

# Extremum-seeking control for periodic steady-state response optimization

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**Abstract**—Extremum-seeking control is a powerful adaptive technique to optimize steady-state system performance. To this date, extremum-seeking control has mainly been used to optimize plants with constant steady-state outputs, whereas the case in which the steady-state outputs are time varying, has received less attention. We propose an extremum-seeking scheme for the optimization of nonlinear plants with periodic steady-state outputs. Extremum-seeking control in this setting is relevant in e.g. the scope of tracking and disturbance rejection problems. We show that under certain assumptions the proposed extremum-seeking controller design guarantees that for an arbitrarily large set of initial conditions the steady-state performance of the plant converges arbitrarily close to its optimal value.

## I. INTRODUCTION

*Extremum-seeking control* is an adaptive control approach that optimizes a performance measure in terms of the steady-state output of a stable or stabilized plant in real time by automated tuning of the system parameters. In many applications, only limited knowledge of the plant dynamics is available and, hence, the steady-state output of the plant (as a function of system parameters) is not analytically known to the designer, but the output can only be measured. The purpose of an extremum-seeking controller is to drive the system parameters to their optimizing values, using only output measurements of the plant. Because extremum-seeking control is model free, it can be applied to many different engineering domains, see e.g. [1]–[4].

Although extremum-seeking control has been used for many decades, it was not until the last decade that *local stability* and *semi-global practical stability* for an extremum-seeking scheme with a general nonlinear plant were demonstrated by Krstić and Wang [5] and Teel and Popović [6], and Tan et al. [7], respectively. In the majority of the works on extremum seeking, the steady-state output of the plant is assumed to be *constant*, see e.g. [5]–[10]. However, in many cases the performance of engineering systems is related to time-varying behavior (think e.g. of tracking or disturbance rejection problems). Examples are repetitive motion tasks in high-tech motion systems, such as e.g. wafer scanners [11],

and the control of sawtooth instabilities in fusion tokamak plasmas [12].

Wang and Krstić [13] designed an extremum-seeking controller to minimize the amplitude of a sinusoidal steady-state output using a detector. The detector contains high-pass and low-pass filters to extract the amplitude of the sinusoidal output. The plant and the amplitude detector can be regarded as one system with the system parameters as input and the detected amplitude as output (which is constant in steady state). Hence, the same extremum-seeking method as for the optimization of plants with constant steady-state outputs can be used to minimize the detected amplitude. This method was applied to instability control of a gas-turbine combustor and a subsonic cavity flow [14], [15]. A similar method was applied for mode matching in vibrating gyroscopes in [16]. Note that the results in [13] are tailored to sinusoidal outputs, whereas in the current paper we develop a more general framework for performance optimization of arbitrary periodic outputs.

Guay et al. [17] developed an extremum-seeking control scheme for the steady-state output optimization of a class of differentially flat periodic nonlinear plants. Flatness is exploited to compute one period of the steady-state output of a plant. Extremum-seeking control is used to optimize the computed output in real time. While the computed steady-state output is optimized, asymptotically stable error dynamics ensure that the actual output of the plant converges to the computed steady-state output and, as a consequence, steady-state performance is optimized. This method requires explicit model knowledge on the relation between the parameters and the steady-state output of the plant, which constrains its applicability to typical extremum seeking problems where the model is unknown. A similar approach is used in [18] for the steady-state output optimization of a class of periodic Hamiltonian systems.

The contribution of this paper is as follows. First, we propose an extremum-seeking control method for steady-state performance optimization of general nonlinear plants with arbitrary periodic steady-state outputs without requiring explicit knowledge of the relation between the parameters and the steady-state output of the plant. Second, we present a novel extremum-seeking controller with moving average filter, which leads to an improved performance. Third, we prove the semi-global practical asymptotic stability of the performance-optimal solution.

## II. PRELIMINARIES

The sets of real numbers and natural numbers (nonnegative integers) are denoted as  $\mathbb{R}$  and  $\mathbb{N}$ , respectively. The sets of

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real numbers larger than zero and larger than or equal to zero are given by  $\mathbb{R}_{>0}$  and  $\mathbb{R}_{\geq 0}$ , respectively. For the vectors  $x, y \in \mathbb{R}^n$ , we use the relations  $>$ ,  $<$ ,  $\geq$ ,  $\leq$  and  $=$  in the elementwise sense, i.e.,  $x > y$  denotes that  $x_i > y_i$  for all  $i \in \{1, 2, \dots, n\}$ . Given two functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ , by  $f \circ g(\cdot)$  we denote  $f(g(\cdot))$ . A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to belong to the class  $\mathcal{K}$  ( $\alpha \in \mathcal{K}$ ) if it is continuous, zero at zero, and strictly increasing. It is said to belong to the class  $\mathcal{K}_\infty$  if it is of class  $\mathcal{K}$  and unbounded (that is,  $\alpha(r) \rightarrow \infty$  as  $r \rightarrow \infty$ ). The following notation is adopted from [19], [20]. Let  $t_d$  be a nonnegative real number. Given a function  $q : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $t \in \mathbb{R}$ , we define  $q_d(t)(\cdot)$  such that  $q_d(t)(\tau) := q(t + \tau)$  for all  $\tau \in [-t_d, 0]$ . We say that  $q_d(t) \in \mathcal{C}([-t_d, 0]; \mathbb{R}^n)$ , where  $\mathcal{C}$  is the Banach space of continuous functions mapping the interval  $[-t_d, 0]$  to  $\mathbb{R}^n$ . Note that  $q_d(t)$  can be considered as the trajectory of  $q$  for the interval  $[t - t_d, t]$ . We define (when it makes sense)  $|q_d(t)| := \max_{s \in [t - t_d, t]} |q(s)|$ , where  $|\cdot|$  denotes the Euclidean norm.

We consider a parameterized family of  $N \in \mathbb{N}$  interconnected systems:

$$\begin{aligned} \dot{x}_i &= f_i(t, x_{1d}, x_{2d}, \dots, x_{nd}, \epsilon), \quad t \geq 0, \\ x_i(t) &= \xi_i(t), \quad t \in [-t_d, 0], \end{aligned} \quad (1)$$

with states  $x_i \in \mathbb{R}^{v_i}$ ,  $v_i \in \mathbb{N}$ , for all  $i \in \{1, 2, \dots, N\}$  and parameter vector  $\epsilon \in \mathbb{R}_{>0}^k$ . The solutions  $x_i(t)$  satisfy the initial conditions  $\xi_i(t)$  for  $t \in [-t_d, 0]$ . For  $x(t) := [x_1^T(t), x_2^T(t), \dots, x_N^T(t)]^T$ , we denote  $x^+(t) := [|x_1(t)|, |x_2(t)|, \dots, |x_N(t)|]^T$ . We state the following definition of *semi-global practical asymptotic stability* (SGPAS) for systems of the form (1).

### Definition 1

The interconnected system in (1) with parameter vector  $\epsilon := [\epsilon_1, \epsilon_2, \dots, \epsilon_k]^T$  is said to be semi-globally practically asymptotically stable (SGPAS), if for any  $\rho^0, \nu \in \mathbb{R}_{>0}^N$  the following holds. There exists an  $\epsilon_1^* \in \mathbb{R}_{>0}$ , such that, for the sequence  $j = 2, 3, \dots, k$ , for all  $\epsilon_{j-1} \in (0, \epsilon_{j-1}^*)$  there exist  $\epsilon_j^* = \epsilon_j^*(\epsilon_1, \epsilon_2, \dots, \epsilon_{j-1}) \in \mathbb{R}_{>0}$ , such that for any  $\epsilon_l \in (0, \epsilon_l^*)$ ,  $\forall l \in \{1, 2, \dots, k\}$  and for all  $x_d^+(0) \leq \rho^0$  the solutions  $x_i(t)$ ,  $\forall i \in \{1, 2, \dots, N\}$ , of (1) are well-defined for all  $t \geq 0$  and satisfy the following properties:

- 1) *uniform boundedness*:  $\sup_{t \geq 0} x^+(t) \leq C$ ;
- 2) *convergence*:  $\limsup_{t \rightarrow \infty} x^+(t) \leq \nu$ ,

where  $C = C(\rho^0, \epsilon) \in \mathbb{R}_{>0}^N$  is a constant vector.

### III. EXTREMUM-SEEKING PROBLEM FOR PERIODIC STEADY STATES

In this section, we formulate the extremum-seeking control problem for periodic steady-states. Consider a nonlinear plant of the following form:

$$\begin{aligned} \dot{x} &= f(x, u, \theta, w(t)), \\ y &= h(x, w(t)), \end{aligned} \quad (2)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}$  are respectively the state, the control input and the output, where  $w(t) \in \mathbb{R}^l$  are input disturbances, and where  $\theta \in \mathbb{R}$  is a scalar parameter. The function  $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^l \rightarrow \mathbb{R}^n$  is twice

continuously differentiable in  $x$ ,  $u$  and  $\theta$ , and continuous in  $w(t)$ . The function  $h : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}$  is twice continuously differentiable in  $x$  and continuous in  $w(t)$ . The disturbances  $w(t)$  correspond to the solution of an *exosystem*:

$$\dot{w} = \varphi(w), \quad (3)$$

where  $\varphi : \mathbb{R}^l \rightarrow \mathbb{R}^l$  is such that the exosystem (3) exhibits the existence and uniqueness of solutions and the continuous dependence of solutions on initial conditions (in backward and forward time). Moreover, we adopt the following assumptions on the exosystem.

### Assumption 1

( [21] ) All solutions of the system (3) are defined for all  $t \in \mathbb{R}$  and for every  $\rho^{w_0} \in \mathbb{R}_{>0}$  there exists a  $\rho^w \in \mathbb{R}_{>0}$  such that  $|w(0)| \leq \rho^{w_0} \Rightarrow |w(t)| \leq \rho^w$ , for all  $t \in \mathbb{R}$ .

### Assumption 2

For any initial condition  $w(0)$ , the solution of the system (3) is periodic with a known constant period  $T_w \in \mathbb{R}_{>0}$ , yielding  $w(t + T_w) \equiv w(t)$  for all  $t \in \mathbb{R}$ .

Consider a state-feedback controller of the following form:

$$u = \alpha(x, \theta), \quad (4)$$

where  $\alpha : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^m$  is twice continuously differentiable in  $x$  and  $\theta$ . We assume that we can find a stabilizing controller (4) such that the following assumption holds.

### Assumption 3

For all fixed  $\theta \in \mathbb{R}$ , there exists a unique, bounded on  $\mathbb{R}$ , uniformly globally asymptotically stable (UGAS) steady-state solution  $\bar{x}_{\theta, w}(t)$  of the stabilized plant in (2), (4). Moreover, there exists a map  $M : \mathbb{R} \times \mathbb{R}^l \rightarrow \mathbb{R}^n$ , twice continuously differentiable in  $\theta$  and continuous in  $w(t)$ , such that

$$\bar{x}_{\theta, w}(t) = M(\theta, w(t)), \quad (5)$$

for fixed values of  $\theta \in \mathbb{R}$  and all  $t \in \mathbb{R}$ . In addition, there exists functions  $\alpha_{x1}, \alpha_{x2} \in \mathcal{K}_\infty$ ,  $\alpha_f \in \mathcal{K}$  and a (smooth) Lyapunov function  $V_x(\tilde{x})$  such that

$$\alpha_{x1}(|\tilde{x}|) \leq V_x(\tilde{x}) \leq \alpha_{x2}(|\tilde{x}|), \quad (6)$$

$$\frac{dV_x}{d\tilde{x}} \tilde{f}(\tilde{x}, M(\theta, w(t)), \theta, w(t)) \leq -\alpha_f(|\tilde{x}|), \quad (7)$$

for all fixed  $\theta \in \mathbb{R}$  and all  $t \geq 0$ , where  $\tilde{x} := x - M(\theta, w(t))$  is the difference between the state  $x$  and the steady-state solution for fixed  $\theta$  given by the map  $M$ , and where  $\tilde{f}(\tilde{x}, M, \theta, w(t)) := f(\tilde{x} + M, \alpha(\tilde{x} + M, \theta), \theta, w(t)) - f(M, \alpha(M, \theta), \theta, w(t))$ .

### Remark 1

The question may rise for which classes of systems such a unique UGAS steady-state solution indeed exists. Suppose that  $\theta \in \mathbb{R}$  is fixed, that Assumption 1 holds and that the stabilized plant in (2), (4) is uniformly convergent, see [21], [22] for a definition of uniform convergence. Under these conditions, there exists a steady-state map  $M$  as in (5) and  $\bar{x}_{\theta, w}(t)$  is UGAS, see [21, Theorem 2]. For some classes

of systems (2) it is possible to design a controller of the form (4) such that the stabilized plant in (2),(4) is uniformly convergent, see [22].

**Remark 2**

Assumption 3 can be seen as the counterpart of the assumption on a unique globally asymptotically stable equilibrium point in the scope of extremum-seeking control in an equilibrium setting, see e.g. [7], or the assumption of a stable limit cycle in the case of limit-cycle minimization in [13].

Next, we introduce a useful property of  $\bar{x}_{\theta,w}(t)$ .

**Property 1**

Suppose that Assumptions 2 and 3 hold and  $\theta \in \mathbb{R}$  is fixed. Then, the corresponding steady-state solution  $\bar{x}_{\theta,w}(t)$  is periodic with period  $T_w \in \mathbb{R}_{>0}$ .

*Proof:* Using the uniqueness and UGAS properties of the steady-state solution in Assumption 3, the proof of the property follows similar steps as the proof of [22, Property 2.23]. ■

Note that from (5) and Property 1, it follows that the output  $\bar{x}_{\theta,w}(t)$  of the map  $M(\theta, w(t))$  is  $T_w$ -periodic if Assumptions 2 and 3 hold and  $\theta \in \mathbb{R}$  is fixed. In addition, from (2) we conclude that if  $\bar{x}_{\theta,w}(t) = M(\theta, w(t))$  and  $w(t)$  are  $T_w$ -periodic, the steady-state output  $y(t) = \bar{y}_{\theta,w}(t)$  of the plant is also  $T_w$ -periodic. Hence, if Assumptions 2 and 3 hold, the steady-state output  $\bar{y}_{\theta,w}(t)$  is  $T_w$ -periodic for fixed  $\theta$ .

We aim to find the (fixed) value of  $\theta \in \mathbb{R}$  that optimizes, in a certain sense, the steady-state output  $\bar{y}_{\theta,w}(t)$  of the stabilized plant in (2),(4). In order to do so, we design a cost function that evaluates the steady-state output of the stabilized plant in (2),(4) for different values of  $\theta \in \mathbb{R}$  and introduce the following performance measures:

$$\begin{aligned} L_p(y_d(t)) &:= \left( \frac{1}{T_w} \int_{t-T_w}^t |y(\tau)|^p d\tau \right)^{\frac{1}{p}} \\ &= \left( \frac{1}{T_w} \int_{-T_w}^0 |y_d(t)(\tau)|^p d\tau \right)^{\frac{1}{p}}, \\ L_\infty(y_d(t)) &:= \max_{\tau \in [t-T_w, t]} |y(\tau)| = \max_{\tau \in [-T_w, 0]} |y_d(t)(\tau)|, \end{aligned} \quad (8)$$

with  $p \in [1, \infty)$ . Note that  $L_p$  and  $L_\infty$  in (8) are respectively the  $\mathcal{L}^p$ -norm and the  $\mathcal{L}^\infty$ -norm for the time interval  $[t - T_w, t]$ . Furthermore, note that  $L_p$  contains a distributed delay and that  $L_\infty$  contains a time-dependent delay, namely the value of  $\tau$  for which the maximum is attained generally changes with time  $t$ . We use one of the performance measures in (8) in the design of the following cost function:

$$Q_i(y_d(t)) := g \circ L_i(y_d(t)), \quad i \in [1, \infty], \quad (9)$$

where  $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is a twice continuously differentiable function chosen by the designer. We say that the steady-state performance of the stabilized plant in (2),(4) is optimized if the steady-state output of the cost  $Q_i$  in (9) is maximized.

The value of the cost function in (9) will be referred to as the performance of the stabilized plant in (2),(4) and is denoted by  $q \in \mathbb{R}$ , i.e.,  $q(t) = Q_i(y_d(t))$  with  $i \in [1, \infty]$ .

**Remark 3**

$L_i(y_d(t))$  is constant if  $y(t)$  is  $T_w$ -periodic. Hence, assuming that Assumptions 2 and 3 hold and  $\theta$  is fixed, from (2), (9) and Property 1, it follows that the steady-state performance  $q = \bar{q}_{\theta,w}$  is constant for each fixed value of  $\theta \in \mathbb{R}$ .

The stabilized plant in (2),(4) and the cost function in (9) can be considered as one lumped plant with the system parameter  $\theta$  and the disturbances  $w(t)$  as input and the performance  $q$  of the plant as output. Combining the equations of the stabilized plant in (2),(4) and the cost function in (9), we have

$$\begin{aligned} \dot{x} &= f(x, \alpha(x, \theta), \theta, w(t)), \\ q &= J_i(x_d, w_d(t)) \end{aligned} \quad (10)$$

with

$$J_i(x_d, w_d(t)) := Q_i \circ h(x_d, w_d(t)) = g \circ L_i \circ h(x_d, w_d(t)) \quad (11)$$

with  $i \in [1, \infty]$ . Herein, for the sake of simplicity we adopted the notation  $y_d = h(x_d, w_d(t))$ . We will refer to  $J_i$  in (11) as the performance function. Considering fixed  $\theta$ , we have that the steady-state solution of the stabilized plant in (2),(4) satisfies  $\bar{x}_{\theta,w}(t) = M(\theta, w(t))$ , see Assumption 3. By adopting Assumptions 2 and 3, we have that the steady-state performance  $\bar{q}_{\theta,w}$  of the stabilized plant in (2),(4) is constant for all fixed  $\theta \in \mathbb{R}$ , see Remark 3. By replacing  $x$  in (11) with the map  $M(\theta, w(t))$  in (5), we obtain that the relation between fixed values of the parameter  $\theta$  and the steady-state performance  $\bar{q}_{\theta,w}$  is given by the following static map:

$$\begin{aligned} J_{sta,p}(\theta) &:= g \circ \left( \frac{1}{T_w} \int_0^{T_w} |h(M(\theta, w(\tau)), w(\tau))|^p d\tau \right)^{\frac{1}{p}}, \\ J_{sta,\infty}(\theta) &:= g \circ \left( \max_{\tau \in [0, T_w]} |h(M(\theta, w(\tau)), w(\tau))| \right). \end{aligned} \quad (12)$$

with  $p \in [1, \infty)$ , where we used the definition of  $L_i$  in (8) and the periodicity of  $w(t)$  and  $M(\theta, w(t))$  to obtain (12).

We assume that the output function  $h$  in (2), the map  $M$  in (5) and/or the input  $w(t)$  are unknown<sup>1</sup> to the designer. Note that this implies that the static map  $J_{sta,i}$  in (12) is also unknown. Nonetheless, we adopt the following assumption on the existence of a unique maximum of  $J_{sta,i}$ .

**Assumption 4**

Consider some  $i \in [1, \infty]$ . It is assumed that the static map  $J_{sta,i}$  in (12) and its first two derivatives with respect to  $\theta$  are continuous and bounded on compact sets of  $\theta$ . Moreover, it is

<sup>1</sup>Note that the period  $T_w$  of the unknown input  $w(t)$  is assumed to be known, since  $w(t)$  satisfies Assumption 2.

assumed that there exists a function  $\alpha_J \in \mathcal{K}$  and a constant  $\theta^* \in \mathbb{R}$ , such that

$$\frac{dJ_{sta,i}}{d\theta}(\theta)[\theta - \theta^*] \leq -\alpha_J(|\theta - \theta^*|), \quad (13)$$

for all  $\theta \in \mathbb{R}$ . In other words, for  $\theta = \theta^*$  the map  $J_{sta,i}$  achieves a unique maximum in  $\mathbb{R}$ .

Note that the static map  $J_{sta,i}$  in (12) relates the (fixed) system parameter  $\theta$  to the steady-state performance  $\bar{q}_{\theta,w}$  of the stabilized plant in (2),(4). Consequently, by finding the maximum of the map  $J_{sta,i}$  at  $\theta = \theta^*$ , we find the value of  $\theta \in \mathbb{R}$  that maximizes the steady-state performance of the stabilized plant. Because the cost function is chosen such that the steady-state output  $\bar{y}_{\theta,w}(t)$  of the stabilized plant is optimized if its performance  $\bar{q}_{\theta,w}$  is maximized, we obtain that the steady-state output  $\bar{y}_{\theta,w}(t)$  is optimized if  $\theta = \theta^*$ . Hence, we can rephrase the objective of finding the value of  $\theta \in \mathbb{R}$  that optimizes the steady-state output  $\bar{y}_{\theta,w}(t)$  by finding the value of  $\theta \in \mathbb{R}$  that corresponds to the maximum of the static map  $J_{sta,i}$ , i.e., by finding  $\theta = \theta^*$ .

In the next section, we introduce an extremum-seeking controller that drives the system parameter  $\theta \in \mathbb{R}$  towards its output-optimizing value  $\theta^*$  using an estimated gradient of the static map  $J_{sta,i}$ . Moreover, it will be shown in Section V that  $\theta$  converges arbitrarily close to its optimal value  $\theta^*$  by proper tuning of the extremum-seeking controller proposed in Section IV.

#### IV. EXTREMUM-SEEKING CONTROLLER DESIGN

Consider the extremum-seeking scheme in Fig. 1. In the spirit of [23], the extremum-seeking scheme consists of the lumped plant in (10) and an extremum-seeking controller consisting of a derivative estimator and an optimizer. The perturbation-based derivative estimator produces an estimate of the gradient of the static map  $J_{sta,i}$  in (12). Subsequently, this estimated gradient is used by the optimizer to steer the nominal value  $\hat{\theta}$  of the parameter  $\theta$  towards the output-optimizing value  $\theta^*$ , with  $\theta = \hat{\theta} + a \sin(\omega t)$ , where  $a \sin(\omega t)$  is the sinusoidal perturbation used by the derivative estimator. The optimizer is given by

$$\dot{\hat{\theta}} = Ke, \quad (14)$$

where  $e$  is the estimate of the gradient  $\frac{dJ_{sta,i}}{d\theta}(\hat{\theta})$ . Here, we propose a novel gradient estimator based on a moving average filter of the following form:

$$e = \frac{\omega}{a\pi} \int_{t-\frac{2\pi}{\omega}}^t q(\tau) \sin(\omega[\tau - \phi]) d\tau. \quad (15)$$

Using  $\theta = \hat{\theta} + a \sin(\omega t)$ , (10), (14) and (15), the closed-loop dynamics are given by

$$\begin{aligned} \dot{x} &= f(x, \alpha(x, \hat{\theta} + a \sin(\omega t)), \hat{\theta} + a \sin(\omega t), w(t)), \\ \dot{\hat{\theta}} &= \frac{\omega K}{a\pi} \int_{t-\frac{2\pi}{\omega}}^t J_i(x_d(\tau), w_d(\tau)) \sin(\omega[\tau - \phi]) d\tau, \end{aligned} \quad (16)$$

with  $i \in [1, \infty]$  and  $\hat{\theta} \in \mathbb{R}$ , where  $a, \omega, K \in \mathbb{R}_{>0}$  are controller parameters and  $\phi \in \mathbb{R}_{\geq 0}$  is a constant. Note

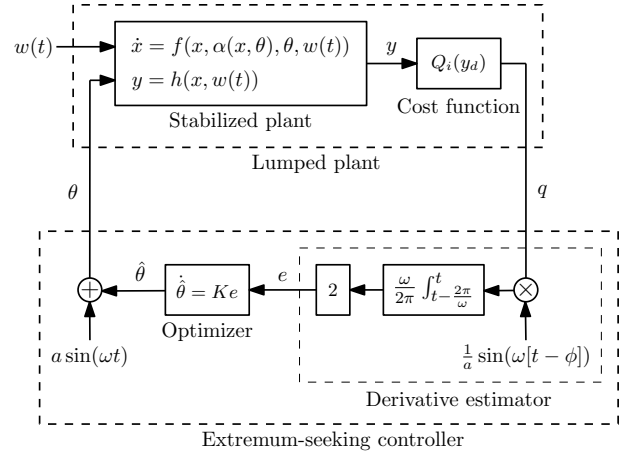


Fig. 1: Extremum-seeking scheme in (16).

that we have  $x_d(t)(\tau) := x(t + \tau)$  for all  $\tau \in [-t_d, 0]$ , where the maximal delay of the extremum-seeking scheme is  $t_d = T_w + \frac{2\pi}{\omega}$ ; the delay  $T_w$  is introduced by the performance measure in (8) while the delay  $\frac{2\pi}{\omega}$  is introduced by the moving average filter in (15).

#### Remark 4

The extremum-seeking controller in Fig. 1 is comparable to the "first-order" extremum-seeking controller in [7]. However, unlike the "first-order" extremum-seeking controller in [7], the extremum-seeking controller in Fig. 1 contains a moving-average filter. The moving-average filter is an alternative for the "higher-order" extremum-seeking controllers with low-pass filter and/or high-pass filter as in [5], [7]. Note that due to the perturbation signal  $a \sin(\omega t)$  the performance  $q$  of the plant contains periodic oscillations with frequency  $\frac{\omega}{2\pi}$  (and higher-order harmonics), see e.g. [23]. Contrary to the use of low-pass and/or high-pass filters, a moving average filter filters out these oscillations completely, which results in a more accurate gradient estimate  $e$ . A second difference is that the sinusoidal perturbation signals  $a \sin(\omega t)$  and  $\frac{1}{a} \sin(\omega[t - \phi])$  in Fig. 1 have a different phase. The phase shift  $\omega\phi$  is introduced to compensate for the delays introduced by the plant dynamics and the performance measure  $L_i$  in (8), which is part of the performance function  $J_i$  in (11). A similar phase shift is introduced in [24].

For analysis purposes, we select

$$K = a^2 \omega \delta, \quad (17)$$

where  $\delta \in \mathbb{R}_{>0}$  is a constant. In addition, the following change of variables is introduced:

$$\tilde{x} := x - M(\theta, w(t)) \quad \text{and} \quad \tilde{\theta} := \hat{\theta} - \theta^*, \quad (18)$$

such that  $\theta = \hat{\theta} + a \sin(\omega t) = \tilde{\theta} + \theta^* + a \sin(\omega t)$ , where  $M$  and  $\theta^*$  are defined in (5) and Assumption 4, respectively. We will apply the change of variables in (18) to the extremum-seeking scheme in (16). To prevent lengthy expressions we will not substitute  $\theta = \hat{\theta} + \theta^* + a \sin(\omega t)$  and we will write

$M$  instead of  $M(\theta, w(t))$ . Using (17) and (18), the system equations in (16) are transformed to

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= \tilde{f}(\tilde{x}, M, \theta, w(t)) - \frac{\partial M}{\partial \theta} \left[ \frac{d\tilde{\theta}}{dt} + a\omega \cos(\omega t) \right], \\ \frac{d\tilde{\theta}}{dt} &= \frac{a\omega^2\delta}{\pi} \int_{t-\frac{2\pi}{\omega}}^t J_i(\tilde{x}_d(\tau) + M_d(\tau), w_d(\tau)) s(\tau) d\tau, \end{aligned} \quad (19)$$

with  $i \in [1, \infty]$ ,  $\tilde{f}(\tilde{x}, M, \theta, w(t)) := f(\tilde{x} + M, \alpha(\tilde{x} + M, \theta), \theta, w(t)) - f(M, \alpha(M, \theta), \theta, w(t))$  as in Assumption 3,  $M_d(t) := M(\theta_d(t), w_d(t))$  and  $s(t) := \sin(\omega[t - \phi])$ .

## V. STABILITY ANALYSIS

We next present conditions under which the extremum-seeking closed-loop dynamics in  $(\tilde{x}, \tilde{\theta})$ -coordinates in (19) is SGPAS as defined in Definition 1, where the corresponding parameter vector is given by  $\epsilon = [a, \omega, \delta]^T$ , see Theorem 1 below. Furthermore, we will show that this stability property is directly related to the achievement of the performance optimization objective. Note that the dynamics of the closed-loop system in (19) are essentially different than those of a comparable closed-loop system for the equilibrium case in e.g. [5], [7] due to the delays introduced by the performance measure in (8) and the moving average filter (15) of the extremum-seeking controller.

### Theorem 1

Suppose that Assumptions 1, 2, 3 and 4 hold. Then, the closed-loop dynamics of the extremum-seeking scheme in (19) is SGPAS, where  $\epsilon = [a, \omega, \delta]^T$ .

*Proof:* For the proof, see [25]. ■

Let us make explicit the implications of this result in terms of achieving the performance optimization objective. Under the conditions of Theorem 1, the state  $x$  of the stabilized plant in (2),(4) converges to an arbitrarily small neighborhood of the steady-state solution given by the map  $M$  in (5) for sufficiently small  $a, \omega, \delta \in \mathbb{R}_{>0}$  since  $x = \tilde{x} + M(\theta, w(t))$  and  $\tilde{x}$  converges to an arbitrarily small neighborhood of the origin, see Definition 1 of SGPAS. Then, from the continuity of  $h$  in (2), it follows that the output  $y$  of the plant converges to a small neighborhood of the steady-state output for sufficiently small  $a, \omega, \delta \in \mathbb{R}_{>0}$ .

Note that from Theorem 1 and  $\hat{\theta} := \tilde{\theta} - \theta^*$  in (18), it also follows that the nominal value  $\hat{\theta}$  of the parameter  $\theta$  converges to an arbitrarily small neighborhood of the performance-optimizing value  $\theta^*$  for sufficiently small  $a, \omega, \delta \in \mathbb{R}_{>0}$ . Moreover, from Theorem 1 and  $\theta = \tilde{\theta} + \theta^* + a \sin(\omega t) = \hat{\theta} + a \sin(\omega t)$ , it also follows that  $\theta$  converges to an arbitrarily small neighborhood of the performance-maximizing (and output-optimizing) value  $\theta^*$  for sufficiently small  $a, \omega, \delta \in \mathbb{R}_{>0}$ . Note that here we use that  $a \sin(\omega t)$  becomes arbitrarily small for sufficiently small  $a \in \mathbb{R}_{>0}$ .

Hence, it follows that the output  $y$  of the stabilized plant in (2),(4) converges arbitrarily close to the optimal steady-state output for sufficiently small  $a, \omega, \delta \in \mathbb{R}$ . Using (17), this implies that the output  $y$  converges arbitrarily closely to the performance optimizing output for sufficiently small parameters  $a, \omega, K \in \mathbb{R}_{>0}$  of the extremum seeking controller.

## VI. ILLUSTRATIVE EXAMPLE

To illustrate the proposed extremum-seeking method, consider a plant of the form:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -25x_1 - b(\theta)x_2 + w_1(t) \quad (20)$$

with  $y = x_1$  and where  $b(\theta) : \mathbb{R} \rightarrow \mathbb{R}_{>0}$  is a nonlinear characteristic given by

$$b(\theta) = 10 + 5(\theta - 10)^2. \quad (21)$$

In (20),  $w_1(t)$  is part of the solution of the exosystem

$$\dot{w}_1 = vw_2, \quad \dot{w}_2 = -vw_1, \quad (22)$$

with  $v = 80$  and initial conditions  $w_1(0) = 0$  and  $w_2(0) = 20$ . The solution of the exosystem in (22) is then given by

$$w_1(t) = 20 \sin(80t), \quad w_2(t) = 20 \cos(80t). \quad (23)$$

Note that for fixed values of  $\theta \in \mathbb{R}$ , the plant in (20)-(21) can be regarded as a linear system for which the steady-state solution is given by

$$\bar{x}_{\theta,w}(t) = M(\theta, w(t)) = \begin{bmatrix} R_1(\theta)w_1(t) + R_2(\theta)w_2(t) \\ 80R_1(\theta)w_2(t) - 80R_2(\theta)w_1(t) \end{bmatrix} \quad (24)$$

with

$$R_i(\theta) = \frac{r_i(\theta)}{6375^2 + 80^2 b^2(\theta)}, \quad i = 1, 2, \quad (25)$$

with  $r_1(\theta) = -6375$  and  $r_2(\theta) = -80b(\theta)$ . To find the value of  $\theta \in \mathbb{R}$  that maximizes the amplitude of the steady-state output of the plant in (20)-(21), we introduce:

$$Q_\infty(y_d) = L_\infty(y_d), \quad (26)$$

where  $L_\infty$  is defined in (8). Note that for fixed  $\theta \in \mathbb{R}$ , the steady-state performance  $q = \bar{q}_{\theta,w}$  of the plant (with  $q = Q_\infty(y_d)$ ) is equal to the amplitude of the steady-state output  $y = \bar{y}_{\theta,w}$  of the plant. Using (23)-(26), the relation between fixed values of the system parameter  $\theta$  and the steady-state performance  $\bar{q}_{\theta,w}$  of the plant is given by the static map  $J_{sta,\infty}$  (i.e.  $\bar{q}_{\theta,w} = J_{sta,\infty}(\theta)$ ), which is defined as  $J_{sta,\infty}(\theta) = 20/(\sqrt{6375^2 + 80^2 b^2(\theta)})$ . Note that the extremum of the map is located at  $\theta = \theta^* = 10$ , see Fig. 2.

The extremum-seeking controller in Fig. 1 is used to optimize the steady-state performance of the plant in (20)-(21). Simulation results in Fig. 2 (in black) show that the system parameter  $\theta$  converges to a small neighborhood of the performance-maximizing value  $\theta^* = 10$ . As  $\theta$  converges to  $\theta^*$ , the performance  $q$  of the plant converges to a small neighborhood of the optimal steady-state performance indicated by the maximum of the static map  $J_{sta,\infty}$ , i.e. the amplitude of the steady-state output  $y$  is maximized.

Fig. 2 also shows simulation results (in grey) for a similar extremum-seeking scheme, where the moving average filter is replaced by a first-order low-pass filter with (properly tuned) angular cutoff frequency  $\omega_l = 1.1$ , see [5], [7]. Fig. 2 clearly shows that using a moving average filter results in a better estimate  $e$  of the gradient  $\frac{dJ_{sta,i}}{d\theta}(\hat{\theta})$ . The main reason for this fact is that the moving average filter filters

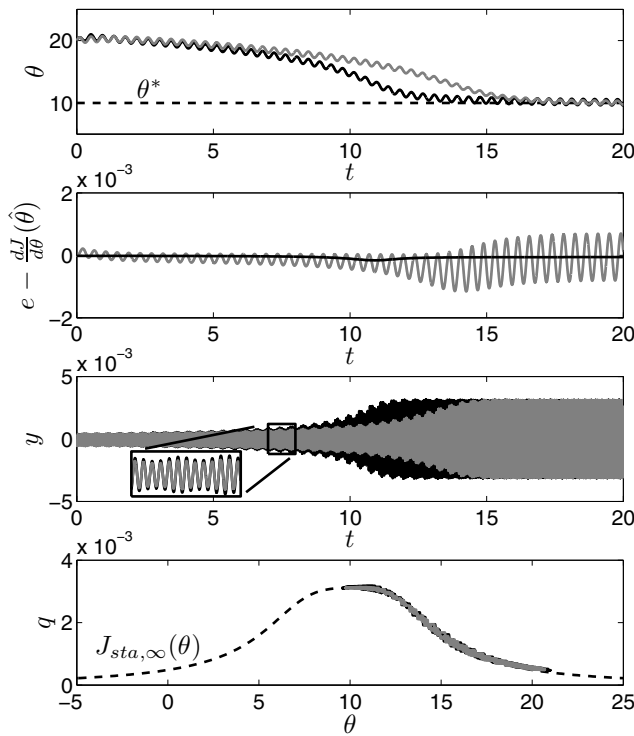


Fig. 2: Simulations results of the extremum-seeking scheme with moving average filter (black) and low-pass filter (grey), for  $a = 0.5$ ,  $K = 4 \times 10^3$ ,  $\omega = 15$  and  $\phi = 0.02$ , with initial conditions  $\hat{\theta}(0) = 20$  and  $x_{1d}(0)(\tau) = -5.4 \times 10^{-4}$ ,  $x_{2d}(0)(\tau) = -6.8 \times 10^{-3}$  for all  $\tau \in [-t_d, 0]$ , with  $t_d = T_w + \frac{2\pi}{\omega} = \frac{\pi}{40} + \frac{2\pi}{15} = \frac{19\pi}{120}$ .

out all oscillations with frequency  $\frac{\omega}{2\pi}$  (and higher-order harmonics), while the low-pass filter does not, as mentioned in Remark 4. Moreover, using the moving average filter results in a smaller estimation delay compared to using the low-pass filter. Because the moving average filter provides a better gradient estimate  $\frac{dJ_{sta,i}}{d\theta}(\hat{\theta})$ , the system parameter  $\theta$  converges faster to the optimal value  $\theta^*$  when the moving average filter is used, see Fig. 2.

## VII. CONCLUSIONS

In this paper, we have presented an extremum-seeking control method for steady-state performance optimization of general nonlinear plants with periodic steady-state outputs. This methodology allows to consider arbitrary periodic steady-state system outputs without requiring explicit knowledge of the relation between the parameters and the steady-state output of the system. Furthermore, we have presented a novel extremum-seeking controller with moving average filter. Moreover, conditions have been presented under which semi-global practical asymptotic stability of the closed-loop system can be guaranteed, which implies the achievement of the performance optimization using extremum seeking.

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