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Output Tracking Control of PWA Systems

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Abstract— **: In this paper, an observer-based output-feedback control design solving the tracking problem for continuous piecewise affine (PWA) systems is proposed. The design of the dynamic controller is based on the idea of, on the one hand, rendering the system convergent by means of feedback (which makes all its solutions converge to each other) and, on the other hand, guaranteeing that the closed-loop system has a bounded solution corresponding to zero tracking error. This implies that all solutions of the closed-loop system converge to this bounded solution with zero tracking error. Using this synthesis approach we solve the state tracking problem for general continuous PWA systems and the output tracking problem for a class of single-input-single-output PWA systems. The results are illustrated by application to mechanical systems with one-sided restoring characteristics and backlash.**

I. Introduction

Currently, PWA systems are receiving wide attention due to the fact that the PWA framework [1] provides a means to describe dynamic systems exhibiting switching between a multitude of linear dynamic regimes, see e.g. [2], [3]. Such switching can, for instance, be due to piecewise-linear characteristics such as dead-zone, saturation, hysteresis or relays. The asymptotic tracking of prescribed reference signals is a central problem in control theory and will be considered here for PWA systems. A common approach in achieving tracking is the stabilisation of the tracking error dynamics. One could then think of translating the tracking problem for PWA systems into some stabilisation problem for PWA systems and subsequently applying known results for the stabilization of PWA systems, see for example [4], [5], [6], [7], [8], [9]. For PWA systems that can be represented in the form of a Lur'e system, stabilising output-feedback control designs are proposed in [10]. As we will illuminate in the next section, the switching nature of the vector-field of general PWA systems seriously complicates such an approach. These difficulties can be recognised in [11], where the tracking control of *bimodal* discontinuous PWA systems is studied.

Here, we propose a different approach towards the tracking problem for general continuous PWA systems. In this approach, the notion of convergence plays a central role. A system, which is excited by an arbitrary bounded input, is called convergent if it has a unique solution (related to the input) that is bounded on the whole time axis and this solution is globally asymptotically stable. Obviously, if such a solution does exist, all other solutions, regardless of their initial conditions, converge to this solution, which can be considered as a steady-state solution [12], [13]. Similar notions describing the property of solutions converging to each other are contraction analysis, incremental stability and incremental input-to-state stability, see [14], [15], [16].

In the scope of the tracking problem we use the convergence property in the following way. The design of the tracking controllers is based on the idea of, on the one hand, rendering the closed-loop system convergent by means of feedback (which means that all its solutions converge to each other) and, on the other hand, guaranteeing that the closed-loop system has a bounded solution corresponding to zero tracking error. This implies that all solutions of the closed-loop system converge to the desired solution.

The paper is structured as follows. In Section II, the tracking problem for PWA systems is stated. The notion of convergence is introduced and sufficient conditions for convergence of PWA systems are presented in Section III. The latter properties are used in Section IV to design (observer-based) dynamic output feedback controllers solving the state tracking problem for general continuous PWA systems and the output tracking problem for a class of single-input-single-output PWA systems. An example illustrating the results is presented in the form of a mechanical system with one-sided restoring characteristics and backlash in Section V. Section VI gives concluding remarks.

II. Problem Formulation

Consider the state space $\mathbb{R}^n$ to be divided into polyhedral cells Λ_i, $i = 1, \ldots, l$, by hyperplanes given by equations of the form $\boldsymbol{H}_j^T \boldsymbol{x} + h_j = 0$, for some $\boldsymbol{H}_j \in \mathbb{R}^n$ and $h_j \in \mathbb{R}$, $j = 1, \ldots, k$. We will consider PWA systems of the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u} \;\text{ for }\; \boldsymbol{x} \in \Lambda_i,\; i = 1, \ldots, l, \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x}. \end{aligned} \tag{1}$$

Here $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}^{q\times n}$, $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b}_i \in \mathbb{R}^n$, $i = 1, \ldots, l$, are constant matrices and vectors, respectively. The vector $\boldsymbol{x} \in \mathbb{R}^n$ is the state, the vector $\boldsymbol{y} \in \mathbb{R}^q$ is the measured output and the vector $\boldsymbol{u} \in \mathbb{R}^m$ is the control input. We assume that the right-hand side of (1) is continuous. This assumption can be formalized in (necessary and sufficient) requirements regarding $\boldsymbol{A}_i$, $\boldsymbol{b}_i$, $\boldsymbol{H}_i$ and h_i (see e.g. [17]).

The problem considered in this work is twofold:

State Tracking: Design a (dynamic) control law for $\boldsymbol{u}$ that, based on information on the desired state trajectory $\boldsymbol{x}_d(t)$ and the measured output $\boldsymbol{y}$, renders $\boldsymbol{x}(t) \to \boldsymbol{x}_d(t)$ as $t \to \infty$

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and the states of the closed-loop system bounded.

Output Tracking: Design a control law for $\boldsymbol{u}$ that, based on information on the desired output trajectory $\boldsymbol{y}_d(t)$ and the measured output $\boldsymbol{y}$, renders $\boldsymbol{y}(t) \to \boldsymbol{y}_d(t)$ for $t \to \infty$ and ensures that the states of the closed-loop system remain bounded.

Many results on tracking for smooth systems do not apply to PWA systems due to the non-smooth (switching) nature of PWA systems. One could then think of translating the tracking problem into some stabilisation problem and subsequently applying known results for the stabilization of PWA systems; a topic which is currently receiving wide attention (see e.g. [4], [5], [6], [7], [8], [9]). Yet, this common way of solving the problem does not lead to tractable solutions. To illustrate this, let us consider the state tracking problem for system (1) for the simplest case of state feedback, i.e. $\boldsymbol{y} = \boldsymbol{x}$, and see how this problem would be approached in a conventional way. The first step in this approach would be to decompose the control law into a feedforward part $\boldsymbol{u}_{ff}(t)$ and a feedback part $\boldsymbol{u}_{fb}(\boldsymbol{x}, \boldsymbol{x}_d(t))$: $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}_d(t), \boldsymbol{u}_{ff}(t)) = \boldsymbol{u}_{ff}(t) + \boldsymbol{u}_{fb}(\boldsymbol{x}, \boldsymbol{x}_d(t))$. This results in the following closed-loop system:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}_{ff}(t) + \boldsymbol{B}\boldsymbol{u}_{fb}(\boldsymbol{x}, \boldsymbol{x}_d(t)), \tag{2}$$

for $\boldsymbol{x} \in \Lambda_i,\ i = 1, \ldots, l$. The feedforward part $\boldsymbol{u}_{ff}(t)$ is such that the desired solution $\boldsymbol{x}_d(t)$ is a solution of the system

$$\dot{\boldsymbol{x}}_d(t) = \boldsymbol{A}_i\boldsymbol{x}_d(t) + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}_{ff}(t), \tag{3}$$

for $\boldsymbol{x}_d \in \Lambda_i,\ i = 1, \ldots, l$. Subsequently, asymptotic tracking is assured by designing the feedback part $\boldsymbol{u}_{fb}(\boldsymbol{x}, \boldsymbol{x}_d(t))$ such that the dynamics of the tracking error $\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{x}_d(t)$ are globally asymptotically stable. These error dynamics follow from (2) and (3):

$$\dot{\boldsymbol{e}} = \boldsymbol{A}_i(\boldsymbol{e} + \boldsymbol{x}_d(t)) - \boldsymbol{A}_j\boldsymbol{x}_d(t) + (\boldsymbol{b}_i - \boldsymbol{b}_j) + \boldsymbol{B}\boldsymbol{u}_{fb}, \tag{4}$$

for $(\boldsymbol{e} + \boldsymbol{x}_d(t)) \in \Lambda_i,\ i = 1, \ldots, l$ and $\boldsymbol{x}_d(t) \in \Lambda_j,\ j = 1, \ldots, l$. The problem in this approach for PWA systems lies in the fact that the error dynamics in (4) not only switch when the state $\boldsymbol{x}$ switches from one polyhedral cell to another but also when the desired trajectory switches from one polyhedral cell to another. Consequently, the error dynamics is described by (potentially) l^2 different vector fields (which vector field applies depends on $\boldsymbol{e}$ and $\boldsymbol{x}_d(t)$, see (4)). Moreover, one should realise that these dynamics are time-varying. This combined switching and time-varying nature seriously complicates the stability analysis of the equilibrium point $\boldsymbol{e} = \boldsymbol{0}$ of (4) for general desired state trajectories $\boldsymbol{x}_d(t)$ and prevents us from applying standard stability analysis methods for PWA systems. The latter exposition aims at clarifying that the tracking control problem for PWA systems, on the one hand, can not be tackled by applying known techniques for tracking of smooth systems due to the non-smooth nature of PWA systems and, on the other hand, it is significantly more complex than the stabilisation problem for PWA systems.

The above discussion motivates our study of the tracking problem for PWA systems. Here, we will propose a new approach to this problem based on the notion of convergent systems [12], [13], which is introduced in the next section.

III. Convergent Systems

Consider systems of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{w}), \tag{5}$$

with state $\boldsymbol{x} \in \mathbb{R}^n$ and input $\boldsymbol{w} \in \mathbb{R}^d$. The function $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{w})$ is locally Lipschitz in $\boldsymbol{x}$ and continuous in $\boldsymbol{w}$. In the scope of our problem setting of tracking control, the input is due to the time-dependent desired trajectory (e.g. $\boldsymbol{x}_d(t)$ and $\boldsymbol{u}_{ff}(t)$ in (2)). In the following we will consider the class $\overline{\mathbb{PC}}_d$ of piecewise continuous inputs $\boldsymbol{w}(t) : \mathbb{R} \to \mathbb{R}^d$ which are bounded on $\mathbb{R}$.

Definition 1

System (5) is said to be

- convergent *if, for every input* $\boldsymbol{w} \in \overline{\mathbb{PC}}_d$, *there exists a solution* $\bar{\boldsymbol{x}}_w(t)$ *satisfying the following conditions:*
  - (i) $\bar{\boldsymbol{x}}_w(t)$ *is defined and bounded for all* $t \in \mathbb{R}$,
  - (ii) $\bar{\boldsymbol{x}}_w(t)$ *is globally asymptotically stable.*
- uniformly convergent *if it is convergent and* $\bar{\boldsymbol{x}}_w(t)$ *is globally uniformly asymptotically stable.*
- exponentially convergent *if it is convergent and* $\bar{\boldsymbol{x}}_w(t)$ *is globally exponentially stable.*

For the definition of Lyapunov stability of solutions of differential equations we refer to [18], [17][1].

The solution $\bar{\boldsymbol{x}}_w(t)$ is called a *steady-state solution*; the subscript w emphasizes its dependency on the input $\boldsymbol{w}(t)$. For *uniformly* convergent systems the steady-state solution is unique, as formulated below.

Property 1 ([17])

If system (5) is uniformly convergent, then the steady-state solution $\bar{\boldsymbol{x}}_w(t)$ *is the only solution defined and bounded for all* $t \in \mathbb{R}$.

The next definition extends the uniform convergence property to the input-to-state stability framework.

Definition 2

System (5) is said to be input-to-state convergent if it is uniformly convergent and for every input $\boldsymbol{w} \in \overline{\mathbb{PC}}_d$ *system (5) is input-to-state stable (ISS) with respect to the steady-state solution* $\bar{\boldsymbol{x}}_w(t)$, *i.e. there exist a* $\mathcal{KL}$*-function* $\beta(r, s)$ *and a* $\mathcal{K}$*-function* $\gamma(r)$ *such that any solution* $\boldsymbol{x}(t)$ *of system (5) corresponding to some input* $\hat{\boldsymbol{w}}(t) := \boldsymbol{w}(t) + \boldsymbol{\Delta w}(t)$ *satisfies*

$$\begin{aligned} |\boldsymbol{x}(t) - \bar{\boldsymbol{x}}_w(t)| \leq & \beta(|\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}_w(t_0)|, t - t_0) \\ & + \gamma(\sup_{t_0 \leq \tau \leq t} |\boldsymbol{\Delta w}(\tau)|). \end{aligned} \tag{6}$$

In general, the functions $\beta(r, s)$ *and* $\gamma(r)$ *may depend on the particular input* $\boldsymbol{w}(t)$.

[1]The solution $\bar{\boldsymbol{x}}_w(t)$ of system (5), which is defined on $t \in (t_*, +\infty)$, is said to be *stable* if for any $t_0 \in (t_*, +\infty)$ and $\epsilon > 0$ there exists $\delta = \delta(\epsilon, t_0) > 0$ such that $|z(t_0) - \bar{z}(t_0)| < \delta$ implies $|z(t) - \bar{z}(t)| < \epsilon$ for all $t \geq t_0$.

Similar to the conventional ISS property [19], the property of input-to-state convergence is especially useful for studying convergence properties of interconnected systems. In [17], it is shown that a series connection of two input-to-state convergent systems is an input-to-state convergent system. The latter property will be used in Section IV to design output feedback tracking controllers.

Sufficient conditions for exponential convergence and input-to-state convergence for the class of continuous PWA systems, given in (1), are stated in the following theorem.

Theorem 1 ([20])
Consider system (1) and assume that its right-hand side is continuous in $\boldsymbol{x}$. If there exists a positive definite matrix $\boldsymbol{P} = \boldsymbol{P}^T > 0$ such that

$$\boldsymbol{P}\boldsymbol{A}_i + \boldsymbol{A}_i^T\boldsymbol{P} < 0, \quad i = 1, \ldots, l, \tag{7}$$

then system (1), with $\boldsymbol{u}$ as input, is exponentially convergent and input-to-state convergent.

Condition (7) is also sufficient for the exponential convergence and input-to-state convergence of system (1) if the polyhedral cells Λ_i (and the corresponding $\boldsymbol{b}_i$) are time-dependent, as long as for any fixed t the right-hand side of (1) is continuous in $\boldsymbol{x}$.

IV. Feedback Control Design

Let us now propose a convergence-based design of the tracking controller that avoids dealing with explicitly investigating the stability of the error dynamics. The main idea of this convergence-based controller design is to find a controller that guarantees two properties: a) the closed-loop system has a trajectory which is bounded for all t and along which the tracking error (either $\boldsymbol{x} - \boldsymbol{x}_d(t)$ or $\boldsymbol{y} - \boldsymbol{y}_d(t)$) is identically zero, b) the closed-loop system is uniformly convergent. Condition b) guarantees that the closed-loop system has a unique bounded UGAS steady-state solution, while condition a) guarantees that, by Property 1, this steady-state solution equals the bounded solution of the closed-loop system with zero tracking error.

A. State Tracking

In the case of state tracking, the desired trajectory is known in terms of the entire state, i.e. $\boldsymbol{x}_d(t)$ is known. We adopt the following assumption on the desired solution $\boldsymbol{x}_d(t)$ and the existence of an appropriate feedforward $\boldsymbol{u}_{ff}(t)$:

Assumption 1
The desired solution $\boldsymbol{x}_d(t)$ is defined and bounded on $\mathbb{R}$. Moreover, there exists a feedforward $\boldsymbol{u}_{ff}(t)$, defined and bounded $\forall t$, that satisfies (3), i.e. $\boldsymbol{u}_{ff}(t)$ can be considered to be a reference control generating $\boldsymbol{x}_d(t)$.

In the following theorem an observer-based control design solving the state tracking problem is proposed.

Theorem 2
Consider the system (1), with a continuous (in $\boldsymbol{x}$) right-hand side, and a desired trajectory $\boldsymbol{x}_d(t)$ satisfying Assumption 1 with $\boldsymbol{u}_{ff}(t)$ being the corresponding feedforward. Suppose the LMIs

$$\begin{gathered} \boldsymbol{\mathcal{P}}_c = \boldsymbol{\mathcal{P}}_c^T > 0, \\ \boldsymbol{A}_i\boldsymbol{\mathcal{P}}_c + \boldsymbol{\mathcal{P}}_c\boldsymbol{A}_i^T + \boldsymbol{B}\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T\boldsymbol{B}^T < 0, \;\; i = 1, \ldots, l, \end{gathered} \tag{8}$$

and the LMIs

$$\begin{gathered} \boldsymbol{\mathcal{P}}_o = \boldsymbol{\mathcal{P}}_o^T > 0, \\ \boldsymbol{\mathcal{P}}_o\boldsymbol{A}_i + \boldsymbol{A}_i^T\boldsymbol{\mathcal{P}}_o + \boldsymbol{\mathcal{X}}\boldsymbol{C} + \boldsymbol{C}^T\,\boldsymbol{\mathcal{X}}^T < 0, \; i = 1, \ldots l, \end{gathered} \tag{9}$$

are feasible. Denote $\boldsymbol{K} := \boldsymbol{\mathcal{Y}}\boldsymbol{\mathcal{P}}_c^{-1}$ and $\boldsymbol{L} := \boldsymbol{\mathcal{P}}_o^{-1}\boldsymbol{\mathcal{X}}$. Then, system (1) in closed loop with the observer-based controller

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}} &= \boldsymbol{A}_i\hat{\boldsymbol{x}} + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\hat{\boldsymbol{y}} - \boldsymbol{y}), \\ \hat{\boldsymbol{y}} &= \boldsymbol{C}\hat{\boldsymbol{x}}, \\ \boldsymbol{u} &= \boldsymbol{K}\left(\hat{\boldsymbol{x}} - \boldsymbol{x}_d(t)\right) + \boldsymbol{u}_{ff}(t), \end{aligned} \tag{10}$$

for $\hat{\boldsymbol{x}} \in \Lambda_i$, $i = 1, \ldots, l$, is input-to-state convergent with respect to the inputs $\boldsymbol{x}_d(t)$ and $\boldsymbol{u}_{ff}(t)$. Moreover, $(\boldsymbol{x}(t), \hat{\boldsymbol{x}}(t)) = (\boldsymbol{x}_d(t), \boldsymbol{x}_d(t))$ is a globally uniformly asymptotically stable solution of the closed-loop system and this solution corresponds to zero tracking error, i.e. asymptotic state tracking is achieved.

Proof: The closed-loop system (1), (10), with inputs $\boldsymbol{x}_d(t)$ and $\boldsymbol{u}_{ff}(t)$, is input-to-state convergent, see Theorem 2 in [20]. The proof of Theorem 2 in [20] is based on Theorem 1 and certain interconnection properties of input-to-state convergent systems [17]. Using Definition 2, we conclude that the closed-loop system (1), (10) therefore is uniformly convergent. Assumption 1 and Property 1 guarantee that $(\boldsymbol{x}(t), \hat{\boldsymbol{x}}(t)) = (\boldsymbol{x}_d(t), \boldsymbol{x}_d(t))$ is the only solution of the closed-loop system that is defined and bounded $\forall t$. Therefore, $(\boldsymbol{x}(t), \hat{\boldsymbol{x}}(t)) = (\boldsymbol{x}_d(t), \boldsymbol{x}_d(t))$ is a globally uniformly asymptotically stable steady-state solution of the closed-loop system (1), (10), i.e. state tracking is achieved asymptotically. ■

In Theorem 2, the proposed control law is decomposed into a feedforward part $\boldsymbol{u}_{ff}(t)$ and a linear tracking error feedback part $\boldsymbol{K}\left(\hat{\boldsymbol{x}} - \boldsymbol{x}_d(t)\right)$. The reconstructed state $\hat{x}$, required in this feedback law, is provided by an observer for the (switching) PWA system that does not require information on the switching moments of the PWA system. This observer consists of a copy of the PWA dynamics and a linear output injection term. Clearly, by requiring the satisfaction of the LMIs (8) and (9) the controller and observer can be designed separately.

B. Output tracking

In the output tracking problem only a desired output $\boldsymbol{y}_d(t)$ is known. A bounded state trajectory $\boldsymbol{x}_d(t)$ corresponding to this desired output as well as the corresponding feedforward term $\boldsymbol{u}_{ff}(t)$ are, in general, unknown. Therefore, the controller design from the previous section is, in general, not applicable for the case of output tracking. Yet, we could use this result if we could asymptotically generate $\boldsymbol{x}_d(t)$ and $\boldsymbol{u}_{ff}(t)$ from the desired output $\boldsymbol{y}_d(t)$ and its derivatives.

We consider single-input-single-output PWA systems ($q = m = 1$) of the form:

$$\begin{aligned} \dot{\xi}_j &= \xi_{j+1}, \qquad \text{for} \quad j = 1, \dots, r-1, \\ \dot{\xi}_r &= \boldsymbol{\alpha}_i^T \boldsymbol{\xi} + \boldsymbol{\beta}_i^T \boldsymbol{\eta} + \gamma_i + \delta u \\ \dot{\boldsymbol{\eta}} &= \boldsymbol{\chi}_i \boldsymbol{\xi} + \boldsymbol{\Psi}_i \boldsymbol{\eta} + \boldsymbol{\sigma}_i + \boldsymbol{\nu} u, \\ y &= \xi_1, \end{aligned} \tag{11}$$

for $\begin{bmatrix} \boldsymbol{\xi}^T & \boldsymbol{\eta}^T \end{bmatrix}^T \in \Lambda_i$, $i = 1, \dots, l$. Note that in (11) the relative degree r is well-defined and fixed since it is the same in all polyhedral cells Λ_i, $i = 1, \dots, l$, and is not affected by the switching between the vectorfields in Λ_i, $i = 1, \dots, l$. Here, $\boldsymbol{\xi} := \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_r \end{bmatrix}^T = \begin{bmatrix} y & \dot{y} & \dots & y^{(r-1)} \end{bmatrix}^T$ is a column of derivatives of the output, where $y^{(r)}$ denotes the r^{th} time derivative of y and $\boldsymbol{\eta} = \begin{bmatrix} \eta_1 & \dots & \eta_{n-r} \end{bmatrix}^T$ a set of additional coordinates. Moreover, $\boldsymbol{\alpha}_i \in \mathbb{R}^r$, $\boldsymbol{\beta}_i \in \mathbb{R}^{n-r}$, $\boldsymbol{\chi}_i \in \mathbb{R}^{(n-r)\times r}$, $\boldsymbol{\Psi}_i \in \mathbb{R}^{(n-r)\times(n-r)}$, $\boldsymbol{\sigma}_i \in \mathbb{R}^{(n-r)}$, for $i = 1, \dots, l$, $\boldsymbol{\nu} \in \mathbb{R}^{n-r}$ and γ_i, for $i = 1, \dots, l$, and $\delta \neq 0$ are scalars. With the definition of the state vector $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{\xi}^T \boldsymbol{\eta}^T \end{bmatrix}^T$, we can write (11) in the form of (1) with

$$\boldsymbol{A}_i = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & \mathbf{0}^T \\ 0 & 0 & 1 & 0 & \dots & 0 & \mathbf{0}^T \\ \dots & \dots & \dots & \dots & \dots & \dots & \mathbf{0}^T \\ 0 & 0 & 0 & \dots & 0 & 1 & \mathbf{0}^T \\ & & & \boldsymbol{\alpha}_i^T & & & \boldsymbol{\beta}_i^T \\ & & & \boldsymbol{\chi}_i & & & \boldsymbol{\Psi}_i \end{bmatrix}, \quad \boldsymbol{b}_i = \begin{bmatrix} 0 \\ \vdots \\ \gamma_i \\ \boldsymbol{\sigma}_i \end{bmatrix}, \tag{12}$$

and $\boldsymbol{B} = \begin{bmatrix} 0 & \dots & \delta & \boldsymbol{\nu}^T \end{bmatrix}^T$, where $\mathbf{0}$ is a zero-column of length $(n-r)$. Once more we assume continuity of the right-hand side of (11) (in $\begin{bmatrix} \boldsymbol{\xi}^T & \boldsymbol{\eta}^T \end{bmatrix}^T$).

In seeking a desired bounded state trajectory $\boldsymbol{x}_d(t) = \begin{bmatrix} \boldsymbol{\xi}_d^T(t) \, \boldsymbol{\eta}_d^T(t) \end{bmatrix}^T$ corresponding to the desired output, we can easily find $\boldsymbol{\xi}_d(t)$. Namely, $\boldsymbol{\xi}_d(t) = \begin{bmatrix} y_d(t) & \dot{y}_d(t) & \dots & y_d^{(r-1)}(t) \end{bmatrix}^T$. To find the remaining part of the desired trajectory $\boldsymbol{\eta}_d(t)$, consider the control u matching the desired $\boldsymbol{\xi}_d(t)$:

$$u = \frac{1}{\delta} \left(y_d^{(r)}(t) - \boldsymbol{\alpha}_i^T \boldsymbol{\xi}_d(t) - \boldsymbol{\beta}_i^T \boldsymbol{\eta} - \gamma_i \right), \tag{13}$$

for $\begin{bmatrix} \boldsymbol{\xi}_d^T(t) & \boldsymbol{\eta}^T \end{bmatrix}^T \in \Lambda_i$, $i = 1, \dots, l$. The tracking dynamics, i.e. the $\boldsymbol{\eta}$-dynamics defined in (11), with $\boldsymbol{\xi}(t) = \boldsymbol{\xi}_d(t)$ and u as in (13) are given by:

$$\dot{\boldsymbol{\eta}} = \boldsymbol{A}_{\eta_i} \boldsymbol{\eta} + \boldsymbol{b}_{\eta_i} + \boldsymbol{B}_{\eta_i} \begin{bmatrix} \boldsymbol{\xi}_d(t) \\ y_d^{(r)}(t) \end{bmatrix}, \tag{14}$$

for $\begin{bmatrix} \boldsymbol{\xi}_d^T(t) & \boldsymbol{\eta}^T \end{bmatrix}^T \in \Lambda_i$, $i = 1, \dots, l$, with $\boldsymbol{A}_{\eta_i} = \boldsymbol{\Psi}_i - \frac{1}{\delta} \boldsymbol{\nu} \boldsymbol{\beta}_i^T$, $\boldsymbol{b}_{\eta_i} = \boldsymbol{\sigma}_i - \frac{\gamma_i}{\delta} \boldsymbol{\nu}$ and $\boldsymbol{B}_{\eta_i} = \begin{bmatrix} (\boldsymbol{\chi}_i - \frac{1}{\delta} \boldsymbol{\nu} \boldsymbol{\alpha}_i^T) & \frac{\boldsymbol{\nu}}{\delta} \end{bmatrix}$, $i = 1, \dots, l$. If we assume that system (14) is input-to-state convergent and the input vector $\begin{bmatrix} \boldsymbol{\xi}_d^T(t) & y_d^{(r)}(t) \end{bmatrix}^T$ is bounded on $\mathbb{R}$, then there exists a unique solution $\boldsymbol{\eta}_d(t)$ of system (14) that is defined and bounded on $\mathbb{R}$ and this solution is UGAS, see Definitions 1, 2 and Property 1. Therefore there exists a unique bounded on $\mathbb{R}$ solution $\boldsymbol{x}_d(t) = \begin{bmatrix} \boldsymbol{\xi}_d^T(t) & \boldsymbol{\eta}_d^T(t) \end{bmatrix}^T$ corresponding to the desired output $y_d(t)$. Moreover, this solution can be asymptotically generated (one could say reconstructed) using the PWA filter

$$\dot{\hat{\boldsymbol{\eta}}} = \boldsymbol{A}_{\eta_i} \hat{\boldsymbol{\eta}} + \boldsymbol{b}_{\eta_i} + \boldsymbol{B}_{\eta_i} \begin{bmatrix} \boldsymbol{\xi}_d(t) \\ y_d^{(r)}(t) \end{bmatrix}, \hat{\boldsymbol{x}}_d(t) = \begin{bmatrix} \boldsymbol{\xi}_d(t) \\ \hat{\boldsymbol{\eta}}(t) \end{bmatrix}, \tag{15}$$

for $\begin{bmatrix} \boldsymbol{\xi}_d^T(t) & \hat{\boldsymbol{\eta}}^T \end{bmatrix} \in \Lambda_i$, $i = 1, \dots, l$, in the sense that for any initial condition $\hat{\boldsymbol{\eta}}(0)$ of system (15) we have $\hat{\boldsymbol{x}}_d(t) \to \boldsymbol{x}_d(t)$ for $t \to \infty$. Then, the feedforward control $u_{ff}(t)$ corresponding to this $\boldsymbol{x}_d(t)$ can be asymptotically generated (reconstructed) by the formula

$$\hat{u}_{ff}(t) = \frac{1}{\delta} \left(y_d^{(r)}(t) - \boldsymbol{\alpha}_i^T \boldsymbol{\xi}_d(t) - \boldsymbol{\beta}_i^T \hat{\boldsymbol{\eta}} - \gamma_i \right) \tag{16}$$

for $\begin{bmatrix} \boldsymbol{\xi}_d^T(t) & \hat{\boldsymbol{\eta}}^T \end{bmatrix}^T \in \Lambda_i$, $i = 1, \dots, l$, in the sense that $\hat{u}_{ff}(t) \to u_{ff}(t)$ as $t \to \infty$, where $u_{ff}(t)$ is defined by (16) with $\hat{\boldsymbol{\eta}} = \boldsymbol{\eta}_d(t)$. Here, we have used the continuity assumption, which, as a consequence, implies that the right-hand side of (16) is continuous. Let us now adopt the following assumption on the tracking dynamics (14):

Assumption 2
$\boldsymbol{B}_{\eta_i} = \boldsymbol{B}_{\eta_{i-1}}$, for $i = 2, \dots, l$, i.e. all input matrices of the tracking dynamics (14) are equal.

Using the latter assumption, the assumption that the tracking dynamics (14) are input-to-state convergent can be checked using Theorem 1, since system (14) is a PWA system with the right-hand side being continuous with respect to $\boldsymbol{\eta}$ for any fixed $t \in \mathbb{R}$ (this follows from the continuity assumption on system (11)). Therefore, system (14) is input-to-state convergent if the following LMI is feasible:

$$\begin{aligned} \boldsymbol{P}_\eta \boldsymbol{A}_{\eta_i} + \boldsymbol{A}_{\eta_i}^T \boldsymbol{P}_\eta < 0, \quad i = 1, \dots, l, \\ \boldsymbol{P}_\eta = \boldsymbol{P}_\eta^T > 0. \end{aligned} \tag{17}$$

The requirement on the convergence of the tracking dynamics is strongly related to the common requirement of minimum-phase tracking dynamics for smooth systems. It should be noted that the approach proposed here for PWA systems follows the nowadays standard approach for solving tracking problems as developed for smooth systems in [21].

The final statement regarding a solution to the output tracking problem is presented in the following theorem:

Theorem 3
Consider the system (11), with a continuous right-hand side, satisfying Assumption 2. Let $y_d(t)$ be the known desired output such that $y_d(t), \dot{y}_d(t), \dots, y_d^{(r)}(t)$ are defined and bounded on $\mathbb{R}$. Suppose the LMIs (8), (9) and (17) are feasible. Denote $\boldsymbol{K} := \boldsymbol{\mathcal{Y}} \boldsymbol{\mathcal{P}}_c^{-1}$ and $\boldsymbol{L} := \boldsymbol{\mathcal{P}}_o^{-1} \boldsymbol{\mathcal{X}}$. Then the overall controller design, consisting of the observer

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}} &= \boldsymbol{A}_i \hat{\boldsymbol{x}} + \boldsymbol{b}_i + \boldsymbol{B} u + \boldsymbol{L}(\hat{y} - y), \\ \hat{y} &= \boldsymbol{C} \hat{\boldsymbol{x}}, \end{aligned} \tag{18}$$

for $\hat{\boldsymbol{x}} \in \Lambda_i$, $i = 1, \dots, l$, the controller

$$u = \boldsymbol{K} \left(\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}_d(t) \right) + \hat{u}_{ff}(t), \tag{19}$$

the trajectory generator (15) and the feedforward controller (16) solves the output tracking problem.

Proof: System (11) in closed loop with (18) and (19) with $\hat{\boldsymbol{x}}_d$ and $\hat{u}_{ff}$ as inputs is input-to-state convergent, see Theorem 2. System (15), (16), with inputs $\boldsymbol{\xi}_d$, $y_d^{(r)}$, is input-to-state convergent since LMI (17) is feasible. Therefore, the overall closed-loop system (11), (18), (19), (15) and (16) with $(\boldsymbol{\xi}_d(t), y_d^{(r)}(t)) = (y_d(t), \dot{y}_d(t), \ldots, y_d^{(r)}(t))$ as inputs is input-to-state convergent, since it is a series connection of input-to-state convergent systems [17]. Moreover, one can easily verify that the closed-loop system has a solution $(\boldsymbol{x}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{\eta}}) = (\boldsymbol{x}_d, \boldsymbol{x}_d, \boldsymbol{\eta}_d)$, where $\boldsymbol{x}_d = \begin{bmatrix}\boldsymbol{\xi}_d^T & \boldsymbol{\eta}_d^T\end{bmatrix}^T$. This solution is defined and bounded on $\mathbb{R}$. Hence, by Property 1, this solution is the UGAS steady-state solution of the closed-loop system. This, in turn, implies that all solutions of the closed-loop system are bounded for $t \geq t_0$ and output tracking is attained, i.e. $y(t) \to y_d(t)$ as $t \to \infty$. ■

Theorems 2 and 3 do not only state that the proposed controllers solve the corresponding tracking problems, but also that the corresponding closed-loop systems are input-to-state convergent with respect to the inputs used by these controllers. This property guarantees a certain robustness of the closed-loop dynamics, since small perturbations of the controller inputs ($(\boldsymbol{x}_d(t), \boldsymbol{u}_{ff}(t))$ in the case of state tracking and $(\boldsymbol{\xi}_d(t), y_d^{(r)}(t))$ in the case of output tracking problem) will result in small deviations from the desired trajectory corresponding to zero tracking error. This, in turn, results in small tracking errors.

V. An Illustrative Example

We consider an output tracking problem for a non-smooth mechanical system as depicted in Figure 1. The displacements of the masses m_1 and m_2 are denoted by z_1 and z_2, respectively, and their respective velocities by $\dot{z}_1$ and $\dot{z}_2$. Mass m_2 is attached to the fixed world by a linear spring-damper combination, with spring stiffness k_1 and damping coefficient d_1. The stiffness coefficients of the two one-sided (linear) springs are given by k_2 and k_3. Moreover, mass m_1 is coupled to mass m_2 by a linear viscous damper with damping coefficient d_2. Mass m_1 experiences a (possibly) asymmetric backlash (with backlash gaps $c_1, c_2 > 0$) before hitting the one-sided springs. Moreover, the control input u actuates mass m_1 and z_1 is measured. Mass m_1 is desired to track a sinusoidal reference trajectory $z_1 = z_d(t) := A_d \sin(\omega t)$, while the remaining states (z_2 and $\dot{z}_2$) remain bounded.

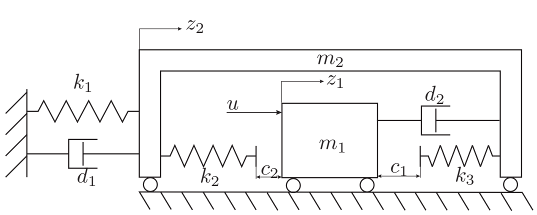

Figure 1: Controlled 2DOF mass-spring-damper system with backlash and one-sided restoring characteristics (depicted configuration is valid for $z_2 - z_1 = 0$).

The dynamics of this system can be formulated in the form (1), with $n = 4$, $k = 2$, $l = 3$, $m = q = 1$, $\boldsymbol{x} = \begin{bmatrix} z_1 & \dot{z}_1 & z_2 & \dot{z}_2\end{bmatrix}^T$, $\boldsymbol{b}_1 = \begin{bmatrix}0 & -\frac{k_2 c_2}{m_1} & 0 & \frac{k_2 c_2}{m_2}\end{bmatrix}^T$, $\boldsymbol{b}_2 = \boldsymbol{0}$, $\boldsymbol{b}_3 = \begin{bmatrix}0 & \frac{k_3 c_1}{m_1} & 0 & -\frac{k_3 c_1}{m_2}\end{bmatrix}^T$, $\Lambda_1 = \{\boldsymbol{x} \mid x_3 \geq x_1 + c_2\}$, $\Lambda_2 = \{\boldsymbol{x} \mid x_3 < x_1 + c_2 \wedge x_3 > x_1 - c_1\}$, $\Lambda_3 = \{\boldsymbol{x} \mid x_3 \leq x_1 - c_1\}$, $\boldsymbol{B} = \begin{bmatrix}0 & \frac{1}{m_1} & 0 & 0\end{bmatrix}^T$, $\boldsymbol{C} = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$ and

$$
\begin{aligned}
\boldsymbol{A}_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_2}{m_1} & -\frac{d_2}{m_1} & \frac{k_2}{m_1} & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{d_2}{m_2} & -\frac{k_2+k_1}{m_2} & -\frac{d_1+d_2}{m_2} \end{bmatrix}, \\
\boldsymbol{A}_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d_2}{m_1} & 0 & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d_2}{m_2} & -\frac{k_1}{m_2} & -\frac{d_1+d_2}{m_2} \end{bmatrix}, \\
\boldsymbol{A}_3 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_3}{m_1} & -\frac{d_2}{m_1} & \frac{k_3}{m_1} & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_3}{m_2} & \frac{d_2}{m_2} & -\frac{k_3+k_1}{m_2} & -\frac{d_1+d_2}{m_2} \end{bmatrix}.
\end{aligned}
\tag{20}
$$

Clearly, this system is of the form (11), with $\boldsymbol{\xi} = \begin{bmatrix}x_1 & x_2\end{bmatrix}^T$ and $\boldsymbol{\eta} = \begin{bmatrix}x_3 & x_4\end{bmatrix}^T$. The PWA filter (15) can now be used to generate the desired trajectory for mass m_2. However, Assumption 2 is not met. Therefore, we propose the following coordinate transformation for the PWA filter: $\bar{\eta}_1 = \hat{\eta}_1 - z_d(t)$ and $\bar{\eta}_2 = \hat{\eta}_2$. In these transformed coordinates, the trajectory generator takes the following form:

$$
\dot{\bar{\boldsymbol{\eta}}} = \boldsymbol{A}_{\bar{\eta}_i}\bar{\boldsymbol{\eta}} + \boldsymbol{b}_{\bar{\eta}_i} + \boldsymbol{B}_{\bar{\eta}}\begin{bmatrix} z_d(t) \\ \dot{z}_d(t)\end{bmatrix}, \tag{21}
$$

for $\begin{bmatrix}\boldsymbol{\xi}_d^T(t) & \bar{\boldsymbol{\eta}}^T - \begin{bmatrix}z_d(t) & 0\end{bmatrix}\end{bmatrix}^T \in \Lambda_i,\ i = 1, \ldots, 3$, with

$$
\begin{aligned}
\boldsymbol{A}_{\bar{\eta}_1} &= \begin{bmatrix} 0 & 1 \\ -\frac{k_2+k_1}{m_2} & -\frac{d_1+d_2}{m_2}\end{bmatrix}, \ \boldsymbol{B}_{\bar{\eta}} = \begin{bmatrix} 0 & -1 \\ -\frac{k_1}{m_2} & \frac{d_2}{m_2}\end{bmatrix} \\
\boldsymbol{A}_{\bar{\eta}_2} &= \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m_2} & -\frac{d_1+d_2}{m_2}\end{bmatrix}, \ \boldsymbol{A}_{\bar{\eta}_3} = \begin{bmatrix} 0 & 1 \\ -\frac{k_3+k_1}{m_2} & -\frac{d_1+d_2}{m_2}\end{bmatrix},
\end{aligned}
\tag{22}
$$

$\boldsymbol{b}_{\bar{\eta}_1} = \begin{bmatrix}0 & -\frac{k_3 c_1}{m_2}\end{bmatrix}^T$, $\boldsymbol{b}_{\bar{\eta}_2} = \boldsymbol{0}$ and $\boldsymbol{b}_{\bar{\eta}_1} = \begin{bmatrix}0 & \frac{k_2 c_2}{m_2}\end{bmatrix}^T$. We require the input-to-state convergence of the $\bar{\boldsymbol{\eta}}$-dynamics (21), which can be checked using the LMIs (17). It should be noted that the convergence properties of a system are preserved under smooth coordinate transformations [17]. Therefore, the PWA filter (15) inherits the convergence properties from (21) (the latter is required in Theorem 3). Now, the generated desired state trajectory is given by $\hat{\boldsymbol{x}}_d(t) = \begin{bmatrix}z_d(t) & \dot{z}_d(t) & \bar{\eta}_1(t) + z_d(t) & \bar{\eta}_2(t)\end{bmatrix}^T$.

We adopt the following parameter setting: $m_1 = m_2 = k_1 = 1$, $d_1 = 0.2$, $d_2 = 0.1$, $k_2 = k_3 = 0.4$, $c_1 = 0.1$, $c_2 = 0.05$, $A_d = 0.15$ and $\omega = 2\pi$. The LMIs (8) are solved, yielding the controller gains $\boldsymbol{K} = \begin{bmatrix}1.6088 & 1.1260 & -0.2282 & 0.1182\end{bmatrix}$ and the LMIs (9) are solved yielding the observer gains $\boldsymbol{L} = \begin{bmatrix}1.1056 & 1.5348 & 0.1027 & -0.1653\end{bmatrix}^T$. Moreover, the

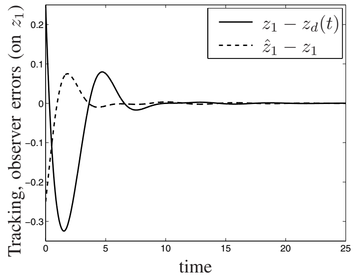

Figure 2: Tracking error and observer error.

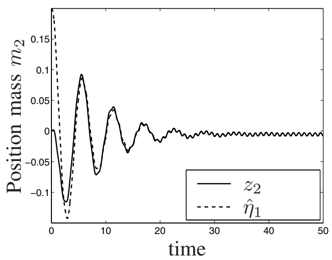

Figure 3: Position z_2 and desired position $\hat{\eta}_1$ of mass m_2.

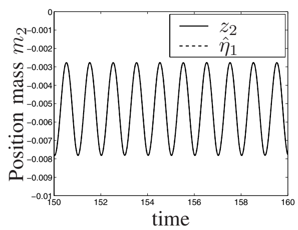

Figure 4: Steady-state behaviour of z_2 and $\hat{\eta}_1$.

LMIs (17) are checked to be feasible for (21), (22). Finally, the feedforward is designed as:

$$u_{ff}(t) = \begin{cases} u_c(t) - k_3(\bar{\eta}_1 + c_1) & \text{if } \bar{\eta}_1 \leq -c_1 \\ u_c(t) & \text{if } c_2 > \bar{\eta}_1 > -c_1 \\ u_c(t) - k_2(\bar{\eta}_1 - c_2) & \text{if } \bar{\eta}_1 \geq c_2 \end{cases} \quad (23)$$

with $u_c(t) = -m_1 A_d \omega^2 \sin(\omega t) - d_2 \left(\bar{\eta}_2 - A_d \omega \cos(\omega t)\right)$. Herewith, all conditions of Theorem 3 are satisfied and asymptotic tracking of the desired trajectory is guaranteed, while all other states remain bounded. In Figure 2, the tracking error and observer error are plotted for the initial condition: $\boldsymbol{x}(0) = \begin{bmatrix} 0.25 & 0 & 0 & 0 \end{bmatrix}^T$ for the controlled system, $\hat{\boldsymbol{x}}(0) = \mathbf{0}$ for the observer and $\bar{\boldsymbol{\eta}}(0) = \begin{bmatrix} 0.2 & 0 \end{bmatrix}^T$ for the trajectory generator. Clearly, asymptotic output tracking is achieved. In Figure 3, the position z_2 of mass m_2 is depicted along with the corresponding desired trajectory $\hat{\eta}_1(t)$ generated by the trajectory generator. Note that indeed z_2 asymptotically tracks $\hat{\eta}_1(t)$ and that it remains bounded. Moreover, note that due to the fact that the trajectory generator (21), (22) is uniformly convergent, with periodic inputs $z_d(t)$ and $\dot{z}_d(t)$, the desired trajectory $\hat{\eta}_1(t)$ is also periodic in steady state with period time $2\pi/\omega(= 1)$ [17], see Figure 4. The latter property adds predictability to the behaviour of the tracking dynamics of the system, while asymptotic tracking of the system is attained.

VI. Conclusions

In this paper, the tracking problem for continuous PWA systems with an arbitrary number of polyhedral cells is addressed. It is shown that due to the switching nature of these systems conventional strategies for tracking, which are commonly based on stabilising the tracking error dynamics, lead to highly complex stabilisation problems.

The tracking control design proposed here is based on the idea of, on the one hand, rendering the system convergent by means of feedback (which makes all its solutions converge to each other) and, on the other hand, guaranteeing that the closed-loop system has a bounded solution corresponding to zero tracking error. This implies that all solutions of the closed-loop system converge to this bounded solution with zero tracking error. Using this synthesis approach, we presented a solution to the state tracking problem for general continuous PWA systems and the output tracking problem for a class of single-input-single-output PWA systems.

The effectiveness of the results is illustrated by application to a mechanical system with one-sided restoring characteristics and backlash.

References

[1] E. Sontag, "Nonlinear regulation: The piecewise linear approach," *IEEE Trans. on Autom. Contr.*, vol. 26, pp. 346–358, 1981.

[2] J. Voros, "Modeling and parameter identification of systems with multisegment piecewise-linear characteristics," *IEEE Trans. on Autom. Contr.*, vol. 47, no. 1, pp. 184–188, 2002.

[3] W. P. M. H. Heemels, M. K. Camlibel, and J. M. Schumacher, "On the dynamic analysis of piecewise-linear networks," *IEEE Trans. on Circ. and Syst. I*, vol. 49, no. 3, pp. 315–327, 2002.

[4] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, no. 3, pp. 407–427, 1999.

[5] A. Rantzer and M. Johansson, "Piecewise linear quadratic optimal control," *IEEE Trans. on Autom. Contr.*, vol. 45, no. 4, pp. 629–637, 2000.

[6] M. Johansson, *Piecewise Linear Control Systems - A Computational Approach*. Heidelberg, Germany: Springer-Verlag, 2002, in Lecture Notes in Control and Information Sciences no. 284.

[7] G. Feng, "Controller design and analysis of uncertain piecewise-linear systems," *IEEE Trans. on Circ. and Syst. I*, vol. 49, no. 2, pp. 242–232, 2002.

[8] L. Rodrigues and J. How, "Observer-based control of piecewise-affine systems," *Int. Journal of Control*, vol. 76, no. 5, pp. 459–477, 2003.

[9] M. Chen, C. Zhu, and G. Feng, "Linear-matrix-inequality-based approach to H_∞ controller synthesis of uncertain continuous-time piecewise linear systems," *IEEE Proceedings-Control Theory and Applications*, vol. 151, no. 3, pp. 295–301, 2004.

[10] M. Arcak and P. Kokotović, "Observer-based control of systems with slope-restricted nonlinearities," *IEEE Trans. on Autom. Contr.*, vol. 46, pp. 1–5, 2001.

[11] K. Sakurama and T. Sugie, "Trajectory tracking control of bimodal piecewise affine systems," *Int. Journal of Control*, vol. 78, no. 16, pp. 1314 – 1326, 2005.

[12] B. Demidovich, *Lectures on stability theory (in Russian)*. Moscow: Nauka, 1967.

[13] A. Pavlov, N. Van de Wouw, and H. Nijmeijer, "Convergent dynamics, a tribute to B.P. Demidovich," *Systems and Control Letters*, vol. 52, no. 3-4, pp. 257–261, 2004.

[14] W. Lohmiller and J.-J. Slotine, "On contraction analysis for nonlinear systems," *Automatica*, vol. 34, pp. 683–696, 1998.

[15] V. Fromion, S. Monaco, and D. Normand-Cyrot, "Asymptotic properties of incrementally stable systems," *IEEE Trans. on Autom. Contr.*, vol. 41, pp. 721–723, 1996.

[16] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Trans. on Autom. Contr.*, vol. 47, pp. 410–421, 2002.

[17] A. Pavlov, N. van de Wouw, and H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: A convergent Dynamics Approach*. Boston: Birkhäuser, 2005.

[18] J. L. Willems, *Stability theory of Dynamical Systems*. London: Thomas Nelson and Sons Ltd., 1970.

[19] E. Sontag, "On the input-to-state stability property," *European J. Control*, vol. 1, pp. 24–36, 1995.

[20] A. Pavlov, N. Van de Wouw, and H. Nijmeijer, "Convergent piecewise affine systems: analysis and design. part i: continuous case," in *Proceedings of the 44th IEEE Conference on Decision and Control and the European Control Conference 2005*, Seville, 2005.

[21] A. Isidori, *Nonlinear control systems, 3rd ed.* London: Springer-Verlag, 1995.