+x_{2}^{2}, \quad y=x_{1}  \tag{15}\\
& \dot{x}_{2}=-x_{2}+u \tag{16}
\end{align*}
$$

excited by the input $u(t)=a \sin (\omega t)$. This system is a series connection of two systems satisfying the conditions of Theorem 1. By the remark to this theorem, system (15), (16) is uniformly convergent with the UBSS property. Consequently, by Theorem 3 the mapping $\alpha\left(w_{1}, w_{2}, \omega\right)$ exists and is unique. We will first find $\alpha_{2}\left(w_{1}, w_{2}, \omega\right)$ (the second component of $\alpha$ ) from (16). Since the $x_{2}$-subsystem is an asymptotically stable LTI system, $\alpha_{2}\left(w_{1}, w_{2}, \omega\right)$ is linear with respect to $w_{1}$ and $w_{2}$ (see Section III), i.e. $\alpha_{2}\left(w_{1}, w_{2}, \omega\right)=b_{1}(\omega) w_{1}+$ $b_{2}(\omega) w_{2}$. Recall that $\alpha_{2}\left(w_{1}(t), w_{2}(t), \omega\right)$ with $w(t)=$ $\left[w_{1}(t) w_{2}(t)\right]^{T}$ being a solution of the linear harmonic oscillator (7) is a solution of system (16) with $u(t)=w_{1}(t)$. Substituting this $\alpha_{2}\left(w_{1}(t), w_{2}(t), \omega\right)$ into equation (16) and equating the corresponding coefficients at $w_{1}$ and $w_{2}$, we obtain $b_{1}(\omega)=1 /\left(1+\omega^{2}\right)$ and $b_{2}(\omega)=-\omega /\left(1+\omega^{2}\right)$. Then, substituting the obtained $\alpha_{2}$ for $x_{2}$ in (15), we compute $\alpha_{1}\left(w_{1}, w_{2}, \omega\right)$. In our case, it is a polynomial of $w_{1}$ and $w_{2}$ of the same degree as the polynomial $\left(\alpha_{2}\left(w_{1}, w_{2}, \omega\right)\right)^{2}$, i.e. of degree 2 (see [2] Lemma 1.2 for details). Therefore, we will seek $\alpha_{1}\left(w_{1}, w_{2}, \omega\right)$ in the form

$$
\begin{equation*}
\alpha_{1}\left(w_{1}, w_{2}, \omega\right)=c_{1}(\omega) w_{1}^{2}+2 c_{2}(\omega) w_{1} w_{2}+c_{3}(\omega) w_{2}^{2} \tag{17}
\end{equation*}
$$

We substitute the steady-state solution $\alpha_{1}\left(w_{1}(t), w_{2}(t), \omega\right)$ for $x_{1}(t)$ into (15) with $x_{2}(t)=\alpha_{2}\left(w_{1}(t), w_{2}(t), \omega\right)$ and then equate the corresponding coefficients at the terms $w_{1}^{2}$, $w_{1} w_{2}$ and $w_{2}^{2}$. This results in $c_{1}(\omega)=\left(2 \omega^{4}+1\right) / \Delta(\omega)$, $c_{2}(\omega)=\left(\omega^{3}-2 \omega\right) / \Delta(\omega)$, and $c_{3}(\omega)=\left(2 \omega^{4}+5 \omega^{2}\right) / \Delta(\omega)$, where $\Delta(\omega):=\left(1+4 \omega^{2}\right)\left(1+\omega^{2}\right)^{2}$. After the function $\alpha\left(w_{1}, w_{2}, \omega\right)$ is computed, one can numerically, though very efficiently, compute the amplification gain $\gamma_{a}(\omega)$ for a range of amplitudes $a$ and frequencies $\omega$, and $\Upsilon_{(0, \bar{a}]}(\omega)$, for some maximal excitation amplitude $\bar{a}$ and all frequencies over the band of interest. Since the output frequency response function $\alpha_{1}\left(w_{1}, w_{2}, \omega\right)$ is a uniform polynomial function of degree 2 with respect to the variables $w_{1}$ and $w_{2}$ (see formula (17)), one can easily check that for arbitrary $a>0$ it holds that $\gamma_{a}(\omega)=a \gamma_{1}(\omega)$. Here we recognize the dependency of the amplification gain on the amplitude of the excitation-an essentially nonlinear phenomenon. Figure 1 shows the graph of numerically computed $\gamma_{1}(\omega)$ over $\omega$-a counterpart of the Bode magnitude plot from linear systems theory.


Fig. 1. The function $\gamma_{1}(\omega)$ (nonlinear Bode plot).

## VII. Conclusions

In this paper we have shown that for a uniformly convergent system with the UBSS property all steady-state solutions corresponding to harmonic excitations at various frequencies and amplitudes can be characterized by one continuous function, which we call a nonlinear frequency response function (FRF). It has been shown that this function extends the notion of FRF from the linear systems theory. In contrast to the describing function method, which provides only approximations of the steady-state solutions corresponding to harmonic excitations, this nonlinear FRF contains exact information on these steady-state solutions. For some systems, as has been illustrated with an example, the nonlinear FRF can be found analytically. If this is not possible, it can always be found numerically or, in case an experimental system is available, measured in experiments by exciting the system with harmonic signals at various amplitudes and frequencies. An extension of the presented results to the case of differential inclusions can be found in [19].

The newly defined nonlinear FRF gives rise to a frequency-dependent amplification gain, which provides information on how a system amplifies harmonic inputs of various frequencies and amplitudes. This information is essential for performance analysis of convergent closed-loop systems since it allows one to quantify the influence of, e.g., the high-frequency measurement noise on the steady-state response of the system, or how close the output of a closedloop system will track low-frequency reference signals. Such information is important in control applications. A plot of this gain versus the harmonic input frequency is a counterpart of the Bode magnitude plot from linear systems theory.

The results presented in this paper may open an interesting direction in nonlinear control systems theory. They provide a potential link between the performance-oriented linear systems thinking dominating in industry and the stabilityoriented nonlinear systems thinking, which is wide-spread in academia.

## Appendix: Proof of Theorem 2

Existence: We prove the existence of $\alpha(w)$ by constructing this mapping. Due to the boundedness assumption BA, for every $w_{0} \in \mathbb{R}^{m}$ the solution $w\left(t, w_{0}\right)$ of system (4) which satisfies the initial condition $w\left(0, w_{0}\right)=w_{0}$ is defined and bounded for all $t \in \mathbb{R}$. Therefore, for each $w_{0} \in \mathbb{R}^{m}$, the
function $w\left(t, w_{0}\right)$, as a function of $t$ belongs to $\overline{\mathbb{P C}}_{m}$. Since system (1) is uniformly convergent, for this $w(t)=w\left(t, w_{0}\right)$ there exists a unique UGAS steady-state solution $\bar{x}_{w}(t)$, which is defined and bounded for all $t \in \mathbb{R}$. For all $w$ lying on the trajectory $w\left(t, w_{0}\right), t \in \mathbb{R}$, define the mapping $\alpha(w)$ in the following way: $\alpha\left(w\left(t, w_{0}\right)\right):=\bar{x}_{w}(t)$. Repeating this process for all trajectories $w\left(t, w_{0}\right)$ of system (4)—notice that these trajectories do not intersect and span the whole $\mathbb{R}^{m}$ )—we uniquely define $\alpha(w)$ for all $w \in \mathbb{R}^{m}$.

Continuity: It remains to show that the mapping $x=$ $\alpha(w)$ constructed above is continuous, i.e. that for any $w_{1} \in$ $\mathbb{R}^{m}$ and any $\varepsilon>0$ there exists $\delta>0$ such that $\mid w_{1}-$ $w_{2} \mid<\delta$ implies $\left|\alpha\left(w_{1}\right)-\alpha\left(w_{2}\right)\right|<\varepsilon$. For simplicity, we will prove continuity in the ball $|w|<r$. Since $r$ can be chosen arbitrarily, this will imply continuity in $\mathbb{R}^{m}$. In what follows, we assume that $w_{1}$ satisfying $\left|w_{1}\right|<r$ and $\varepsilon>0$ are fixed and the point $w_{2}$ varies in the ball $\left|w_{2}\right|<r$.

As a preliminary observation, notice that $\left|w_{1}\right| \leq r$ and $\left|w_{2}\right| \leq r$ imply, due to the boundedness assumption BA, that $\left|w\left(t, w_{i}\right)\right| \leq \rho$ for $i=1,2$ and for all $t \in \mathbb{R}$. This, in turn, due to the UBSS property of system (1) (see (2)) and due to the construction of $\alpha(w)$, implies

$$
\begin{equation*}
\left|\alpha\left(w\left(t, w_{i}\right)\right)\right| \leq \mathcal{R}, \quad \forall t \in \mathbb{R}, \quad i=1,2 \tag{18}
\end{equation*}
$$

In order to prove continuity of $\alpha(w)$, we introduce the function $\varphi_{T}\left(w_{1}, w_{2}\right):=\chi\left(0,-T, \alpha\left(w\left(-T, w_{2}\right)\right), w_{1}\right)$, where the number $T>0$ will be specified later and $\chi\left(t, t_{0}, x_{0}, w_{*}\right)$ is the solution of the time-varying system

$$
\begin{equation*}
\dot{\chi}=F\left(\chi, w\left(t, w_{*}\right)\right) \tag{19}
\end{equation*}
$$

with the initial condition $\chi\left(t_{0}, t_{0}, x_{0}, w_{*}\right)=x_{0}$. The function $\varphi_{T}\left(w_{1}, w_{2}\right)$ has the following meaning, see Fig. 2. First,


Fig. 2. The construction of the function $\varphi_{T}\left(w_{1}, w_{2}\right)$.
consider the steady-state solution $\alpha\left(w\left(t, w_{2}\right)\right)$, which is a solution of system (19) with the input $w\left(t, w_{2}\right)$ and initial condition $\alpha\left(w\left(0, w_{2}\right)\right)=\alpha\left(w_{2}\right)$. We shift along $\alpha\left(w\left(t, w_{2}\right)\right)$ to time $t=-T$ and appear in $\alpha\left(w\left(-T, w_{2}\right)\right)$. Then we switch the input to $w\left(t, w_{1}\right)$, shift forward to the time instant $t=0$ along the solution $\chi(t)$ corresponding to this $w\left(t, w_{1}\right)$ and starting in $\chi(-T)=\alpha\left(w\left(-T, w_{2}\right)\right)$ and appear in $\chi(0)=\varphi_{T}\left(w_{1}, w_{2}\right)$. Notice, that $\varphi_{T}\left(w_{0}, w_{0}\right)=\alpha\left(w_{0}\right)$, for all $w_{0} \in \mathbb{R}^{m}$, because there is no switch of inputs and we just shift back and forth along the same solution $\alpha\left(w\left(t, w_{0}\right)\right)$.

Thus,

$$
\begin{align*}
\alpha\left(w_{1}\right)-\alpha\left(w_{2}\right)= & \varphi_{T}\left(w_{1}, w_{1}\right)-\varphi_{T}\left(w_{2}, w_{2}\right) \\
= & \varphi_{T}\left(w_{1}, w_{1}\right)-\varphi_{T}\left(w_{1}, w_{2}\right)  \tag{20}\\
& +\varphi_{T}\left(w_{1}, w_{2}\right)-\varphi_{T}\left(w_{2}, w_{2}\right)
\end{align*}
$$

By the triangle inequality, this implies

$$
\begin{aligned}
\left|\alpha\left(w_{1}\right)-\alpha\left(w_{2}\right)\right| & \leq\left|\varphi_{T}\left(w_{1}, w_{1}\right)-\varphi_{T}\left(w_{1}, w_{2}\right)\right| \\
& +\left|\varphi_{T}\left(w_{1}, w_{2}\right)-\varphi_{T}\left(w_{2}, w_{2}\right)\right|
\end{aligned}
$$

As follows from Lemma 1 (see below), there exists $T>0$ such that

$$
\begin{equation*}
\left|\varphi_{T}\left(w_{1}, w_{1}\right)-\varphi_{T}\left(w_{1}, w_{2}\right)\right|<\varepsilon / 2 \quad \forall\left|w_{2}\right|<r \tag{21}
\end{equation*}
$$

It follows from Lemma 2 (see below), that given a number $T>0$, there exists $\delta>0$ such that

$$
\begin{align*}
\left|\varphi_{T}\left(w_{1}, w_{2}\right)-\varphi_{T}\left(w_{2}, w_{2}\right)\right| & <\varepsilon / 2  \tag{22}\\
\forall w_{2}:\left|w_{1}-w_{2}\right| & <\delta
\end{align*}
$$

Unifying inequalities (21) and (22), we obtain $\mid \alpha\left(w_{1}\right)-$ $\alpha\left(w_{2}\right) \mid<\varepsilon$ for all $w_{2}$ satisfying $\left|w_{1}-w_{2}\right|<\delta$. Due to the arbitrary choice of $\varepsilon>0$ and $\left|w_{1}\right|<r$, this proves continuity of $\alpha(w)$ in the ball $|w|<r$. Due to the arbitrary choice of $r>0$, this implies continuity of $\alpha(w)$ in $\mathbb{R}^{m}$.

Lemma 1: There is $T>0$ such that inequality (21) holds.
Proof: In order to prove inequality (21), notice that $\varphi_{T}\left(w_{1}, w_{1}\right)=\chi_{1}(0)$ and $\varphi_{T}\left(w_{1}, w_{2}\right)=\chi_{2}(0)$, where $\chi_{1}(t)$ and $\chi_{2}(t)$ are solutions of system (19) with the input $w\left(t, w_{1}\right)$ satisfying the initial conditions $\chi_{1}(-T)=$ $\alpha\left(w\left(-T, w_{1}\right)\right)$ and $\chi_{2}(-T)=\alpha\left(w\left(-T, w_{2}\right)\right)$. By the construction of $\alpha(w), \chi_{1}(t)=\alpha\left(w\left(t, w_{1}\right)\right)$ is a UGAS solution of system (19). This implies that for $\mathcal{R}>0$ and $\varepsilon>0$ there exists $\tilde{T}_{\varepsilon}(\mathcal{R})>0$ such that for any solution $\chi(t)$ of system (19) the inequality $\left|\chi_{1}\left(t_{0}\right)-\chi\left(t_{0}\right)\right| \leq 2 \mathcal{R}$ implies

$$
\begin{equation*}
\left|\chi_{1}(t)-\chi(t)\right|<\varepsilon / 2, \quad \forall t \geq t_{0}+\tilde{T}_{\varepsilon}(\mathcal{R}), t_{0} \in \mathbb{R} \tag{23}
\end{equation*}
$$

Set $T:=\tilde{T}_{\varepsilon}(\mathcal{R})$. By the definition of $\chi_{1}(t)$ and $\chi_{2}(t)$, we have $\chi_{1}(-T)=\alpha\left(w\left(-T, w_{1}\right)\right)$ and $\chi_{2}(-T)=$ $\alpha\left(w\left(-T, w_{2}\right)\right)$. By the inequality (18) and the triangle inequality, we conclude that $\left|\chi_{1}(-T)-\chi_{2}(-T)\right| \leq 2 \mathcal{R}$. Thus, for $t_{0}=-T$ and $t=0$ formula (23) implies

$$
\begin{equation*}
\left|\chi_{1}(0)-\chi_{2}(0)\right|<\varepsilon / 2 \tag{24}
\end{equation*}
$$

which is equivalent to (21).
Lemma 2: Given a number $T>0$ there exists a number $\delta>0$ such that inequality (22) is satisfied.

Proof: In order to show (22), notice that for a fixed $T>0$, the function $\chi\left(0,-T, x_{0}, w_{0}\right)$ is continuous with respect to $x_{0}$ and $w_{0}$. Thus, $\chi\left(0,-T, x_{0}, w_{0}\right)$ is uniformly continuous over the compact set $J:=\left\{\left(x_{0}, w_{0}\right):\left|x_{0}\right| \leq \mathcal{R},\left|w_{0}\right| \leq r\right\}$. Hence, there exists $\delta>0$ such that if $\left|x_{0}\right| \leq \mathcal{R},\left|w_{1}\right| \leq r$, $\left|w_{2}\right| \leq r$ and $\left|w_{1}-w_{2}\right|<\delta$, then

$$
\begin{equation*}
\left.\mid \chi\left(0,-T, x_{0}, w_{1}\right)\right)-\chi\left(0,-T, x_{0}, w_{2}\right) \mid \leq \varepsilon / 2 \tag{25}
\end{equation*}
$$

Recall, that by the definition of $\varphi_{T}\left(w_{1}, w_{2}\right)$

$$
\begin{align*}
& \varphi_{T}\left(w_{1}, w_{2}\right)-\varphi_{T}\left(w_{2}, w_{2}\right)= \\
& \chi\left(0,-T, x_{0}, w_{1}\right)-\chi\left(0,-T, x_{0}, w_{2}\right) \tag{26}
\end{align*}
$$

where $x_{0}:=\alpha\left(w\left(-T, w_{2}\right)\right)$. Notice, that $\left|w_{1}\right| \leq r,\left|w_{2}\right| \leq r$ and $\left|\alpha\left(w\left(-T, w_{2}\right)\right)\right| \leq \mathcal{R}$. Hence, as follows from (25) and (26), $\left|w_{1}-w_{2}\right|<\delta$ implies $\left|\varphi_{T}\left(w_{1}, w_{2}\right)-\varphi_{T}\left(w_{2}, w_{2}\right)\right|<$ $\varepsilon / 2$. Thus, we have shown (22).

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