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Generalized incremental balanced truncation for nonlinear systems

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Abstract— **The method of generalized incremental balanced truncation is introduced in this paper, providing a technique for model reduction of nonlinear systems in which the autonomous part of the vector field is anti-symmetric. This approach differs from existing balancing-like reduction techniques in the definition of two novel, incremental energy functions, which provides several advantages. First, stability properties of the reduced-order model can be guaranteed, hereby considering the stability of trajectories for both zero and nonzero input. Second, a computable bound on the reduction error is derived. The reduction technique is illustrated by means of application to an example of a nonlinear electronic circuit.**

I. Introduction

Model reduction represents an important tool for the analysis of complex high-order systems. For (asymptotically stable) linear systems, balanced truncation, introduced in [15], is a popular technique. Balanced truncation is based on the definition of energy functions providing a characterization of the amount of observability and controllability of the system (see, e.g., [12], [2]). Based on these energy functions, a reduced-order model is constructed that corresponds to the parts of the high-order system with the largest contribution to the input-output behavior. This reduced-order model is guaranteed to be asymptotically stable (see [18]) and satisfies an a priori error bound [9], [12]. Optimal Hankel norm approximation [12] relies on the same energy functions and also features the desirable properties of stability preservation and the availability of a computable error bound.

In [22] and [10], [11], an extension of balanced truncation towards nonlinear systems is developed, hereby using the same energy functions. However, this method of nonlinear balanced truncation only guarantees *local* stability properties of the equilibrium point for *zero input* for the reduced-order system. Stability properties of trajectories for nonzero input are not guaranteed and, consequently, no bound on the reduction error is available. Results on the preservation of local stability properties for the equilibrium are also available for the reduction method of moment matching for nonlinear systems, as developed in [3], but, again, no error bound is available. Other approaches for the reduction of nonlinear systems do generally not guarantee stability preservation or an error bound either. Examples include trajectory piecewise linear approximation [19] and data-based methods such as proper orthogonal decomposition [23]. Some exceptions are obtained for systems with low-order nonlinearities (see [5], [6]), where results on stability preservation and error bounds are available for a reduced-order model that is obtained by linear reduction techniques.

In this paper, a novel approach towards the reduction of nonlinear systems is presented, hereby addressing the preservation of stability properties (also for trajectories for nonzero input) and the derivation of a computable error bound. In particular, the method of *generalized incremental balanced truncation* is introduced, which provides an extension of balanced truncation for linear systems to the nonlinear case. Here, nonlinear systems are considered in which the input and output terms are linear and the (nonlinear) autonomous part of the vector field is anti-symmetric.

Contrary to nonlinear balanced truncation as in [22], the current method is based on the introduction of two novel energy functions, which can be interpreted as incremental versions of the observability and controllability functions of balanced truncation. Namely, these so-called generalized incremental energy functions are based on the comparison of two system trajectories, rather than on the analysis of energy associated to a single trajectory. Nonetheless, for linear systems, these generalized incremental energy functions can be directly related to the observability and controllability Gramians. The model reduction procedure based on balancing these generalized incremental energy functions provides several advantages when applied to nonlinear systems. First, the preservation of stability properties is guaranteed. In particular, the property of incremental stability is considered, which provides a stability notion for all system trajectories, both for zero and nonzero input. Second, a computable error bound can be derived in terms of the $\mathcal{L}_2$ signal norm.

The remainder of this paper is organized as follows. In Section II, the generalized incremental energy functions are introduced and their relation to stability properties is given in Section III. The model reduction procedure of generalized incremental balanced truncation, based on these energy functions, is discussed in Section IV, whereas stability properties of the reduced-order system and the error bound are given in Section V. The method is illustrated by means of an example of a nonlinear electronic circuit in Section VI before stating conclusions in Section VII.

An extended and more detailed discussion of the results in this paper can be found in [7], which, in particular, provides a

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thorough interpretation of the incremental energy functions, hereby relating them to observability and reachability properties. Moreover, in [7], an additional approach towards model reduction for nonlinear systems is presented and a detailed evaluation of the conservatism in the error bound is given.

Notation. The field of real numbers is denoted by $\mathbb{R}$. For a vector $x \in \mathbb{R}^n$, its Euclidian norm is given as $|x| = \sqrt{x^{\mathrm{T}}x}$. A function $x : \mathcal{T} \to \mathbb{R}^n$ is said to be in $\mathcal{L}_2^n(\mathcal{T})$ if $\int_{\mathcal{T}} |x(t)|^2 \, \mathrm{d}t < \infty$. Finally, $\|x\|_2^2$ denotes the (squared) corresponding norm for $\mathcal{T} = [0, \infty)$.

II. Generalized incremental energy functions

Nonlinear systems of the form

$$\mathbf{\Sigma} : \begin{cases} \dot{x} = f(x) + Bu, \\ y = Cx, \end{cases} \tag{1}$$

are considered, with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Here, the function f is assumed to be locally Lipschitz continuous and satisfies $f(0) = 0$. The model reduction procedure developed in this paper is based on two novel energy functions, which are introduced as follows.

Definition 1: If there exists a matrix $\tilde{Q} = \tilde{Q}^{\mathrm{T}} \succ 0$ such that

$$(x - \bar{x})^{\mathrm{T}} \tilde{Q} \big(f(x) - f(\bar{x}) \big) \leq -\tfrac{1}{2} (x - \bar{x})^{\mathrm{T}} C^{\mathrm{T}} C (x - \bar{x}), \tag{2}$$

holds for all $x, \bar{x} \in \mathbb{R}^n$, then the function

$$\tilde{E}_o(x, \bar{x}) = (x - \bar{x})^{\mathrm{T}} \tilde{Q} (x - \bar{x}) \tag{3}$$

is said to be a generalized incremental observability function of the system $\mathbf{\Sigma}$ as in (1).

Definition 2: If there exists a matrix $\tilde{R} = \tilde{R}^{\mathrm{T}} \succ 0$ such that

$$(x + \bar{x})^{\mathrm{T}} \tilde{R} \big(f(x) + f(\bar{x}) \big) \leq -\tfrac{1}{2} (x + \bar{x})^{\mathrm{T}} \tilde{R} B B^{\mathrm{T}} \tilde{R} (x + \bar{x}), \tag{4}$$

holds for all $x, \bar{x} \in \mathbb{R}^n$, then the function

$$\tilde{E}_c(x, \bar{x}) = (x + \bar{x})^{\mathrm{T}} \tilde{R} (x + \bar{x}) \tag{5}$$

is said to be a generalized incremental controllability function function of the system $\mathbf{\Sigma}$ as in (1).

The inequalities (2) and (4) are based upon two copies of the vector field f, evaluated at x and $\bar{x}$, respectively. These energy functions are thus related by the comparison of *two* trajectories of the nonlinear system $\mathbf{\Sigma}$, as opposed to the energy functions computed on the basis of a single trajectory as in [22], [11]. In Section V, it will be shown that this *incremental* nature of the energy functions $\tilde{E}_o$ and $\tilde{E}_c$ is crucial for the development of a model reduction procedure guaranteeing stability preservation and an error bound.

Remark 1: It can be shown that, if a solution $\tilde{R}$ exists to (4), the vector field f is necessarily odd, i.e., it satisfies $f(x) = -f(-x)$. Details can be found in [4]. ◁

Remark 2: When $\mathbf{\Sigma}$ is linear (with $f(x) = Ax$), it is readily checked that (2) and (4) reduce to the inequalities

$$A^{\mathrm{T}} \tilde{Q} + \tilde{Q} A \preccurlyeq -C^{\mathrm{T}} C, \tag{6}$$

$$\tilde{P} A^{\mathrm{T}} + A \tilde{P} \preccurlyeq -B B^{\mathrm{T}}, \tag{7}$$

with $\tilde{P} = \tilde{R}^{-1}$. Assuming that A is Hurwitz, the solutions $\tilde{Q}$ to (6) and $\tilde{P}$ to (7) exist and are known as the *generalized* observability and controllability Gramians (see [8]), which are applied in structure-preserving model reduction techniques (see, e.g., [20]). In particular, $\tilde{Q}$ and $\tilde{P}$ reduce to the observability Gramian Q and controllability Gramian P, respectively, when equality holds in (6) and (7). ◁

The Gramians P and Q as discussed in Remark 2 form the basis of balanced truncation for linear system (see, e.g., [15], [2]), a model reduction technique that is known to provide accurate reduced-order models. This thus further motivates the use of the energy functions $\tilde{E}_o$ and $\tilde{E}_c$ as in Definitions 1 and 2 in the scope of model reduction for nonlinear systems.

Remark 3: The solutions to (2) and (4) are not unique. However, the linear case discussed in Remark 2 suggests that $\tilde{Q}$ and $\tilde{R}$ should be chosen as "small" and as "large" as possible, respectively, as this would imply that they are close to the Gramians Q and P. Details are given in Section V. ◁

III. Incremental energy functions and stability

The relation between the energy functions $\tilde{E}_o$ and $\tilde{E}_c$ as in Definitions 1 and 2 and stability properties of $\mathbf{\Sigma}$ as in (1) are analyzed in the current section. First, boundedness of solutions is guaranteed, as stated in the following lemma.

Lemma 1: Let there exist a positive definite solution $\tilde{R}$ to (4). Then, the state trajectory $x(t)$, $t \geq 0$, of system $\mathbf{\Sigma}$ as in (1) is bounded for any initial condition $x(0) = x_0 \in \mathbb{R}^n$ and any input function $u \in \mathcal{L}_2^m([0, \infty))$.

Proof: Denote $V(x) = \tilde{E}_c(x, 0) = x^{\mathrm{T}} \tilde{R} x$. Then, the time differentiation of V along trajectories of (1) gives

$$\dot{V}(x) = 2x^{\mathrm{T}} \tilde{R} f(x) + x^{\mathrm{T}} \tilde{R} B u + u^{\mathrm{T}} B^{\mathrm{T}} \tilde{R} x, \tag{8}$$

$$\leq -x^{\mathrm{T}} \tilde{R} B B^{\mathrm{T}} \tilde{R} x + x^{\mathrm{T}} \tilde{R} B u + u^{\mathrm{T}} B^{\mathrm{T}} \tilde{R} x, \tag{9}$$

where the latter is obtained by applying (4) for $\bar{x} = 0$. Completion of the squares in (9) yields

$$\dot{V}(x) \leq |u|^2 - |u - B^{\mathrm{T}} \tilde{R} x|^2 \leq |u|^2, \tag{10}$$

which can be integrated to obtain

$$V(x(t)) \leq V(x_0) + \int_0^t |u(\tau)|^2 \, \mathrm{d}\tau. \tag{11}$$

By boundedness of the initial condition, $V(x_0)$ is bounded. Moreover, as $u \in \mathcal{L}_2^m([0, \infty))$, the integral in (11) is bounded as well, such that $V(x(t))$ is bounded for all $t \geq 0$. As V is radially unbounded (due to $\tilde{R} \succ 0$), this implies boundedness of $x(t)$ for all $t \geq 0$, proving the lemma. ■

The generalized incremental observability function is related to certain incremental stability properties, as stated next.

Lemma 2: Let there exist a positive definite solution $\tilde{Q}$ to (2). Then, the system $\mathbf{\Sigma}$ as in (1) is incrementally stable, i.e., there exists a function α of class $\mathcal{K}$ such that, for any two solutions x and $\bar{x}$ corresponding to an input function $u \in \mathcal{L}_2^m([0, \infty))$ and initial conditions $x(0) = x_0 \in \mathbb{R}^n$ and $\bar{x}(0) = \bar{x}_0 \in \mathbb{R}^n$, respectively, the inequality

$$|x(t) - \bar{x}(t)| \leq \alpha(|x_0 - \bar{x}_0|) \tag{12}$$

holds for all $t \geq 0$. Moreover, for any bounded input function u, the corresponding outputs converge, i.e.,

$$\lim_{t \to \infty} |Cx(t) - C\bar{x}(t)| = 0. \tag{13}$$

Proof: Incremental stability will be proven by using $\tilde{E}_o$ as in (1) as a Lyapunov function candidate. As $\tilde{E}_o$ is a quadratic function and $\tilde{Q} \succ 0$, there exists functions α_1 and α_2 of class $\mathcal{K}_\infty$ such that

$$\alpha_1(|x-\bar{x}|) \leq \tilde{E}_o(x,\bar{x}) \leq \alpha_2(|x-\bar{x}|). \tag{14}$$

Moreover, (2) implies that the time derivative of $\tilde{E}_o$ satisfies

$$\dot{\tilde{E}}_o(x,\bar{x}) \leq -|Cx - C\bar{x}|^2 \leq 0, \tag{15}$$

such that $\tilde{E}_o(x(t),\bar{x}(t)) \leq \tilde{E}_o(x_0,\bar{x}_0) \leq \alpha_2(|x_0-\bar{x}_0|)$. Here, it is remarked that the linearity of the input function in (1) guarantees that (15) holds for any input. Then, the result in (12) follows with $\alpha = \alpha_1^{-1} \circ \alpha_2$.

Output convergence can be proven by integrating (15), which yields, for any initial condition $x_0, \bar{x}_0$ and any input function u,

$$\int_0^\infty |Cx(t) - C\bar{x}(t)|^2 \,\mathrm{d}t \leq \tilde{E}_o(x_0,\bar{x}_0) < \infty. \tag{16}$$

Then, Barbalat's lemma (see, e.g., [14]) gives the desired result, where it is noted that boundedness of the input signal u and continuity of f guarantees uniform continuity of $|Cx(t) - C\bar{x}(t)|^2$. ■

The definition of incremental stability in the statement of Lemma 2 is related to that in [1], where incremental *asymptotic* stability properties are studied. Incremental stability provides a notion for stability of *all* trajectories of a nonlinear system $\mathbf{\Sigma}$ and thus includes trajectories for nonzero input as well as stability of the origin for zero input.

The existence of the energy functions in Definitions 1 and 2 is not guaranteed a priori. Therefore, the following result is stated, which, firstly, gives a sufficient condition for the existence of solutions to (2) and (4) and, secondly, provides a relation to existing stability properties.

Theorem 3: Consider the system $\mathbf{\Sigma}$ as in (1) and assume that f is differentiable. If there exist matrices $M = M^{\mathrm{T}} \succ 0$ and $N = N^{\mathrm{T}} \succ 0$ such that

$$M\frac{\partial f}{\partial x}(x) + \frac{\partial^{\mathrm{T}} f}{\partial x}(x)M \preccurlyeq -N \tag{17}$$

holds for all $x \in \mathbb{R}^n$, then the generalized incremental observability function as in Definition 1 exists. If, in addition, f in (1) satisfies $f(x) = -f(-x)$ for all $x \in \mathbb{R}^n$, then the generalized incremental controllability function as in Definition 2 exists.

Proof: As a first step, it will be shown that (17) implies

$$(x-\bar{x})^{\mathrm{T}} M\big(f(x) - f(\bar{x})\big) \leq -\tfrac{1}{2}(x-\bar{x})^{\mathrm{T}} N(x-\bar{x}) \tag{18}$$

for all $x, \bar{x} \in \mathbb{R}^n$, hereby following a result in [16]. Thereto, the function $\phi(\lambda) := (x-\bar{x})^{\mathrm{T}} M f(\bar{x} + \lambda(x-\bar{x}))$ is introduced, such that the left-hand side of (18) equals $\phi(1) - \phi(0)$. Then, by the mean value theorem, there exists a $\lambda^* \in [0,1]$ such that $\phi(1) - \phi(0) = \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}(\lambda^*)$. As a result,

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}(\lambda^*) &= \tfrac{1}{2}(x-\bar{x})^{\mathrm{T}}\left(M\frac{\partial f}{\partial x}(x^*) + \frac{\partial^{\mathrm{T}} f}{\partial x}(x^*)M\right)(x-\bar{x}),\\ &\leq -\tfrac{1}{2}(x-\bar{x})^{\mathrm{T}} N(x-\bar{x}),\end{aligned} \tag{19}$$

where (17) is used and with $x^* = \bar{x} + \lambda^*(x-\bar{x})$. Combining (19) with the fact that $\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}(\lambda^*) = (x-\bar{x})^{\mathrm{T}} M(f(x)-f(\bar{x}))$ proves (18). Since N is positive definite, there exists a parameter $\varepsilon > 0$ such that $\varepsilon C^{\mathrm{T}} C \preccurlyeq N$. Then, it immediately follows from (18) that (2) holds with $\tilde{Q} = \varepsilon^{-1} M$.

To prove the result on the incremental controllability function, it is noted that replacing $\bar{x}$ by $-\bar{x}$ in (18) yields

$$(x+\bar{x})^{\mathrm{T}} M\big(f(x) + f(\bar{x})\big) \leq -\tfrac{1}{2}(x+\bar{x})^{\mathrm{T}} N(x+\bar{x}), \tag{20}$$

where the property $f(x) = -f(-x)$ is exploited. Similar to before, positive definiteness of N implies the existence of some $\varepsilon > 0$ such that $\varepsilon M B B^{\mathrm{T}} M \preccurlyeq N$. Then, choosing $\tilde{R} = \varepsilon M$ gives the condition (4), proving the theorem. ■

Remark 4: In the literature, the condition (17) is referred to as the Demidovich condition, see [17], which is related to stability properties of systems with nonzero inputs. ◁

IV. Generalized Incremental Balanced Truncation

Based on the generalized incremental energy functions, a realization can be defined that enables model reduction.

Definition 3: A realization (1) of the system $\mathbf{\Sigma}$ is said to be a generalized incrementally balanced realization if there exists a diagonal matrix Σ as

$$\Sigma = \begin{bmatrix} \sigma_1 I_{m_1} & 0 & \cdots & 0 \\ 0 & \sigma_2 I_{m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_q I_{m_q} \end{bmatrix}, \tag{21}$$

where the parameters σ_i satisfy $\sigma_1 > \sigma_2 > \cdots > \sigma_q > 0$ and have multiplicity m_i with $\sum_{i=1}^q m_i = n$, such that (2) and (4) hold with $\tilde{Q} = \Sigma$ and $\tilde{R} = \Sigma^{-1}$.

Then, the following result is immediate.

Theorem 4: Let the system $\mathbf{\Sigma}$ as in (1) be such that there exist positive definite symmetric matrices $\tilde{Q}$ and $\tilde{R}$ satisfying (2) and (4), respectively, thus characterizing the generalized incremental observability function as in (3) and generalized incremental controllability function as in (5). Then, there exists a coordinate transformation $x = Tz$ such that the system $\mathbf{\Sigma}$ is a generalized incrementally balanced realization in the new coordinates z. Moreover, the parameters σ_i^2 in (21) equal the eigenvalues of the product $\tilde{Q}\tilde{R}^{-1}$.

Proof: The simultaneous diagonalization of two positive definite symmetric matrices is a standard result in linear algebra, see [13]. In particular, there exists a nonsingular matrix T such that $T^{\mathrm{T}}\tilde{Q}T = (T^{\mathrm{T}}\tilde{R}T)^{-1} = \Sigma$, with Σ of the form (21). This matrix T is the desired transformation. ■

Remark 5: The transformation T in Theorem 4 is essentially the same as the transformation used in balanced truncation for linear systems, for which $\tilde{Q}$ and $\tilde{R}^{-1}$ can be chosen equal to the observability and controllability Gramian. In this case, the parameters σ_i correspond to the Hankel singular values of the linear system (see [12], [2]). ◁

In the remainder of this section, it is assumed that the realization (1) of $\mathbf{\Sigma}$ is a generalized incrementally balanced realization as in Definition 3. For this realization, the matrix Σ as in (21) can be partitioned as

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \tag{22}$$

with $\Sigma_1 \in \mathbb{R}^{k\times k}$ and $\Sigma_2 \in \mathbb{R}^{(n-k)\times(n-k)}$. It is assumed that k is chosen such that Σ is split according to the multiplicities of the parameters σ_i, i.e., there exists r such that $k = \sum_{i=1}^{r} m_i$. Then, Σ_1 and Σ_2 in are given as

$$\Sigma_1 = \mathrm{blkdiag}\{\sigma_1 I_{m_1}, \sigma_2 I_{m_2}, \ldots, \sigma_r I_{m_r}\}, \tag{23}$$
$$\Sigma_2 = \mathrm{blkdiag}\{\sigma_{r+1} I_{m_{r+1}}, \sigma_{r+2} I_{m_{r+2}}, \ldots, \sigma_q I_{m_q}\}. \tag{24}$$

The partitioning (22) directly implies a partitioning of the generalized incremental observability and generalized incremental controllability function as

$$\tilde{E}_o(x,\bar{x}) = \tilde{E}_o^1(x_1,\bar{x}_1) + \tilde{E}_o^2(x_2,\bar{x}_2), \tag{25}$$
$$\tilde{E}_c(x,\bar{x}) = \tilde{E}_c^1(x_1,\bar{x}_1) + \tilde{E}_c^2(x_2,\bar{x}_2), \tag{26}$$

where, for $i \in \{1,2\}$,

$$\tilde{E}_o^i(x_i,\bar{x}_i) := (x_i - \bar{x}_i)^{\mathrm{T}}\Sigma_i(x_i - \bar{x}_i), \tag{27}$$
$$\tilde{E}_c^i(x_i,\bar{x}_i) := (x_i + \bar{x}_i)^{\mathrm{T}}\Sigma_i^{-1}(x_i + \bar{x}_i). \tag{28}$$

Here, $x^{\mathrm{T}} = [\, x_1^{\mathrm{T}} \;\, x_2^{\mathrm{T}} \,]$ and $\bar{x}^{\mathrm{T}} = [\, \bar{x}_1^{\mathrm{T}} \;\, \bar{x}_2^{\mathrm{T}} \,]$ with $x_1, \bar{x}_1 \in \mathbb{R}^k$ and $x_2, \bar{x}_2 \in \mathbb{R}^{n-k}$. When the function f and matrices B and C as in (1) are partitioned according to (25) and (26) as

$$f(x) = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix},\; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\; C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{29}$$

an approximiation of $\mathbf{\Sigma}$ can be obtained by truncation (i.e., setting $x_2 = 0$ and discarding the dynamics of x_2), leading to the reduced-order system $\hat{\mathbf{\Sigma}}_k$ as

$$\hat{\mathbf{\Sigma}}_k : \begin{cases} \dot{\xi} = f_1(\xi, 0) + B_1 u, \\ \hat{y} = C_1\xi, \end{cases} \tag{30}$$

with $\xi \in \mathbb{R}^k$ an approximation for x_1. Motivated by earlier definitions, this reduction procedure will be referred to as generalized incremental balanced truncation.

V. Stability preservation and error bound

To analyze the properties of the reduced-order model $\hat{\mathbf{\Sigma}}_k$ as in (30), the following lemma can be exploited.

Lemma 5: Let (1) be a generalized incrementally balanced realization of the system $\mathbf{\Sigma}$ and let $\hat{\mathbf{\Sigma}}_k$ as in (30) be a reduced-order system obtained by generalized incremental balanced truncation. Then, the functions $\tilde{E}_o^1$ as in (27) and $\tilde{E}_c^1$ as in (28) are a generalized incremental observability function and a generalized incremental controllability function for $\hat{\mathbf{\Sigma}}_k$, respectively.

Proof: The proof for the generalized incremental observability function will be given first. Thereto, the inequality (2) with $\tilde{Q} = \Sigma$ is considered in the partitioned coordinates $x^{\mathrm{T}} = [\, x_1^{\mathrm{T}} \;\, x_2^{\mathrm{T}} \,]$ and $\bar{x}^{\mathrm{T}} = [\, \bar{x}_1^{\mathrm{T}} \;\, \bar{x}_2^{\mathrm{T}} \,]$, leading to

$$\begin{aligned}(x_1 - \bar{x}_1)^{\mathrm{T}}\Sigma_1\big(f_1(x_1,x_2) - f_1(\bar{x}_1,\bar{x}_2)\big)\\ + (x_2 - \bar{x}_2)^{\mathrm{T}}\Sigma_2\big(f_2(x_1,x_2) - f_2(\bar{x}_1,\bar{x}_2)\big)\\ \le -\tfrac{1}{2}\big|C_1(x_1 - \bar{x}_1) + C_2(x_2 - \bar{x}_2)\big|^2.\end{aligned} \tag{31}$$

Here, the partitioning (29) is used, whereas Σ_1 and Σ_2 are given by (23) and (24). Setting $x_2 = \bar{x}_2 = 0$ in (31) yields

$$\begin{aligned}(x_1 - \bar{x}_1)^{\mathrm{T}}\Sigma_1\big(f_1(x_1,0) - f_1(\bar{x}_1,0)\big)\\ \le -\tfrac{1}{2}|C_1(x_1 - \bar{x}_1)|^2.\end{aligned} \tag{32}$$

Thus, the function $\tilde{E}_o^1$ as in (25) represents a generalized incremental observability function for the reduced-order system $\hat{\mathbf{\Sigma}}_k$ as in (30), as follows from Definition 1.

The statement for the generalized incremental controllability function $\tilde{E}_c^1$ can be proven similarly, hereby exploiting the inequality (4) with $\tilde{R} = \Sigma^{-1}$ in partitioned coordinates as

$$\begin{aligned}(x_1 + \bar{x}_1)^{\mathrm{T}}\Sigma_1^{-1}\big(f_1(x_1,x_2) + f_1(\bar{x}_1,\bar{x}_2)\big)\\ + (x_2 + \bar{x}_2)^{\mathrm{T}}\Sigma_2^{-1}\big(f_2(x_1,x_2) + f_2(\bar{x}_1,\bar{x}_2)\big)\\ \le -\tfrac{1}{2}\big|B_1^{\mathrm{T}}\Sigma_1^{-1}(x_1 + \bar{x}_1) + B_2^{\mathrm{T}}\Sigma_2^{-1}(x_2 + \bar{x}_2)\big|^2.\end{aligned} \tag{33}$$

As before, setting $x_2 = \bar{x}_2 = 0$ gives the desired result. ■

This lemma can be used to guarantee stability properties of the reduced-order model, as stated in the following theorem.

Theorem 6: Let (1) be a generalized incrementally balanced realization of the system $\mathbf{\Sigma}$ and let $\hat{\mathbf{\Sigma}}_k$ as in (30) be a reduced-order system obtained by generalized incremental balanced truncation. Then, the following statements hold:

1) Any state trajectory ξ of the reduced-order system (30) with initial condition $\xi(0) = \xi_0 \in \mathbb{R}^k$ and input function $u \in \mathcal{L}_2^m([0,\infty))$ is bounded for all $t \ge 0$;
2) The reduced-order nonlinear system $\hat{\mathbf{\Sigma}}_k$ is incrementally stable for the class of inputs $\mathcal{L}_2^m([0,\infty))$;
3) For any bounded input function $u \in \mathcal{L}_2^m([0,\infty))$ and initial conditions $\xi(0) = \xi_0$, $\bar{\xi}(0) = \bar{\xi}_0$ the output trajectories converge, i.e.,

$$\lim_{t\to\infty} |C_1\xi(t) - C_1\bar{\xi}(t)| = 0. \tag{34}$$

Proof: By Lemma 5, the generalized incremental observability and controllability functions exist for the reduced-order system $\hat{\mathbf{\Sigma}}_k$. Consequently, boundedness of solutions follows directly from Lemma 1. Similarly, incremental stability and output convergence follow from Lemma 2. ■

Theorem 6 thus guarantees that the reduced-order system obtained by incremental balanced truncation has bounded solutions, is incrementally stable and output trajectories converge asymptotically to each other. By comparing this reduced-order system $\hat{\mathbf{\Sigma}}_k$ to the original system $\mathbf{\Sigma}$, an error bound in terms of the $\mathcal{L}_2$ signal norm can be stated, as formalized in the following theorem.

Theorem 7: Let (1) be a generalized incrementally balanced realization of the system $\mathbf{\Sigma}$. In addition, let $\hat{\mathbf{\Sigma}}_k$ as in (30) be a reduced-order system obtained by generalized incremental balanced truncation such that the order k satisfies $\sum_{i=1}^{r} m_i = k$ for some r. Then, for trajectories x and ξ of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}_k$, respectively, for a common input signal $u \in \mathcal{L}_2^m([0,\infty))$ and initial conditions $x(0) = 0$ and $\xi(0) = 0$, respectively, the corresponding outputs y and $\hat{y}$ satisfy the error bound

$$\|y - \hat{y}\|_2 \le \left(2 \sum_{i=r+1}^{q} \sigma_i\right)\|u\|_2, \tag{35}$$

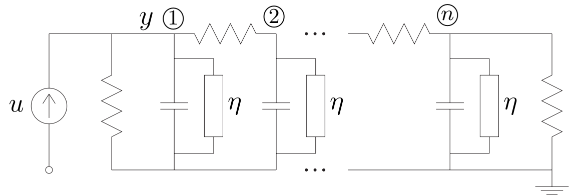

Fig. 1. Electronic circuit with nonlinear resistors η.

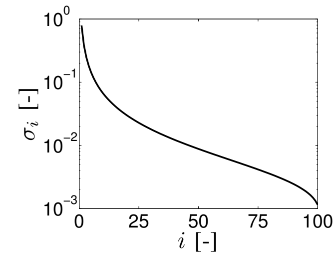

Fig. 2. Parameters σ_i characterizing the generalized incrementally balanced realization.

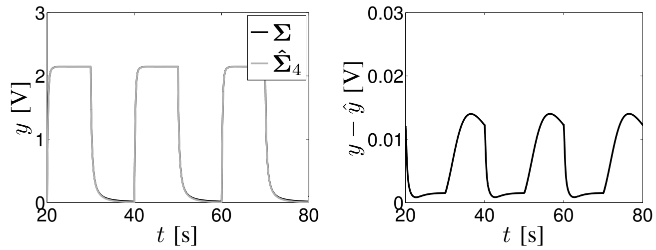

Fig. 3. Comparison of the high-order system $\mathbf{\Sigma}$ and the reduced-order system $\hat{\mathbf{\Sigma}}_4$ for $\eta(v) = \mathrm{sign}(v)v^2$ and $u(t) = 4(\mathrm{sign}(\sin(2\pi\frac{1}{20}t)) + 1)$ (left) and the corresponding error (right).

with σ_i as in Definition 3.

Proof: The proof is given in Appendix A. ■

When applied to linear systems, the error bound corresponds to that of balanced truncation as obtained in [9] and [12].

Remark 6: The error bound in Theorem 7 directly depends, through the incrementally balanced realization as in Definition 3, on the generalized incremental observability and controllability functions $\tilde{E}_o$ and $\tilde{E}_c$ of the high-order system $\mathbf{\Sigma}$. Thus, the error bound can be evaluated a priori and no analysis of the reduced-order system is required. ◁

Remark 7: Clearly, a small error bound and (possibly) more accurate reduced-order model is obtained when the (discarded) parameters σ_i are small. As these parameters depend on the generalized energy functions through Definition 3 and Theorem 4, solutions to (2) and (4) might be sought for which the eigenvalues of the product $\tilde{Q}\tilde{R}^{-1}$ are minimized. Rather than the direct minimization, a more practical approach is obtained when the energy functions are considered separately, hereby minimizing the eigenvalues of the matrix $\tilde{Q}$ satisfying (2) and maximizing the eigenvalues of $\tilde{R}$ satisfying (4). Here, it is remarked that this approach corresponds to the intuition in Remark 3. ◁

VI. Illustrative Example

The proposed reduction method is illustrated by means of application to the example of a nonlinear electronic circuit in Figure 1, which is taken from [19]. All resistors and capacitors have unit resistance and capacitance, respectively. Moreover, the elements marked by η represent nonlinear resistors, where the function η is assumed to be nondecreasing and satisfies $\eta(v) = -\eta(-v)$, leading to the dynamics

$$f(x) = Ax + \varphi(x), \tag{36}$$

with

$$A = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}, \quad \varphi(x) = -\begin{bmatrix} \eta(x_{(1)}) \\ \vdots \\ \eta(x_{(n)}) \end{bmatrix}. \tag{37}$$

Here, $x_{(i)}$, $i \in \{1, 2, \ldots, n\}$ denote the components of the state $x \in \mathbb{R}^n$ (with $n = 100$), which represent the voltages at the nodes indicated with ① to ⓝ in Figure 1. Next, the input $u \in \mathbb{R}$ represents the source current, whereas $y \in \mathbb{R}$ is the voltage at node 1, such that the input and output matrices in (1) read $B^{\mathrm{T}} = C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$.

In order to enable the reduction of the nonlinear circuit model, solutions $\tilde{Q}$ to (3) and $\tilde{R}$ to (5) are sought. As η is nondecreasing, it can be shown that $\tilde{Q}$ can be chosen as a diagonal matrix $\Sigma = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\}$, where the entries σ_i are given in Figure 2. Moreover, due to the properties $A = A^{\mathrm{T}}$ and $C = B^{\mathrm{T}}$, $\tilde{R}$ can be characterized as $\tilde{R} = \Sigma^{-1}$. Details can be found in [4]. Thus, by Definition 3, (36-37) is an incrementally balanced realization and a reduced-order model can be obtained by truncation. Considering the numerical values of σ_i in Figure 2, a reduced-order system of order $k = 4$ is chosen. By Theorem 6, $\hat{\mathbf{\Sigma}}_4$ has bounded trajectories and is incrementally stable, whereas Theorem 7 yields an error bound with $\varepsilon = 2\sum_{i=5}^{100} \sigma_i = 3.402$. It is recalled that such properties cannot be guaranteed by balanced truncation for nonlinear systems as in [22].

The quality of the reduced-order model is assessed by means of simulation, as depicted in Figure 3. The reduced-order model provides an accurate approximation of the high-order model. For this simulation, the error bound ε is conservative. This is largely due to the fact that the solutions $\tilde{Q}$ to (3) and $\tilde{R}$ to (5) are required to be diagonal, which limits the freedom in "minimizing" (resp. "maximizing") these solutions. More details on conservatism can be found in [4].

VII. Conclusions

In this paper, an approach for model reduction of a class of nonlinear systems is proposed. The method of generalized incremental balanced truncation is based on the introduction of two novel incremental energy functions and features the desirable properties of the stability preservation and the availability of a computable error bound. Herein, stability properties for both zero and nonzero input are guaranteed.

Appendix

A. Proof of Theorem 7

In order to prove the theorem, a one-step reduction is considered first. In this reduction, the state-space dimension is reduced according to the multiplicity m_q of the smallest parameter σ_q as in Definition 3. In particular, dissipativity theory (see [24]) will be used to show that the error system $\mathbf{\Sigma}-\hat{\mathbf{\Sigma}}_{n-m_q}$ is dissipative with respect to the supply rate

$$s(u,y,\bar{y})=(2\sigma_q)^2|u|^2-|y-\hat{y}|^2 \tag{38}$$

providing an upper bound on the $\mathcal{L}_2$ gain of the error system (see [21]). Thereto, a candidate storage function V is introduced as

$$\begin{aligned}V(x_1,x_2,\xi)=\tilde{E}_o^1(x_1,\xi)+\tilde{E}_o^2(x_2,0)&\\ +\sigma_q^2\big(\tilde{E}_c^1(x_1,\xi)+\tilde{E}_c^2(x_2,0)\big),&\end{aligned} \tag{39}$$

where $\tilde{E}_o^i$ and $\tilde{E}_c^i$ represent the partitioned incremental observablity and controllability functions as in (27) and (28). Then, the time differentiation of (39) along trajectories of $\mathbf{\Sigma}$ as in (1) and $\hat{\mathbf{\Sigma}}_{n-m_q}$ as in (30) yields

$$\begin{aligned}\dot{V}(x_1,x_2,\xi)=&\,2(x_1-\xi)^{\mathrm{T}}\Sigma_1\big(f_1(x_1,x_2)-f_1(\xi,0)\big)\\ &+2x_2^{\mathrm{T}}\Sigma_2\big(f_2(x_1,x_2)+B_2u\big)\\ &+2\sigma_q^2(x_1+\xi)^{\mathrm{T}}\Sigma_1^{-1}\big(f_1(x_1,x_2)+f_1(\xi,0)+2B_1u\big)\\ &+2\sigma_q^2x_2^{\mathrm{T}}\Sigma_2^{-1}\big(f_2(x_1,x_2)+B_2u\big).\end{aligned} \tag{40}$$

Here, it is noted that $\Sigma_2=\sigma_qI_{m_q}$, as only a one-step reduction is considered. As a result, the equality

$$\begin{aligned}&-2x_2^{\mathrm{T}}\Sigma_2\big(f_2(\xi,0)+B_2u\big)\\ &\quad+2\sigma_q^2x_2^{\mathrm{T}}\Sigma_2^{-1}\big(f_2(\xi,0)+B_2u\big)=0\end{aligned} \tag{41}$$

holds. Then, adding the left-hand side of (41) to (40) gives

$$\begin{aligned}\dot{V}(x_1,x_2,\xi)=&\quad 2(x_1-\xi)^{\mathrm{T}}\Sigma_1\big(f_1(x_1,x_2)-f_1(\xi,0)\big)\\ &+2(x_2-0)^{\mathrm{T}}\Sigma_2\big(f_2(x_1,x_2)-f_2(\xi,0)\big)\\ +2\sigma_q^2&(x_1+\xi)^{\mathrm{T}}\Sigma_1^{-1}\big(f_1(x_1,x_2)+f_1(\xi,0)+2B_1u\big)\\ +2\sigma_q^2&(x_2+0)^{\mathrm{T}}\Sigma_2^{-1}\big(f_2(x_1,x_2)+f_2(\xi,0)+2B_2u\big),\end{aligned} \tag{42}$$

where it can be observed that the first two lines correspond to the left-hand side of the partitioned inequality describing the generalized incremental observability function (see (31) in the proof of Lemma 2). Similarly, (parts of) the final two lines (42) can be related to the generalized incremental controllability function (see (33)). In particular, the application of (31) and (33) with $\bar{x}_1=\xi$ and $\bar{x}_2=0$ yields

$$\begin{aligned}\dot{V}(x_1,x_2,\xi)\le&-|C_1(x_1-\xi)+C_2x_2|^2\\ &-\sigma_q^2\big|B_1^{\mathrm{T}}\Sigma_1^{-1}(x_1+\xi)+B_2^{\mathrm{T}}\Sigma_2^{-1}x_2\big|^2\\ &+4\sigma_q^2(x_1+\xi)^{\mathrm{T}}\Sigma_1^{-1}B_1u+4\sigma_q^2x_2^{\mathrm{T}}\Sigma_2^{-1}B_2u,\end{aligned} \tag{43}$$

Rewriting (43) using completion of the squares leads to

$$\begin{aligned}\dot{V}(x_1,x_2,\xi)\le&-|C_1(x_1-\xi)+C_2x_2|^2+\sigma_q^2|2u|^2\\ &-\sigma_q^2\big|2u-B_1^{\mathrm{T}}\Sigma_1^{-1}(x_1+\xi)-B_2^{\mathrm{T}}\Sigma_2^{-1}x_2\big|^2\\ \le&\,(2\sigma_q)^2|u|^2-|y-\hat{y}|^2.\end{aligned} \tag{44}$$

Consequently, V as in (39) is a storage function for the supply rate (38), proving that the $\mathcal{L}_2$ norm of the error system $\mathbf{\Sigma}-\hat{\mathbf{\Sigma}}_{n-m_q}$ is bounded by $2\sigma_q$. As, by Lemma 5, the one-step reduced-order system $\hat{\mathbf{\Sigma}}_{n-m_q}$ is again a generalized incrementally balanced realization, the above result can be repeated to remove more state components. Then, application of the triangle inequality gives the error bound (35). ■

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