Frequency domain performance analysis of nonlinearly controlled motion systems

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Abstract—At the heart of the performance analysis of linear motion control systems lie essential frequency domain characteristics such as sensitivity and complementary sensitivity functions. For a class of nonlinear motion control systems called convergent systems, generalized versions of these sensitivity functions can be defined. Incorporating such effective means in nonlinear motion control design requires efficient computation or estimation methods. In this paper we present a computationally efficient method for estimating these sensitivity functions for a class of convergent Lur'e type systems. The results are illustrated by application to an industrial example of a variablegain controlled optical storage drive.

I. INTRODUCTION

A common way to analyze the behavior of a linear (closedloop) dynamical system is to investigate its response to harmonic excitations. For linear (motion) control systems, the information on responses to harmonic excitations, which is contained in the sensitivity and complementary sensitivity functions, is essential for performance evaluation of the closed-loop system. These functions allow us to quantify the sensitivity of the closed-loop system to measurement noise and external perturbations and its tracking properties. These characteristics are essential for many control applications. Commonly used controller design and tuning approaches, such as based on loop shaping techniques and H_{∞} techniques, extensively use these functions, see, e.g., [13].

These sensitivity functions, however, can not be straightforwardly extended to the case of nonlinear systems. If the plant or the controller are nonlinear, the definition of these functions through the notion of a transfer function is not applicable any more. Still, some counterparts of these functions would facilitate performance analysis of the closedloop system, in particular for performance-based controller design. Such counterparts of the sensitivity and complementary sensitivity functions can be defined for the class of convergent systems [9], [11]. Convergent systems, even though they may be nonlinear, have the property that being excited by a bounded input, they have a unique bounded globally asymptotically stable steady-state solution (see also [8], [1], [2] for similar notions of incremental stability and contraction). The notion of convergent systems has proved to be very convenient when dealing with nonlinear

time-varying systems, such as in tracking or disturbance rejection problems [11], [15]. Moreover, the existence of a unique steady-state response allows us to define frequency response functions for convergent systems [12]. The fact that for convergent systems we can extend the sensitivity and complementary sensitivity functions was used in [4], [14] for performance evaluation and performance-based design of variable-gain controllers for optical storage drives. Yet the computation of such generalized sensitivity and complementary sensitivity functions requires extensive simulations of the closed-loop system, which is clearly unfavorable in the process of nonlinear controller design.

In this paper we present a computationally efficient method for finding upper bounds on these generalized sensitivity and complementary sensitivity functions for convergent Lur'e systems. The proposed method does not require simulations of the system, which makes it attractive for performance-based controller design. We illustrate the presented results by estimating the generalized sensitivity functions for a variable-gain controlled optical storage drive. The proposed numerical method is based on recent developments on the application of the describing function method to Lur'e systems presented in [16].

The paper is organized as follows. In Section II we discuss an extension of the sensitivity and complementary sensitivity functions to the case of nonlinear convergent systems. In Section III we provide a numerical procedure that allows one to find efficient and computationally inexpensive estimates of these functions for convergent Lur'e systems. An application of this procedure to a nonlinearly controlled optical storage drive is presented in Section IV. Finally, Section V contains conclusions.

II. PERFORMANCE ANALYSIS OF CONTROL SYSTEMS: FROM LINEAR TO NONLINEAR

Consider a linear closed-loop (motion) control system shown in Figure 1. Here \mathcal{P} is a linear plant, \mathcal{C} is a linear controller, $p \in \mathbb{R}$ is the output, $n \in \mathbb{R}$ is the measurement noise, $\tilde{p} = p + n$ is the measured output, $r \in \mathbb{R}$ is the reference signal, e = r - p is the tracking error, and $\tilde{e} = r - \tilde{p}$. Crucial performance characteristics of this closedloop system are contained in the sensitivity and complementary sensitivity functions S(s) and T(s), respectively. The sensitivity function S(s) is defined as the transfer function from the reference signal r to the error e. The complementary sensitivity function T(s) is defined as the transfer function from the measurement noise -n to the output p. The sensitivity function $|S(i\omega)|$ provides a frequency-dependent

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Fig. 1. Block diagram of a linearly controlled motion system.

characteristic of how the reference trajectory r affects the tracking error e at various frequencies. In the same way, the complementary sensitivity function $|T(i\omega)|$ allows one to establish how the measurement noise n affects the output p at various frequencies. In many applications this information is essential in the performance evaluation of (motion) control systems. While tuning controller parameters we usually want to achieve small values of $|T(i\omega)|$ in the frequency range of the measurement noise (the high frequency range), and small values of $|S(i\omega)|$ in the frequency range of the reference signals r (usually the low frequency range). Linear controller design and tuning based on these sensitivity and complementary sensitivity functions is a standard technique widely used in practice.

The definitions of the sensitivity and the complementary sensitivity functions $S(i\omega)$ and $T(i\omega)$ essentially rely on the linearity of the plant model and the controller. If the plant model or the controller contain nonlinearities, then these $S(i\omega)$ and $T(i\omega)$ cannot be defined as before without neglecting the nonlinearities. Yet it may still be possible to define an analog of $|S(i\omega)|$ and $|T(i\omega)|$. In the linear case, if the closed-loop system is excited by $n(t) = a \sin \omega t$, the corresponding steady-state output response equals $\bar{p}_{a\omega}(t) =$ $|T(i\omega)|a\sin(\omega t + \psi)$ where $\psi = -\arg(T(i\omega))$. One can easily check that $|T(i\omega)|$ can be defined as the ratio between the root mean square (rms)value of $\bar{p}_{a\omega}(t)$ and the rms value of the harmonic excitation $n(t) = a \sin \omega t$. Recall that for a τ -periodic signal y(t) the rms value is defined as $\|y\|_2 := \left(\frac{1}{\tau} \int_0^\tau |y(t)|^2 dt\right)^{\frac{1}{2}}$. In the same way, for the sensitivity function, $|S(i\omega)|$ can be defined as the ratio between the rms value of the steady-state response $\bar{e}_{a\omega}(t)$ corresponding to the excitation $r(t) = a \sin(\omega t)$ and the rms value of this r(t).

Although such a definition of $|S(i\omega)|$ and $|T(i\omega)|$ avoids dealing with transfer functions, it is still not applicable to all nonlinear systems due to the fact that a general nonlinear system can have multiple coexisting periodic and even chaotic responses to a harmonic excitation. Hence, the notion of a steady-state response is undefined for general nonlinear systems. Yet, there is a class of nonlinear systems for which such a definition is acceptable, namely the class of convergent systems. Consider a nonlinear system of the form

$$\dot{x} = f(x, v(t)),\tag{1}$$

with state $x \in \mathbb{R}^n$ and input $v(t) \in \mathbb{R}^m$. It is assumed that the inputs v(t) are piecewise-continuous functions of time defined on \mathbb{R} .

Definition 1 ([10]): System (1) with a given input v(t) is called

- convergent if
 - (i) there exists a solution $\bar{x}_v(t)$ that is defined and bounded on \mathbb{R} ,
 - (ii) $\bar{x}_v(t)$ is globally asymptotically stable,
- uniformly convergent if $\bar{x}_v(t)$ is uniformly globally asymptotically stable,
- exponentially convergent if $\bar{x}_v(t)$ is globally exponentially stable.

The system is called (uniformly, exponentially) convergent for a class of bounded piecewise-continuous inputs, if it is (uniformly, exponentially) convergent for any input v(t) from this class. The solution $\bar{x}_v(t)$ is called a *steady-state solution*. It is known that for uniformly convergent systems the steadystate solution $\bar{x}_v(t)$ is the only solution that is bounded on \mathbb{R} . Moreover, if v(t) is τ -periodic, then $\bar{x}_v(t)$ is also τ -periodic, see [10].

Now suppose a closed-loop system like the one shown in Figure 1, but with a nonlinear plant and/or controller, is modeled by

$$\dot{x} = f(x, r, n),$$

$$p = h(x), \quad e = r - p,$$
(2)

and it is uniformly convergent for the class of piecewisecontinuous bounded inputs r and n. For the inputs $n_{a\omega}(t) = a \sin \omega t$ and r(t) = 0 system (2) has a *unique* periodic steady-state solution with the corresponding response $\bar{p}_{a\omega}(t)$. For the inputs $r_{a\omega}(t) = a \sin \omega t$ and n(t) = 0 the corresponding unique steady-state response e equals $\bar{e}_{a\omega}(t)$. Then we can define the generalized sensitivity and complementary sensitivity functions as follows.

Definition 2: The functions

$$S(a,\omega) = \frac{\|\bar{e}_{a\omega}\|_2}{\|r_{a\omega}\|_2}, \quad \mathcal{T}(a,\omega) = \frac{\|\bar{p}_{a\omega}\|_2}{\|n_{a\omega}\|_2}$$
(3)

are called, respectively, the generalized sensitivity and the generalized complementary sensitivity functions of the convergent system (2).

Notice that due to nonlinearity of the system, the functions $S(a, \omega)$ and $T(a, \omega)$ depend not only on the excitation frequency, but also on the amplitude *a*. For nonlinear systems that have exponentially stable linearization at the origin (therefore they are locally convergent, see [11]), a similar gain linking rms values of the periodic excitation and the corresponding steady-state response was proposed in [6]. Yet this gain is local in the sense that it is defined only for excitations of small amplitudes.

For linear systems, due to the superposition principle, the functions $|S(i\omega)|$ and $|T(i\omega)|$ characterize frequency dependent amplification properties of the system not only for harmonic inputs, but for arbitrary inputs. For nonlinear systems the superposition principle does not hold and therefore the functions $S(a, \omega)$ and $T(a, \omega)$ only provide us with information on responses to harmonic inputs of the form $a \sin \omega t$. Yet, for performance evaluation and performancebased controller design or tuning, even this limited information is valuable. In the case of optical storage drives considered in Section IV, industrial performance tests involve testing the behavior of the closed-loop system subject to harmonic excitations at various frequencies and amplitudes, see [3].

For practical application of these generalized sensitivity and complementary sensitivity functions we need to find efficient ways to compute or estimate them. It is known that under some mild additional assumptions all steady-state solutions of a uniformly convergent system corresponding to harmonic excitations are characterized by one function, the so-called state frequency response function (FRF) [12], [11]. Although the computation of $S(a, \omega)$ and $T(a, \omega)$ can be based on this FRF, at the moment there are no general recipes on how to find it analytically or efficiently compute it numerically. A method for estimating steady-state responses to harmonic excitations was proposed in [5]. To use the theory presented in that paper we need to represent our system as a linear differential inclusion. Although this can be done for the class of systems considered in the subsequent sections, the estimates of the generalized sensitivity and complementary sensitivity functions resulting from that paper will be independent of the excitation amplitude a, which makes them very conservative. Another way of computing these functions would be to simulate system (2) excited by the corresponding harmonic input until its state converges to the steady-state solution. Knowing the steady-state solution, we can numerically compute the generalized sensitivity functions. Although we can always find $\mathcal{S}(a,\omega)$ and $\mathcal{T}(a,\omega)$ via this approach, the computational costs make it inefficient for design purposes. If one needs to compute $\mathcal{S}(a,\omega)$ and $\mathcal{T}(a,\omega)$ not only for a range of the excitation frequencies and amplitudes, but also for a range of controller parameters that need to be tuned, the corresponding computations become prohibitive. For certain classes of nonlinear systems one can try to find these sensitivity functions approximately via the describing function method. Although in some cases this method provides rather accurate results, its accuracy is not guaranteed. And in some cases this method may provide not one, but several possible approximations of the periodic output (depending on the number of solutions of the corresponding harmonic balance equation). In this case it is not clear which approximation should be used.

In the next section we provide a computational procedure that, for the case of convergent Lur'e systems, allows us to find an upper bound on the ratio between the rms value of the steady-state response of a system excited by a harmonic input and the rms value of this input. It is based on recent results on the application of the describing function method to Lur'e systems [16].

III. ANALYSIS OF HARMONICALLY EXCITED CONVERGENT LUR'E SYSTEMS

In this section we consider nonlinear Lur'e systems of the form

$$\dot{x} = Ax + B\varphi(z) + Fu$$

$$z = -Cx + Du, \quad y = C_y x + D_y u,$$
(4)

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the input and $\varphi(z)$ is a scalar nonlinearity of a scalar argument z. We assume that

A1 the matrix A is Hurwitz;

A2 the nonlinearity $\varphi(z)$ is odd and satisfies

$$0 \le \frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2} \le \mu,\tag{5}$$

for all $z_1, z_2 \in \mathbb{R}$ and some $\mu > 0$; A3 the circle criterion condition holds:

$$\operatorname{Re}(C(i\omega I - A)^{-1}B) > -\frac{1}{\mu}, \quad \forall \omega \in \mathbb{R}.$$
 (6)

It is known that under Assumptions A1-A3 system (4) is exponentially convergent for the class of bounded piecewise continuous inputs, see, e.g., [17]. Therefore, for the input $u_{a\omega}(t) = a \sin \omega t$ this system has a unique periodic steadystate solution $\bar{x}_{a\omega}(t)$ with the corresponding output $\bar{y}_{a\omega}(t)$ with period $\tau = 2\pi/\omega$. We are interested in finding an upper bound on $\|\bar{y}_{a\omega}\|_2 / \|u_{a\omega}\|_2$. Since $\|u_{a\omega}\|_2 = a/\sqrt{2}$, we only need to find an upper bound on $\|\bar{y}_{a\omega}\|_2$. This will be done in two steps. First, we will find $\bar{\nu}_{a\omega}(t)$ —an approximation of $\bar{y}_{a\omega}(t)$ based on the describing function method. It will be shown that under assumptions A1-A3 this $\bar{\nu}_{a\omega}(t)$ is uniquely defined. Second, we will find an upper bound of $\|\bar{y}_{a\omega} - \bar{\nu}_{a\omega}\|_2$. Then, since we can easily compute $\|\bar{\nu}_{a\omega}\|_2 (\bar{\nu}_{a\omega}(t)$ is sinusoidal), from the triangular inequality

$$\|\bar{y}_{a\omega}\|_{2} \le \|\bar{\nu}_{a\omega}\|_{2} + \|\bar{y}_{a\omega} - \bar{\nu}_{a\omega}\|_{2} \tag{7}$$

we will obtain an upper bound on $\|\bar{y}_{a\omega}\|_2$.

Since the analysis in the next two subsections will be done for a fixed pair of the excitation amplitude a and frequency ω , for the sake of simplicity of notations we will omit the subscripts $a\omega$ and, for example, instead of writing $\bar{y}_{a\omega}$ and $\bar{\nu}_{a\omega}$ we will write \bar{y} and $\bar{\nu}$, respectively.

A. Finding $\|\bar{\nu}\|_2$

To find $\bar{\nu}(t)$ we will approximate the nonlinear system (4) by a linear system of the form

$$\dot{\xi} = A\xi + BK\zeta + Fu$$

$$\zeta = -C\xi + Du, \quad \nu = C_y\xi + D_yu.$$
(8)

If the matrix A - BKC does not have eigenvalues on the imaginary axis, then for the harmonic input $u(t) = a \sin \omega t$ system (8) has a unique periodic solution with the corresponding output ζ equal to $\overline{\zeta}(t) = b \sin(\omega t + \psi)$ for some $b = b(a, \omega) > 0$ and some ψ . Choose the gain K to minimize the following cost functional

$$J = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} |\varphi(\bar{\zeta}(t)) - K\bar{\zeta}(t)|^2 dt.$$

The optimal gain can be derived from the condition $\frac{dJ}{dK} = 0$. This condition leads to the following expression

$$K = K(b) = \frac{2}{\pi b} \int_0^{\pi} \varphi(b\sin\theta)\sin\theta d\theta.$$
 (9)

The function K(b) is the describing function of the nonlinearity φ . Examples of its computation can be found in many textbooks, see, e.g. [7]. Next, we need to find $b = b(a, \omega)$ such that $\overline{\zeta}(t) = b\sin(\omega t + \psi)$ is the steady-state output of system (8) with K(b) given in (9). Notice that for $s = \frac{d}{dt}$ we have

$$\bar{\zeta}(t) = -C(sI_n - A)^{-1}BK(b)\bar{\zeta}(t) + (D - C(sI_n - A)^{-1}F)u(t)$$
(10)

Substituting $u(t) = a \sin \omega t$ and $\overline{\zeta}(t) = b \sin(\omega t + \psi)$, we obtain the following harmonic balance equation [16]

$$|1+K(b)C(i\omega I_n - A)^{-1}B|^2b^2 = |D-C(i\omega I_n - A)^{-1}F|^2a^2.$$
(11)

It has been shown in [16] that under assumptions A1-A3, for any $a \ge 0$ and $\omega > 0$, this harmonic balance equation with K(b) defined in (9) has a unique real nonnegative solution $b = b(a, \omega)$. The algebraic equation (11) can be solved numerically. The uniqueness of the solution of the harmonic balance equation (11), which has been addressed in [16], is essential for our procedure since it allows us to *uniquely* determine a linear approximation (8) of the nonlinear system (4) excited by a harmonic input.

Having found $b(a, \omega)$ and, therefore, $K = K(b(a, \omega))$, we can find the steady-state output $\bar{\nu}(t)$ of system (8) corresponding to the input $u(t) = a \sin \omega t$. It is known that for a nonlinearity $\varphi(z)$ satisfying the sector condition $0 \leq \varphi(z)/z \leq \mu$ (this is the case for our nonlinearity because of assumption A2), the corresponding K defined in (9) satisfies $0 \leq K \leq \mu$, see [7]. Therefore, from the circle criterion (assumptions A1 and A3) we conclude that the matrix A - BKC is Hurwitz, see [7]. Hence for the input $u(t) = a \sin \omega t$ system (8) has a unique periodic steady-state solution with the corresponding output $\bar{\nu}(t) = \alpha \sin(\omega t + \tilde{\psi})$ for some $\tilde{\psi}$ and $\alpha = a |C_y(i\omega I - (A - BKC))^{-1}(F + BKD) + D_y|$. Hence the corresponding rms value of $\bar{\nu}$ equals $\|\bar{\nu}\|_2 = \frac{a}{\sqrt{2}}|C_y(i\omega I - (A - BKC))^{-1}(F + BKD) + D_y|$,

$$\sqrt{2} \tag{12}$$

with $K = K(b(a, \omega))$.

B. Estimating $\|\bar{y} - \bar{\nu}\|_2$

To estimate the rms value of the approximation error $\sigma := \bar{y} - \bar{\nu}$, notice that the difference $\epsilon = \bar{x} - \bar{\xi}$ between the periodic steady-state solution of the nonlinear system (4) and the steady-state solution of the linear system (8) corresponding to the harmonic input $u(t) = a \sin \omega t$ is a τ -periodic ($\tau = 2\pi/\omega$) solution of the system

$$\dot{\epsilon} = A\epsilon + B(\varphi(\bar{z}) - \varphi(\zeta)) + B\Delta(t)$$

$$\sigma = C_u \epsilon$$
(13)

where $\bar{z}(t) = Du(t) - C\bar{x}(t)$, $\bar{\zeta}(t) = Du(t) - C\bar{\xi}(t)$, and $\Delta(t) = \varphi(\bar{\zeta}(t)) - K(b(a,\omega))\bar{\zeta}(t)$. Equivalently, this system can be written in the form

$$\dot{\epsilon} = \hat{A}\epsilon + B[\hat{\varphi}(\bar{\zeta},\lambda) + \Delta(t)],$$

$$\lambda = -C\epsilon, \quad \sigma = C_y\epsilon,$$
(14)

where
$$\hat{A} = A - \frac{\mu}{2}BC$$
, $\lambda = \bar{z} - \bar{\zeta}$ and
 $\hat{\varphi}(\bar{\zeta}, \lambda) = \varphi(\bar{\zeta} + \lambda) - \varphi(\bar{\zeta}) - \frac{\mu}{2}\lambda.$

We will find an upper bound on $\|\sigma\|_2$ based on the fact that $\sigma(t)$ is an output corresponding to the τ -periodic solution $\epsilon(t) = \bar{x}(t) - \bar{\xi}(t)$ of system (14).

Since the original nonlinearity φ satisfies the incremental sector condition (5), then the modified nonlinearity $\hat{\varphi}$ satisfies the condition

$$|\hat{\varphi}(\bar{\zeta},\lambda)| \le \frac{\mu}{2} |\lambda|, \quad \forall \ \bar{\zeta} \in \mathbb{R}, \quad \lambda \in \mathbb{R}.$$
(15)

To proceed further, we need to define the linear operators W_{σ} and W_{λ} that map τ -periodic inputs v into τ -periodic steady-state outputs σ and λ of the system

$$\dot{\epsilon} = A\epsilon + Bv,$$

 $\sigma = C_v \epsilon, \quad \lambda = -C\epsilon.$
(16)

Notice that due to Assumptions A1 and A3, by the circle criterion the matrix $\hat{A} = A - \frac{\mu}{2}BC$ is Hurwitz and, therefore, for any τ -periodic input v(t) system (16) has a unique τ -periodic steady-state solution. Hence, the operators W_{σ} and W_{λ} are well defined. Using Parseval equality it can be verified that for any τ -periodic input v(t) with the spectrum $\{\omega, 2\omega, \ldots\}$, the corresponding steady-state outputs $W_{\lambda}v$ and $W_{\sigma}v$ satisfy

$$||W_{\sigma}v||_{2} \leq \sup_{k=1,2,\dots} |G_{\sigma}(ik\omega)|||v||_{2},$$

$$||W_{\lambda}v||_{2} \leq \sup_{k=1,2,\dots} |G_{\lambda}(ik\omega)|||v||_{2},$$

(17)

where

$$G_{\sigma}(s) = C_y \left(sI - \left(A - \frac{\mu}{2}BC\right) \right)^{-1} B,$$

$$G_{\lambda}(s) = -C \left(sI - \left(A - \frac{\mu}{2}BC\right) \right)^{-1} B$$
(18)

are the transfer functions from v to σ and from v to λ , respectively. Denote

$$\|W_{\sigma}\| := \sup_{k=1,2,\dots} |G_{\sigma}(ik\omega)|, \quad \|W_{\lambda}\| := \sup_{k=1,2,\dots} |G_{\lambda}(ik\omega)|.$$
(19)

From (14) we conclude that τ -periodic signals $\sigma(t)$, $\overline{\zeta}(t)$, $\lambda(t)$ and $\Delta(t)$ with the period $\tau = 2\pi/\omega$ satisfy the relation

$$\sigma = W_{\sigma}\hat{\varphi}(\bar{\zeta},\lambda) + W_{\sigma}\Delta. \tag{20}$$

Taking into account the triangular inequality and (17), we conclude that

$$\|\sigma\|_{2} \leq \|W_{\sigma}\|\|\hat{\varphi}(\bar{\zeta},\lambda)\|_{2} + \|W_{\sigma}\|\|\Delta\|_{2}.$$
 (21)

From (15) we obtain $\|\hat{\varphi}(\bar{\zeta},\lambda)\|_2 \leq \frac{\mu}{2} \|\lambda\|_2$. This leads to

$$\|\sigma\|_{2} \leq \|W_{\sigma}\|\frac{\mu}{2}\|\lambda\|_{2} + \|W_{\sigma}\|\|\Delta\|_{2}.$$
 (22)

In the same way we obtain

$$\|\lambda\|_{2} \leq \|W_{\lambda}\|\frac{\mu}{2}\|\lambda\|_{2} + \|W_{\lambda}\|\|\Delta\|_{2}.$$
 (23)

Notice that the transfer function $G_{\lambda}(s)$ defined in (18) satisfies the condition

$$|G_{\lambda}(i\omega)| < \frac{2}{\mu}, \quad \forall \omega \in \mathbb{R}.$$
 (24)

It can be verified that this inequality is equivalent to the inequality (6) from assumption A3. Hence, from the definition of $||W_{\lambda}||$ (see (19)) we conclude that $||W_{\lambda}|| < \frac{2}{\mu}$. Taking this into account, from (23) we obtain

$$\|\lambda\|_{2} \leq \frac{\|W_{\lambda}\|}{1 - \|W_{\lambda}\|_{2}^{\underline{\mu}}} \|\Delta\|_{2}.$$
 (25)

After substituting this estimate into (22), we obtain

$$\|\bar{y} - \bar{\nu}\|_2 = \|\sigma\|_2 \le \frac{\|W_\sigma\|}{1 - \|W_\lambda\|_2^{\underline{\mu}}} \|\Delta\|_2.$$
(26)

The rms value of Δ can be computed numerically since it equals

$$\|\Delta\|_{2} = \left(\frac{\omega}{2\pi} \int_{0}^{\frac{2\pi}{\omega}} |\varphi(b\sin\omega t) - K(b)b\sin\omega t|^{2}dt\right)^{\frac{1}{2}}$$
$$= \left(\frac{1}{2\pi} \int_{0}^{2\pi} |\varphi(b\sin\theta) - K(b)b\sin\theta|^{2}d\theta\right)^{\frac{1}{2}},$$
(27)

where $b = b(a, \omega)$ is the solution of the harmonic balance equation (11).

C. Final statement

Uniting inequality (7), relations (12), (26), (27) and the fact that for $u_{a\omega}(t) = a \sin \omega t$ it holds that $||u_{a\omega}||_2 = a/\sqrt{2}$, we summarize the analysis from the previous subsections in the following statement.

Theorem 1: Consider system (4) satisfying assumptions A1-A3. System (4) is exponentially convergent and the steady-state output $\bar{y}_{a\omega}(t)$ corresponding to the harmonic input $u_{a\omega}(t) = a \sin \omega t$ satisfies

$$\frac{\|\bar{y}_{a\omega}\|_2}{\|u_{a\omega}\|_2} \le d(a,\omega) + g(\omega)h(a,\omega), \tag{28}$$

where

$$d(a,\omega) = |C_y(i\omega I - (A - BK(b)C))^{-1}(F + BK(b)D) + D_y|,$$
(29)

$$g(\omega) = \frac{\sup_{k=1,2,\dots} |G_{\sigma}(ik\omega)|}{1 - \frac{\mu}{2} \sup_{k=1,2,\dots} |G_{\lambda}(ik\omega)|},$$
(30)

with $G_{\sigma}(s)$ and $G_{\lambda}(s)$ defined in (18), and

$$h(a,\omega) := \frac{1}{a\sqrt{\pi}} \left(\int_0^{2\pi} |\varphi(b\sin\theta) - K(b)b\sin\theta|^2 d\theta \right)^{\frac{1}{2}}.$$
(31)

Here K(b) is the describing function defined in (9) and $b = b(a, \omega)$ is the unique real nonnegative solution of the harmonic balance equation (11).

Remark 1: In the same way as we obtained the upper bound on $\|\bar{y}_{a\omega}\|_2 / \|u_{a\omega}\|_2$, we can obtain its lower bound:

$$\frac{\|\bar{y}_{a\omega}\|_2}{\|u_{a\omega}\|_2} \ge d(a,\omega) - g(\omega)h(a,\omega),\tag{32}$$

with the same functions $d(a, \omega)$, $g(\omega)$ and $h(a, \omega)$ as in Theorem 1. Here, instead of the inequality (7) we use the inequality $\|\bar{y}_{a\omega}\|_2 \ge \|\bar{\nu}_{a\omega}\|_2 - \|\bar{y}_{a\omega} - \bar{\nu}_{a\omega}\|_2$ and apply the analysis from Subsections III-A and III-B.

IV. APPLICATION TO A NONLINEARLY CONTROLLED OPTICAL STORAGE DRIVE

In this section we apply Theorem 1 to estimate the generalized sensitivity and complementary sensitivity functions for an optical storage drive (like a CD or DVD drive) controlled by a variable gain controller. These sensitivity functions can then be used to evaluate the frequency domain performance of the closed-loop system and to tune the variable gain controller, see [4], [14] for examples of such analysis and controller design.

In optical storage drives the information is read from disc tracks by a lens in a so-called optical pick-up unit. A model for the lens dynamics in radial direction is depicted schematically in Figure 2. Herein, r represents the position of the track to be read, since the turntable with the disc is mounted on the base frame. The two-stage control strategy of the optical pick-up unit consists of a so-called long-stroke motion of a sledge containing the lens (p_{ls}) and a shortstroke motion of the lens with respect to the sledge (p_{ss}) . We primarily focus on the control of the short-stroke motion (i.e. p_{ls} is assumed fixed). The lens dynamics in the sledge are modeled by a mass-spring-damper system with mass m, stiffness k and damping b.



Fig. 2. Model of the lens dynamics in radial direction.

In Figure 3, a variable gain strategy is depicted schematically. Herein, \mathcal{P} is the linear lens and actuator dynamics given by the transfer function

$$H_{\mathcal{P}}(s) = \frac{\omega_a}{(ms^2 + bs + k)(s + \omega_a)}, \ s \in \mathbb{C}.$$

Note that the actuator dynamics are modeled using a lowpass filter, where ω_a is the breakpoint of the filter. C represents a PID controller with the transfer function

$$H_{\mathcal{C}}(s) = \frac{u(s)}{\tilde{e}(s)} = \frac{k_p \omega_{lp}^2 \left(s^2 + (\omega_d + \omega_i)s + \omega_d \omega_i\right)}{\omega_d (s^3 + 2\beta \omega_{lp} s^2 + \omega_{lp}^2 s)},$$

where ω_i is the breakpoint of the integral action, ω_d is the breakpoint of the differential action, ω_{lp} and β denote the breakpoint and the damping parameter of the low-pass filter, respectively, and k_p is a gain. The measurement noise is denoted by n and u is the control action. Since measured error e is corrupted by the measurement noise, it is denoted by \tilde{e} . The parameter values related to the lens dynamics, actuator dynamics and the control design (the latter being based on loop shaping arguments) for a typical DVD player

are $m = 7.0 \cdot 10^{-4}$ kg, $k_p = 9.0 \cdot 10^3$ N/m, $b = 2.0 \cdot 10^{-2}$ Ns/m, $\omega_i = 1.3 \cdot 10^3$ rad/s, k = 32.2 N/m, $\omega_d = 1.8 \cdot 10^3$ rad/s, $\omega_a = 1.3 \cdot 10^5$ rad/s, $\omega_{lp} = 2.8 \cdot 10^4$ rad/s and $\beta = 0.7$, see [3], [4] for related experimental validation results.

The variable gain block $\phi(\tilde{e})$ and the output of the variable gain block $\gamma(\tilde{e})$ are depicted in Figure 4. The output of the variable gain element is given by

$$\gamma(\tilde{e}) = \begin{cases} \alpha(\tilde{e} - \operatorname{sign}(\tilde{e})\delta), & |\tilde{e}| > \delta \\ 0, & |\tilde{e}| \le \delta. \end{cases}$$
(33)

It is characterized by two design parameters: the size of the deadzone δ and the additional gain ratio α . The purpose of the variable gain block is to improve the performance of the closed-loop system. In order to quantify the performance, we need to find counterparts of the sensitivity and complementary sensitivity functions as known from the linear control systems theory. To this end we will use the results from the previous section.

The closed-loop system shown in Figure 3 can be represented in state-space notations by the equation

$$\dot{x} = A_{cl}x + B\gamma(\tilde{e}) + B(r+n),$$

$$p = Cx, \quad e = r - Cx, \quad \tilde{e} = r + n - Cx,$$
(34)

with $r, n \in \mathbb{R}$, the state vector $x \in \mathbb{R}^6$, the measured radial error signal $\tilde{e} \in \mathbb{R}$ and the scalar nonlinearity $\gamma(\tilde{e})$ due to the variable-gain element. In the state x, the variables x_1, x_2 and x_3 correspond to the derivative, proportional and integral action of the PID controller, all filtered by the low-pass filter installed in series with the PID controller; x_4 denotes the force that actuates the lens mass; x_5 and x_6 represent the radial position and the radial velocity of the lens mass, respectively. In (34), $C = [0 \ 0 \ 0 \ 0 \ 1 \ 0],$

$$A_{cl} = \begin{bmatrix} -2\beta\omega_{lp} & -\omega_{lp}^2 & 0 & 0 & -k_p\omega_{lp}^2 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ \frac{\omega_a}{\omega_d} & \omega_a(1+\frac{\omega_i}{\omega_d}) & \frac{\omega_a}{\omega_i} & -\omega_a & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & \frac{1}{m} & -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$$

and $B = \begin{bmatrix} k_p \omega_{lp}^2 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. System (34) is a Lur'e system of the form (4). The matrix A_{cl} of the closedloop system is Hurwitz due to the stabilizing PID control design. The nonlinearity $\gamma(\tilde{e})$ satisfies the incremental sector condition (5) for $\mu = \alpha$. Finally, for the parameters



Fig. 3. Block diagram of a variable-gain controlled optical storage drive.

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 $\gamma(\tilde{e})$ [-] $\phi(\tilde{e})$ [-] $\frac{-\delta \circ \delta}{\tilde{e} [m]}$ $-\delta \circ \delta$ \tilde{e} [m]

Fig. 4. Variable gain $\phi(\tilde{e})$ and the output of the variable gain block $\gamma(\tilde{e})$.

of the lens dynamics and of the PID controller specified above, the circle criterion condition (6) holds provided that $\alpha \in [0, 1.82)$. Therefore, for these α and arbitrary $\delta > 0$ the closed-loop system (34) is exponentially convergent, see Section III. Therefore, the generalized sensitivity and complementary sensitivity functions $\mathcal{S}(a,\omega)$ and $\mathcal{T}(a,\omega)$ from Definition 2 are well defined. In particular, $S(a, \omega) =$ $\|\bar{e}_{a\omega}\|_2/\|r_{a\omega}\|_2$, where $\bar{e}_{a\omega}(t)$ is the periodic steady-state error response corresponding to the input $r_{a\omega}(t) = a \sin \omega t$ and n(t) = 0; $T(a, \omega) = \|\bar{p}_{a\omega}\|_2 / \|n_{a\omega}\|_2$, where $\bar{p}_{a\omega}(t)$ is the periodic steady-state position output corresponding to the input $n_{a\omega}(t) = a \sin \omega t$ and r(t) = 0. Therefore, these functions can be estimated from above using the numerical procedure resulting from Theorem 1. In this procedure, we first need to find the describing function K(b) for the nonlinearity $\gamma(\tilde{e})$. For the deadzone nonlinearity given in (33) this K(b) is given by

$$K(b) = \frac{2\alpha}{\pi} \left(\frac{\pi}{2} - \arcsin\left(\frac{\delta}{b}\right) - \frac{\delta}{b} \sqrt{1 - \left(\frac{\delta}{b}\right)^2} \right), \quad (35)$$

for $b > \delta$ and K(b) = 0 for $b \in [0, \delta]$. Then, for every a and ω from the range of interest, we find $b = b(a, \omega)$ the unique solution of the corresponding harmonic balance equation (11). This can be done numerically, for example with the MATLAB function fzero. Finally, we compute the corresponding values of $d(a, \omega)$, $g(\omega)$ and $h(a, \omega)$ from the formulas given in (29)-(31) and obtain the corresponding upper bounds on $\mathcal{S}(a,\omega)$ and $\mathcal{T}(a,\omega)$.

The upper bounds on $\mathcal{S}(a,\omega)$ and $\mathcal{T}(a,\omega)$ computed in this way as well as their actual values obtained via simulations for $\alpha = 1.5, \ \delta = 8 \cdot 10^{-8}$ and two values of the excitation amplitude a are shown in Figures 5 and 6 $(a = 1.6 \cdot 10^{-7})$ and in Figures 7 and 8 $(a = 8 \cdot 10^{-7})$. In all figures the upper bounds are tight in the regions where the system dynamics are close to linear and show deviation in the regions where the nonlinearity plays a significant role.

The obtained upper bounds on $\mathcal{S}(a,\omega)$ and $\mathcal{T}(a,\omega)$ are rather close to the real values. At the same time, finding these upper bounds is computationally far less demanding than computing $\mathcal{S}(a,\omega)$ and $\mathcal{T}(a,\omega)$ via simulations. This demonstrates the effectiveness of the approach presented in this paper and its strong potential for problems in which the generalized sensitivity and complementary sensitivity functions need to be evaluated for wide ranges of the excitation amplitudes, frequencies and/or controller parameters.



Fig. 5. Generalized sensitivity function $S(a, \omega)$ for $a = 1.6 \cdot 10^{-7}$



Fig. 6. Generalized complementarity sensitivity function $\mathcal{T}(a,\omega)$ for $a=1.6\cdot 10^{-7}$

V. CONCLUSIONS

We have presented an extension of the sensitivity and the complementary sensitivity functions for nonlinear convergent motion control systems. For a class of convergent Lur'e systems we have presented a computationally efficient numerical algorithm that allows one to estimate these generalized sensitivity functions from above. The efficiency of the proposed approach has been demonstrated by application to the industrial example of an optical storage drive controlled by a variable gain controller. The presented results can be used for performance-based nonlinear controller design for motion control systems.

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Fig. 7. Generalized sensitivity function $S(a, \omega)$ for $a = 8 \cdot 10^{-7}$



Fig. 8. Generalized complementarity sensitivity function $\mathcal{T}(a,\omega)$ for $a = a = 8 \cdot 10^{-7}$

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