

An error bound for model reduction of Lur'e-type systems

Bart Besselink, Nathan van de Wouw, Henk Nijmeijer

Abstract—In general, existing model reduction techniques for stable nonlinear systems lack a guarantee on stability of the reduced-order model, as well as an error bound. In this paper, a model reduction procedure for absolutely stable Lur'e-type systems is presented, where conditions to ensure absolute stability of the reduced-order model as well as an error bound are given. The proposed model reduction procedure exploits linear model reduction techniques for the reduction of the linear part of the Lur'e-type system. Hence, the proposed model reduction strategy is computationally attractive. Moreover, both stability and the error bound for the obtained reduced-order model hold for an entire class of nonlinearities. The results are illustrated by application to a nonlinear mechanical system.

I. INTRODUCTION

In the design of complex high-tech systems, predictive models are typically of high order. Model reduction can be used to obtain a low-order approximation of these models, allowing for efficient analysis or control design. Balanced truncation [11] is among the most popular methods for the reduction of stable linear systems, since it guarantees stability of the reduced-order model [14] and provides an error bound [4]. An alternative method for the reduction of linear systems that shares these properties is optimal Hankel norm approximation [6]. Both balanced truncation and Hankel norm approximation require the solutions of Lyapunov equations for the calculation of gramians and are therefore only applicable on models of orders up to $\mathcal{O}(1000)$. For higher-order models, efficient numerical techniques such as moment matching using Krylov subspaces [7], [1] might be used. However, these methods do not provide an error bound for the reduced-order model, nor guarantee stability.

Model reduction for nonlinear systems has received extensive attention in literature, but is less well-developed than for linear systems. An approach exploiting linear model reduction techniques in the scope of nonlinear systems is trajectory piecewise-linear (TPWL) approximation, where model reduction is performed in two steps. First, the nonlinear model is approximated as a collection of linear systems. Second, linear model reduction techniques as moment matching [16] or balancing [19] are applied to the linear subsystems to find a projection basis for the nonlinear system. However, this method does not guarantee stability of the reduced-order model, nor provides an error bound. Local stability of the reduced-order model is guaranteed by the extension of balancing to stable nonlinear systems [17],

[5]. However, computation of the reduced-order model is not straightforward, since analytical manipulations of equations are required to find a suitable coordinate transformation. This method does not provide an error bound either. A computable approximation to this method is given by balancing using empirical gramians [8], [10], where impulse responses of the full nonlinear system are calculated and analyzed. Finally, proper orthogonal decomposition (POD) [2], [15] also uses data generated by the nonlinear system to find a reduced-order model. Here, stability of the reduced-order model is not guaranteed and an error bound is not available.

Hence, these methods for model reduction of nonlinear systems have in common that they lack an error bound on the reduced-order system. Further, stability of the reduced-order model is not guaranteed (except for nonlinear balancing). In this paper, a model reduction procedure for a class of Lur'e-type systems is presented, for which conditions on stability of the reduced-order model as well as an error bound are given. Lur'e-type systems represent an important class of nonlinear systems, consisting of linear dynamics with static output-dependent nonlinearities in the feedback loop. Models of many relevant nonlinear engineering (control) systems, such as mechanical motion systems with friction, one-sided flexibilities or backlash or certain variable-gain control systems, can be cast in Lur'e-type form. The proposed model reduction procedure exploits linear model reduction techniques for the reduction of the linear part of the Lur'e-type system and is therefore computationally attractive. Further, the conditions for stability and the error bound can be used to select the order of the reduced-order model such that stability of the reduced-order model is guaranteed and the approximation error satisfies a predefined error bound.

This paper is organized as follows. The class of Lur'e-type systems will be introduced and some results on absolute stability theory will be reviewed in Section II. In Section III, a model reduction strategy for Lur'e-type systems is presented. Moreover, conditions for stability of the reduced-order model as well as an error bound are derived. These results are independent of the procedure used to reduce the linear part of the Lur'e-type system. As a relevant candidate, balanced truncation is discussed in Section IV, as well as its application to Lur'e-type systems. In Section V, the proposed model reduction strategy is applied to an example of a nonlinear mechanical system: namely a perturbed flexible beam with a one-sided flexible support. Finally, in Section VI conclusions and directions for future research will be given.

Notation: Standard notation is used throughout the paper. Given a matrix A , its transpose is denoted as A^T . A symmetric positive definite matrix is denoted as $A = A^T \succ 0$.

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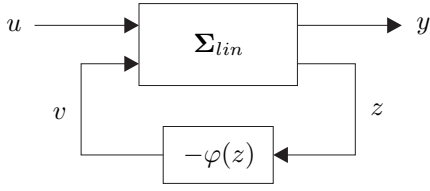


Fig. 1: Lur'e-type system.

The \mathcal{H}_∞ -norm is written as $\|\cdot\|_\infty$ and is defined by

$$\|G(s)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(i\omega)), \quad (1)$$

where $\bar{\sigma}$ denotes the largest singular value, $i = \sqrt{-1}$ and $s \in \mathbb{C}$. The \mathcal{L}_2 -norm on time signals is defined by

$$\|x(t)\|_2 = \sqrt{\int_0^\infty x^T(t)x(t) dt}, \quad (2)$$

and denoted by $\|\cdot\|_2$.

II. LUR'E-TYPE SYSTEMS

A Lur'e-type system $\Sigma = (\Sigma_{lin}, \varphi)$ consists of a linear part Σ_{lin} and a continuous static output-dependent nonlinearity $\varphi(z)$, as schematically depicted in Fig. 1. The linear dynamics is given by

$$\Sigma_{lin} : \begin{cases} \dot{x} = Ax + B_u u + B_v v, \\ y = C_y x, \\ z = C_z x, \end{cases} \quad (3)$$

which is assumed to be a minimal realization with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $v, z \in \mathbb{R}$. The corresponding matrix of transfer functions from inputs u, v to outputs y, z is given by

$$G(s) = \begin{bmatrix} G_{yu}(s) & G_{yv}(s) \\ G_{zu}(s) & G_{zv}(s) \end{bmatrix} = \begin{bmatrix} C_y \\ C_z \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} B_u & B_v \end{bmatrix}. \quad (4)$$

The linear dynamics is connected to a scalar static nonlinearity $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ in the feedback loop according to

$$v = -\varphi(z). \quad (5)$$

The nonlinearity $\varphi(z)$ is assumed to satisfy the following incremental sector condition:

$$-\mu \leq \frac{\varphi(z_2) - \varphi(z_1)}{z_2 - z_1} \leq \mu, \quad \forall z_1, z_2 \in \mathbb{R}, \quad (6)$$

with $\mu > 0$ and $\varphi(0) = 0$. For smooth nonlinearities ($\varphi(\cdot) \in C^1$), the incremental sector condition (6) implies that the derivative of $\varphi(z)$ with respect to z is bounded by μ (i.e. $|d\varphi/dz| \leq \mu$). Further, it has to be noted that Lur'e-type systems with arbitrary incremental sector condition bounds can always be written in the form (3-6) by using loop transformations [9].

Since the nonlinearity $\varphi(z)$ satisfies the incremental sector condition (6), the sector condition

$$(\varphi(z) + \mu z)(\varphi(z) - \mu z) \leq 0 \quad (7)$$

holds as well, such that conditions for stability are given by the circle criterion [9]. The system (3-5) is assumed to satisfy

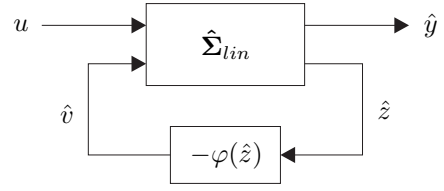


Fig. 2: Reduced-order Lur'e-type system.

the conditions for absolute stability (i.e. $x = 0$ is a globally asymptotically stable equilibrium of (3-5) for $u = 0$ and for any $\varphi(z)$ satisfying (6)), which read

Assumption 1

- 1) A is Hurwitz, and
- 2) the transfer function $G_{zv}(s)$ satisfies

$$\|G_{zv}(s)\|_\infty < \frac{1}{\mu}. \quad (8)$$

Further, since the incremental sector condition (6) is assumed to be satisfied, these conditions for stability also imply the so-called convergence property [20]. Convergence is a stability property of the system with non-zero input $u(t)$ guaranteeing that for any bounded input $u(t)$, there exists a unique, bounded on \mathbb{R} , solution that is globally asymptotically stable. In the current paper, the above conditions on the Lur'e-type systems are exploited in the scope of model reduction (guaranteeing both stability and an error bound for the reduced system).

Clearly, these conditions limit the class of Lur'e-type systems for which the reduction technique may be employed. It should be noted, however, that in many cases these conditions can be imposed by means of feedback. Clearly, absolute stability and convergence are favorable stability properties commonly desired in the scope of many control problems such as stabilization, output regulation, tracking, disturbance rejection, see e.g. [12], [13]. The proposed method is therefore relevant in the analysis of closed-loop systems, providing a low-order closed-loop model that can be simulated efficiently to assess performance of the designed controller. On the other hand, the ideas presented in this work might be extended to facilitate the design of low-order controllers with guaranteed performance bounds.

III. STABILITY GUARANTEE AND ERROR BOUND

For the reduction of Lur'e-type systems, a strategy based on linear model reduction techniques is proposed. First, the linear dynamics can be reduced using linear techniques by discarding the nonlinearity, yielding a reduced-order linear part. Second, the nonlinearity can be reconnected to the reduced-order linear dynamics to obtain a reduced-order Lur'e-type system. Here, any linear model reduction technique can be used, as long as it provides a stable reduced-order model and an error bound. In Section IV, more details will be given on the application of balanced truncation for the reduction of the linear part of the Lur'e-type system.

Before stating the main result of this paper, a reduced-order Lur'e-type system $\hat{\Sigma} = (\hat{\Sigma}_{lin}, \varphi)$ is defined which approximates the input-output behavior of the high-order system Σ . Here, the linear dynamics Σ_{lin} of the original Lur'e-type system (3) is reduced to

$$\hat{\Sigma}_{lin} : \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}_u u + \hat{B}_v \hat{v}, \\ \hat{y} = \hat{C}_y \hat{x}, \\ \hat{z} = \hat{C}_z \hat{x}, \end{cases} \quad (9)$$

with $\hat{x} \in \mathbb{R}^k$, $k < n$ and corresponding transfer function $\hat{G}(s)$, defined similar to (4). The number of inputs and outputs remains unchanged in the reduction. The original scalar nonlinearity $\varphi(z)$ (5) is reconnected to obtain the reduced-order Lur'e-type system as depicted in Fig. 2 (i.e. $\hat{v} = -\varphi(\hat{z})$). Now, the main result can be stated.

Theorem 1 *Let $\Sigma = (\Sigma_{lin}, \varphi)$ be a Lur'e-type system of the form (3-5) satisfying the incremental sector condition (6) and Assumption 1. Let $\hat{\Sigma} = (\hat{\Sigma}_{lin}, \varphi)$ be a reduced order Lur'e-type system of the same form, with $\hat{\Sigma}_{lin}$ as in (9) and \hat{A} Hurwitz. If the error bound $\|G - \hat{G}\|_\infty \leq \varepsilon$ holds for some ε , $0 < \varepsilon < \mu^{-1}$, then*

a) the reduced-order system $\hat{\Sigma}$ is absolutely stable if the original system satisfies

$$\|G_{zv}(s)\|_\infty < \frac{1}{\mu} - \varepsilon; \quad (10)$$

b) for absolutely stable $\hat{\Sigma}$, the error $\delta y(t) = y(t) - \hat{y}(t)$ is bounded as

$$\|\delta y(t)\|_2 \leq \gamma \varepsilon \|u(t)\|_2, \quad (11)$$

with

$$\gamma = \left(1 + \frac{\mu \|\hat{G}_{yv}(s)\|_\infty}{1 - \mu \|\hat{G}_{zv}(s)\|_\infty}\right) \left(1 + \frac{\mu \|G_{zu}(s)\|_\infty}{1 - \mu \|G_{zv}(s)\|_\infty}\right) \quad (12)$$

Proof: First, statement *a)* of the theorem is proven. The reduced-order Lur'e-type system is absolutely stable if \hat{A} is Hurwitz and the frequency-domain condition

$$\|\hat{G}_{zv}(s)\|_\infty < \frac{1}{\mu}, \quad (13)$$

holds. Here, stability of \hat{A} holds by assumption. Next, the error bound $\|G(s) - \hat{G}(s)\|_\infty \leq \varepsilon$ on the reduced-order linear system $\hat{\Sigma}_{lin}$ implies bounds on the individual transfer functions as well, such that $\|G_{zv}(s) - \hat{G}_{zv}(s)\|_\infty \leq \varepsilon$. This expression implies an upper bound on $\|\hat{G}_{zv}(s)\|_\infty$ as

$$\|\hat{G}_{zv}(s)\|_\infty \leq \|G_{zv}(s)\|_\infty + \varepsilon \quad (14)$$

which, together with (10), proves the validity of (13). Hence, statement *a)* is proven.

Next, statement *b)* of the theorem is proven. Here, the \mathcal{L}_2 -gain of the linear dynamics and the static nonlinearity will be analyzed and applied in combination with a contraction property of the nonlinear loop. First, Σ_{lin} is considered in

Laplace domain, where the input to the nonlinearity z can be written as

$$z(s) = G_{zu}(s)u(s) + G_{zv}(s)v(s), \quad s \in \mathbb{C}. \quad (15)$$

This linear input-output relation implies a bound on $\|z(t)\|_2$ as follows:

$$\|z(t)\|_2 \leq \|G_{zu}(s)\|_\infty \|u(t)\|_2 + \|G_{zv}(s)\|_\infty \|v(t)\|_2. \quad (16)$$

Since $\varphi(z)$ satisfies the sector condition (7), $\|v(t)\|_2$ is bounded by

$$\|v(t)\|_2 \leq \mu \|z(t)\|_2, \quad (17)$$

such that substitution of (17) in (16) gives

$$\|z(t)\|_2 \leq \frac{\|G_{zu}(s)\|_\infty}{1 - \mu \|G_{zv}(s)\|_\infty} \|u(t)\|_2. \quad (18)$$

The error variable $\delta z(t) = z(t) - \hat{z}(t)$ is introduced, which can be expressed in Laplace domain as follows:

$$\begin{aligned} \delta z(s) &= G_{zu}(s)u(s) + G_{zv}(s)v(s) \\ &\quad - \hat{G}_{zu}(s)u(s) - \hat{G}_{zv}(s)(v(s) - \delta v(s)). \end{aligned} \quad (19)$$

Here, the relation $\delta v(t) = v(t) - \hat{v}(t)$ is used. Clearly, (19) implies that $\|\delta z(t)\|_2$ is bounded as

$$\begin{aligned} \|\delta z(t)\|_2 &\leq \|G_{zu}(s) - \hat{G}_{zu}(s)\|_\infty \|u(t)\|_2 \\ &\quad + \|G_{zv}(s) - \hat{G}_{zv}(s)\|_\infty \|v(t)\|_2 \\ &\quad + \|\hat{G}_{zv}(s)\|_\infty \|\delta v(t)\|_2. \end{aligned} \quad (20)$$

By assumption $\|G_{ji}(s) - \hat{G}_{ji}(s)\|_\infty \leq \varepsilon$ with $j \in \{y, z\}$, $i \in \{u, v\}$. Further, the incremental sector condition (6) implies that the following inequality holds:

$$\|\delta v(t)\|_2 \leq \mu \|\delta z(t)\|_2, \quad (21)$$

such that (20) can be rewritten to

$$\|\delta z(t)\|_2 \leq \frac{\varepsilon \|u(t)\|_2 + \varepsilon \|v(t)\|_2}{1 - \mu \|\hat{G}_{zv}(s)\|_\infty}. \quad (22)$$

Next, using (17) and (18) to find a bound for $\|v(t)\|_2$ in terms of $\|u(t)\|_2$ gives

$$\|v(t)\|_2 \leq \frac{\mu \|G_{zu}(s)\|_\infty}{1 - \mu \|G_{zv}(s)\|_\infty} \|u(t)\|_2, \quad (23)$$

which in combination with (22) yields

$$\|\delta z(t)\|_2 \leq \frac{\varepsilon \|u(t)\|_2}{1 - \mu \|\hat{G}_{zv}(s)\|_\infty} \left(1 + \frac{\mu \|G_{zu}(s)\|_\infty}{1 - \mu \|G_{zv}(s)\|_\infty}\right). \quad (24)$$

Finally, the output error variable $\delta y(t) = y(t) - \hat{y}(t)$ is considered. In Laplace domain, the equality

$$\begin{aligned} \delta y(s) &= (G_{yu}(s) - \hat{G}_{yu}(s))u(s) \\ &\quad + (G_{yv}(s) - \hat{G}_{yv}(s))v(s) + \hat{G}_{yv}(s)\delta v(s) \end{aligned} \quad (25)$$

holds, leading to the following error bound on $\|\delta y(t)\|_2$:

$$\begin{aligned} \|\delta y(t)\|_2 &\leq \varepsilon \|u(t)\|_2 + \varepsilon \|v(t)\|_2 \\ &\quad + \|\hat{G}_{yv}(s)\|_\infty \|\delta v(t)\|_2. \end{aligned} \quad (26)$$

Here, $\|\delta v(t)\|_2$ can be bounded by using (21) and (24), which gives

$$\|\delta v(t)\|_2 \leq \frac{\mu \varepsilon \|u(t)\|_2}{1 - \mu \|\hat{G}_{zv}(s)\|_\infty} \left(1 + \frac{\mu \|G_{zu}(s)\|_\infty}{1 - \mu \|G_{zv}(s)\|_\infty} \right), \quad (27)$$

Substitution of (23) and (27) in (26) gives the bound on $\|\delta y(t)\|_2$ in terms of $\|u(t)\|_2$ as in (11-12), which proves statement b). This completes the proof. ■

Remark 1 Obviously, the error bound (12) is dependent on the size of the (incremental) sector μ . For increasing μ (i.e. increasing incremental sector for the nonlinearity), the error bound increases as well. On the other hand, the error bound decreases for decreasing μ and equals the error bound for linear model reduction for $\mu \rightarrow 0$. Hence, for $\mu = 0$, linear model reduction is recovered for the linear model with input u and outputs y and z .

Remark 2 As can be seen in (12), the gain γ in the error bound is dependent on norms of transfer functions of the reduced-order system $\|\hat{G}_{jv}\|_\infty$, $j \in \{y, z\}$. In this form, (12) provides an a posteriori error bound (i.e. after the reduction has been employed). If an a priori error bound specification is required to be met, the norms $\|\hat{G}_{jv}\|_\infty$ can be bounded as $\|\hat{G}_{jv}\|_\infty \leq \|G_{jv}\|_\infty + \varepsilon$, such that the gain on the error bound in (12) only depends on properties of the original high-order system, which is denoted by $\bar{\gamma}$.

Remark 3 The terms $1 - \mu \|G_{zv}\|_\infty$ and $1 - \mu \|\hat{G}_{zv}\|_\infty$ appear in the denominator of (12). It has to be noted that these terms are positive when the condition for absolute stability of the reduced-order system (10) is guaranteed to be satisfied, which reads $\|G_{zv}\|_\infty < \mu^{-1} - \varepsilon$ and implies $\|\hat{G}_{zv}\|_\infty < \mu^{-1}$. Consequently, the error bound is finite. Nonetheless, the error bound may be conservative.

Remark 4 Since the nonlinearity is not explicitly taken into account in the model reduction procedure, the results in Theorem 1 hold for all nonlinearities satisfying the incremental sector condition (6). Hence, the result is also applicable when the nonlinearity is not exactly known, as is relevant in many practical applications, where nonlinearities are typically hard to model and are subject to uncertainty. From this perspective, this approach is a natural application of absolute stability theory for Lur'e-type systems for the purpose of model reduction.

Remark 5 In practice, it might be useful to select ε such that the output error in (11) is bounded by a predefined gain α , i.e. $\gamma \varepsilon < \alpha$. This can be achieved by replacing γ by its a priori counterpart $\bar{\gamma} = \bar{\gamma}(\varepsilon)$ as discussed in Remark 2. Since $\gamma < \bar{\gamma}$, a reduction of the linear part with $\bar{\gamma}(\varepsilon) \varepsilon < \alpha$ gives an a priori guarantee on $\gamma \varepsilon < \alpha$.

IV. MODEL REDUCTION FOR LUR'E-TYPE SYSTEMS

The conditions for stability and the error bound as given in Theorem 1 require a stable reduced-order model of the linear dynamics Σ_{lin} of the Lur'e-type system. Further, an error bound on the linear dynamics is assumed to be known. Hence, any linear model reduction technique that provides these properties can be used to obtain the linear reduced-order model. More specifically, balanced truncation [11], [14], [4] is a good candidate, since it provides an error bound that is easy to compute. In order to illustrate the model reduction of Lur'e-type systems using balanced truncation, this model reduction technique is briefly reviewed first.

A. Balanced truncation

Associated to the minimal and stable linear system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \quad (28)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the controllability and observability gramians $P = P^T \succ 0$ and $Q = Q^T \succ 0$, which are the unique solutions of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad (29)$$

$$A^T Q + QA + C^T C = 0, \quad (30)$$

respectively. The gramians lead to the definition of the Hankel singular values σ_i as

$$\sigma_i(\Sigma) = \sqrt{\lambda_i(PQ)}, \quad \sigma_1 \geq \sigma_2 \geq \dots \sigma_n > 0, \quad (31)$$

which are system invariants (i.e. basis-independent). With $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, balancing amounts to finding a coordinate transformation $\bar{x} = Tx$ such that the transformed gramians are equal:

$$TPT^T = T^{-T}QT^{-1} = \Sigma. \quad (32)$$

In the balanced realization, the states are ordered according to their contribution to the input-output behavior, such that a reduced-order model $\bar{\Sigma}$ can be obtained by truncation, where stability of the reduced-order system is guaranteed. Truncating the balanced state to order $k < n$ yields the following error bound:

$$\|G(s) - \hat{G}(s)\|_\infty \leq 2 \sum_{i=k+1}^n \sigma_i = \varepsilon, \quad (33)$$

where $G(s)$ and $\hat{G}(s)$ denote the transfer functions of the full-order and reduced-order system, respectively.

B. Model Reduction for Lur'e-type Systems

Balanced truncation can be applied to Lur'e-type systems by combining the inputs u and v and the outputs y and z , yielding the input matrix $B = [B_u \ B_v]$ and output matrix $C = [C_y^T \ C_z^T]^T$. Reconnecting the reduced-order linear part to the original nonlinearity (5) yields a reduced-order Lur'e-type system, as depicted in Fig. 2. For this system, conditions for stability and an error bound are given by Theorem 1, where ε can be computed according to (33).

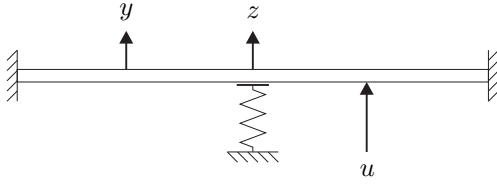
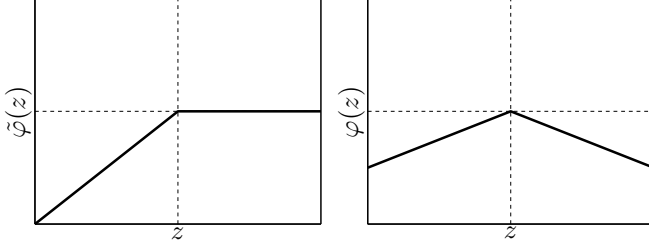


Fig. 3: Flexible beam system with a one-sided support.

Fig. 4: Original nonlinearity $\tilde{\varphi}(z)$ (left) and nonlinearity $\varphi(z)$ after loop transformation (right).

Since linear model reduction techniques are used to find the reduced-order Lur'e-type system, the proposed method is computationally attractive and does not require simulations of the full-order model. Further, balanced truncation allows direct control over the reduction error ε as in (33) of the linear dynamics by selecting the order k of the reduced model. Then, ε can be chosen to ensure stability of the reduced-order model by using (10) and/or ε can be chosen in order to meet a pre-specified error bound for the reduced-order Lur'e-type system in (11-12).

V. ILLUSTRATIVE EXAMPLE

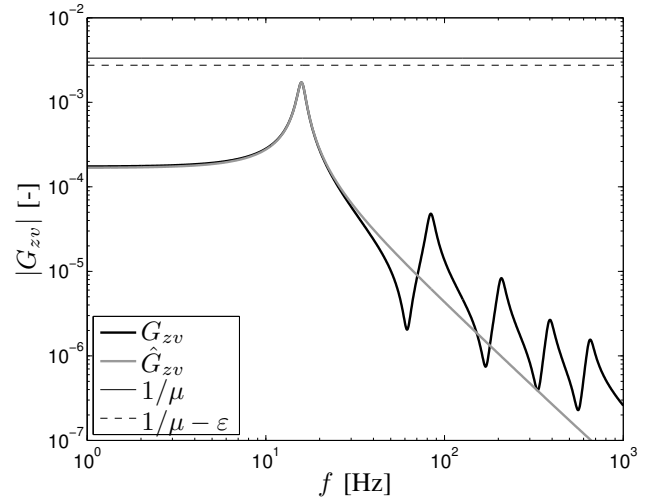
To illustrate the model reduction procedure for Lur'e-type systems, an example of a flexible beam with a one-sided flexible support as in [18] is considered, see Fig. 3. The beam, which is modeled using Euler beam elements, yields a high-order linear model with $x \in \mathbb{R}^{40}$ and a Hurwitz system matrix \tilde{A} . Here, it is assumed that the error resulting from spatial discretization is small compared to the error introduced by model reduction of the discretized model. The input $u \in \mathbb{R}$ is a force on the beam, whereas the output $y \in \mathbb{R}$ is a vertical displacement of a point on the beam, as in Fig. 3. In the center, the beam is supported by a one-sided spring, whose force \tilde{v} as a function of the vertical displacement z of the center of the beam is given by

$$-\tilde{v} = \tilde{\varphi}(z) = \begin{cases} k_{nl}z, & z < 0 \\ 0, & z \geq 0, \end{cases} \quad (34)$$

as is schematically depicted in the left graph of Fig. 4. Here, k_{nl} is the stiffness of the one-sided spring. Even though the system is of the form (3-5), a loop transformation is performed to minimize the sector μ . Thereto, the nonlinearity is written as

$$\varphi(z) = \tilde{\varphi}(z) - \frac{1}{2}k_{nl}z = \begin{cases} \frac{1}{2}k_{nl}z, & z < 0, \\ -\frac{1}{2}k_{nl}z, & z \geq 0, \end{cases} \quad (35)$$

as is schematically depicted in the right graph of Fig. 4. Accordingly, the linear dynamics is transformed as $A = \tilde{A} -$

Fig. 5: Frequency response function G_{zv} for original and reduced-order beam model and bounds on stability for $k_{nl} = 600$ N/m.TABLE I: Error bounds γ and $\bar{\gamma}$ for varying stiffness k_{nl} .

k_{nl}	γ	$\bar{\gamma}$
600	2.67	4.19
800	5.15	19.15
1000	17.78	-

$\frac{1}{2}k_{nl}B_vC_z$, yielding a Lur'e-type model of the form (3-6) with $\mu = \frac{1}{2}k_{nl}$.

For $k_{nl} = 600$ N/m, the balancing procedure of Section IV is applied to the system to obtain a reduced order Lur'e-type system with $\hat{x} \in \mathbb{R}^2$, which yields an error bound $\varepsilon = 5.84 \cdot 10^{-4}$. Figure 5 shows the magnitude of the transfer function G_{zv} as well as the line μ^{-1} , indicating that the full-order model is absolutely stable. Further, since $\mu^{-1} - \varepsilon > \|G_{zv}\|_\infty$, absolute stability of the reduced-order model is guaranteed by Theorem 1, where it is noted that the balancing procedure guarantees stability of the linear dynamics of the reduced-order model. Absolute stability is confirmed by the observation that the absolute value of the reduced-order frequency response function \hat{G}_{zv} is under the line μ^{-1} .

The error bound γ as given in (12) is shown in Table I for different stiffness values of the one-sided spring, where it is recalled that $\mu = \frac{1}{2}k_{nl}$. Next, an error bound $\bar{\gamma}$ is shown, where the terms $\|\hat{G}_{jv}\|_\infty$, $j \in \{y, z\}$ in (12) are replaced by $\|G_{jv}\|_\infty + \varepsilon$, yielding an a priori error bound dependent on the properties of the high-order system only. Obviously, γ gives a tighter bound than $\bar{\gamma}$. Further, the error bound increases for increasing nonlinearity since the denominator terms in (12) approach 0 for increasing μ . Finally, for $k_{nl} = 1000$ N/m, stability of the reduced-order system can not be guaranteed a priori such that the error bound $\bar{\gamma}$ is meaningless.

It has to be noted that the total error bound is determined by both ε and γ (see (11)), where γ can be considered an additional uncertainty on the error bound caused by the feedback loop containing the static nonlinearity.

In the left graph of Fig. 6, the output y of the reduced-order

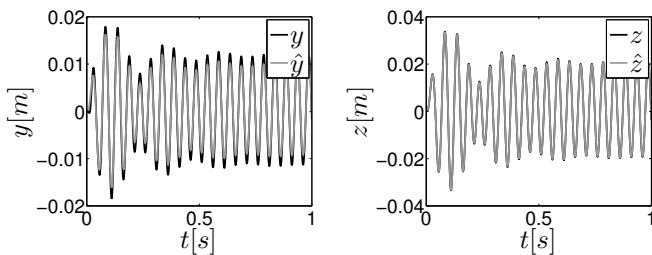


Fig. 6: Output y (left) and nonlinearity input z (right) for $k_{nl} = 600$ N/m and input $u = 100 \sin(2\pi 20t)$.

and original system are compared for zero initial condition and a sinusoidal input signal, ensuring that the nonlinearity is encountered. Clearly, the output of the reduced-order model matches that of the full-order system closely. The right graph of Fig. 6, which depicts the input to the nonlinearity $\varphi(z)$, also shows a good match, indicating that the nonlinearity similarly influences the dynamics of the reduced-order model and original system.

It is noted that for higher values of the one-sided stiffness the satisfaction of the conditions in Theorem 1 can still be guaranteed by means of feedback, see e.g. [3], where it is argued that the satisfaction of such conditions is desirable from a control perspective. Consequently, the proposed model reduction technique may also be fruitfully employed in the context of controlled Lur'e-type systems.

VI. CONCLUSIONS

A model reduction procedure for absolutely stable Lur'e-type systems is presented, where conditions for stability of the reduced-order model as well as an error bound are given. Since linear model reduction techniques are used for the reduction of the linear part of the Lur'e-type system, the approach is computationally attractive. Although the requirement of absolute stability (with an incremental sector condition) limits the class of nonlinear systems to which this model reduction procedure can be applied, it generally is a desirable property in control systems which can be enforced by feedback.

Hence, future research may focus on the application of the obtained results for designing low-order controllers for Lur'e-type systems.

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