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Optimal control for non-exponentially stabilizable spatially invariant systems with an application to vehicular platooning

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Abstract— **This paper considers the optimal control problem for a class of infinite-dimensional systems, namely spatially invariant systems. A common assumption in the scope of such optimal control problem is the exponential stabilizability of the infinite-dimensional plant. We propose sufficient conditions for the optimizability of spatially invariant systems that are not exponentially stabilizable. The practical significance of this problem setting is motivated by vehicular platooning, for which it is desired to design controllers that attenuate the effect of disturbances, both in time and space, i.e., over the vehicle index.**

Keywords:Vehicular platooning, spatially invariant systems, optimal control, stabilization

I. Introduction

Linear quadratic (LQ) optimal control for linear finite-dimensional systems is one of the corner stones of modern control theory, and therefore can be found in almost any textbook on this subject, [1], [2]. Soon after its birth, the theory was extended to infinite-dimensional systems, [3], [4]. There it is showed that the LQ-theory carries over from finite- to infinite-dimensional systems provided the system is exponentially stabilizable, i.e., there exists a state feedback law $u(t) = Fx(t)$ such that the trajectories of the closed-loop system converges exponentially to zero as t goes to infinity. For infinite-dimensional systems there is a distinction between exponential and asymptotic stability. In the latter situation, the state converges to zero as time goes to infinity, but the convergence rate needs not to be exponential. Hence, already from a purely scientific point of view, it is interesting to study the LQ problem for systems that are not exponentially stabilizable, but may be only asymptotically stabilizable.

In this paper, we focus on a practically important class of non-exponentially stabilizable systems, namely, a standard spatially invariant system model for an infinite string of vehicles. This application domain is relevant in view of automated vehicle platooning to increase road throughput and/or to reduce fuel consumption [5], [6]. An important control design objective for this application is that the effects of disturbances, introduced by, e.g., initial condition errors or velocity variations of the lead vehicle, are attenuated along the string of vehicles. This requirement is formalized using the notion of string stability [7], which is very closely related to asymptotic stability of infinite-dimensional interconnected systems.

This paper is structured as follows. In the remainder of this section we, firstly, present preliminaries on spatially invariant systems, secondly, present a motivating example concerning a vehicular platooning system which is not exponentially stabilizable and, thirdly, formulate the problem statement. In Section II, we present the main results concerning conditions (involving both necessity and sufficiency) for the optimizability of the class of spatially invariant systems under study. In Section III, we apply the main results to the vehicular platooning model and Section IV closes with concluding remarks.

A. Spatially invariant systems

A spatially invariant system is described by the following set of equations, for $k \in \mathbb{Z}$,

$$\frac{d}{dt}x_k(t) = \sum_{j=-\infty}^{\infty} A_{k-j}x_j(t) + B_{k-j}u_j(t), \tag{1}$$

$$y_k(t) = \sum_{j=-\infty}^{\infty} C_{k-j}x_j(t), \qquad k \in \mathbb{Z}, \tag{2}$$

with initial condition $x_k(0) = x_{k0}$. The vectors $x_k(t) \in \mathbb{C}^n$, $u_k(t) \in \mathbb{C}^m$, and $y_k(t) \in \mathbb{C}^p$ denote the state, the control input, and the output of the k'th subsystem, $k \in \mathbb{Z}$, at time $t \geq 0$, respectively. The matrices $A_{k-j} \in \mathbb{C}^{n\times n}$, $B_{k-j} \in \mathbb{C}^{n\times m}$, and $C_{k-j} \in \mathbb{C}^{p\times n}$ describe the influence of subsystem j on subsystem k. The system (1)–(2) is called spatially invariant since the influence between the subsystems depends only on the difference between their indices, or in other words, a shift in the numbering of the subsystems does not change the overall dynamics of the system.

Note that we used complex-valued vectors for the state, input and output spaces. We could also have chosen real-valued vectors, but since later (after employing the z-transform) these signals will become complex, see (4)–(5) we have chosen to work with complex-valued signals from the beginning.

The properties of the system (1)–(2) are more conveniently studied in the z-domain. Therefore, we apply the bilateral z-transformation, which transforms the sequence $x(t) := (x_k(t))_{k=-\infty}^{\infty}$ to the function $\check{x}(z,t)$, as

$$\check{x}(z,t) = \mathcal{Z}(x(t)) = \sum_{k=-\infty}^{\infty} x_k(t)z^{-k}. \tag{3}$$

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Here, the z variable is not used as a time shift operator, but as an index shift operator. Now, knowing that $\mathcal{Z}(x(t)) = \check{x}(z,t)$, where $z = e^{j\theta}$ for $\theta \in [0, 2\pi]$, the state-space system (1)–(2) is transformed into the z-domain leading to the following system

$$\frac{\partial \check{x}}{\partial t}(z,t) = \check{A}(z)\check{x}(z,t) + \check{B}(z)\check{u}(z,t), \tag{4}$$

$$\check{y}(z,t) = \check{C}(z)\check{x}(z,t), \tag{5}$$

with initial condition $\check{x}(z,0) = \check{x}_0(z)$. The functions $\check{A}$, $\check{B}$, and $\check{C}$ are the z-transforms of the sequences A_k, B_k, and C_k, respectively. For instance, $\check{A}(z) = \sum_{k=-\infty}^{\infty} A_k z^{-k}$.

After the z-transform, the system (1)–(2) has become a (parametrized) system, with the parameter z lying on the unit circle, $\partial\mathbb{D}$. As state space of (4)–(5) we choose $L^2(\partial\mathbb{D};\mathbb{C}^n)$, i.e., the space consisting of all functions $f : \partial\mathbb{D} \to \mathbb{C}^n$ which satisfy $\int_0^{2\pi} \|f(e^{i\theta})\|^2 d\theta < \infty$. The input and output spaces are $L^2(\partial\mathbb{D};\mathbb{C}^m)$ and $L^2(\partial\mathbb{D};\mathbb{C}^p)$, respectively. We denote the system (4)–(5) with these state, input and output space by $\Sigma\left(\check{A}, \check{B}, \check{C}\right)$.

So the state space system (1)–(2) is transformed into the z-domain leading to (4)–(5), see also [8] and [9]. It is well-known that the z-transform is an isometry between $\ell^2(\mathbb{Z})$ (square summable sequences) and $L^2(\partial\mathbb{D})$, i.e.,

$$\begin{aligned} \|(f_k)_{k\in\mathbb{Z}}\| &:= \sum_{k=-\infty}^{\infty} \|f_k\|^2 \\ &= \frac{1}{2\pi}\int_0^{2\pi} \|\check{f}(e^{i\theta})\|^2 d\theta \\ &=: \|\check{f}\|. \end{aligned} \tag{6}$$

This implies that any system property of (1)–(2) can be formulated into an equivalent system property for (4)–(5) and visa versa. A similar statement holds for control problems formulated for (1)–(2). An example of such a property is exponentially stabilizability.

The system (1)–(2) is exponentially stabilizable if there exists a linear mapping F from the state space $\ell^2(\mathbb{Z};\mathbb{C}^n)$ to the input space $\ell^2(\mathbb{Z};\mathbb{C}^m)$ such that $\|F((x_k)_{k\in\mathbb{Z}})\| \leq c\|(x_k)_{k\in\mathbb{Z}}\|$ for some $c > 0$ and all $(x_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{C}^n)$, and the resulting closed-loop system is exponentially stable. The equivalent definition of exponentially stabilizability for the system (4)–(5) is that there exists a linear mapping $\check{F}$ from the state space $L^2(\partial\mathbb{D};\mathbb{C}^n)$ to the input space $L^2(\partial\mathbb{D};\mathbb{C}^m)$ such that $\|\check{F}\check{z}\| \leq c\|\check{z}\|$ for some $c > 0$ and all $\check{z} \in L^2(\partial\mathbb{D};\mathbb{C}^n)$, and that the resulting closed-loop system is exponentially stable.

For the parametrized system (4)–(5) this property can be checked point-wise, i.e. for all $z \in \partial\mathbb{D}$, as formalized in the following theorem [9].

Theorem 1.1: Consider the system (4)–(5), and assume that $\check{A}, \check{B}$, and $\check{C}$ are continuous functions on the unit circle $\partial\mathbb{D}$.

The system $\Sigma(\check{A}, \check{B}, -)$ is exponentially stabilizable if and only if for all $z \in \partial\mathbb{D}$ the finite-dimensional system $\Sigma(\check{A}(z), \check{B}(z), -)$ is (exponentially) stabilizable.

The system $\Sigma(\check{A}, -, \check{C})$ is exponentially detectable if and only if for all $z \in \partial\mathbb{D}$ the finite-dimensional system $\Sigma(\check{A}(z), -, \check{C}(z))$ is (exponentially) detectable. □

In the following section, we present a motivating example that shows that exponential stabilizability can be a too stringent assumption for practically relevant problems. The lack of exponentially stabilizable/detectability for some platoon systems has also been pointed out in [10].

B. Motivating example: vehicular platooning

Consider a string of (identical) vehicles, where the model for the longitudinal dynamics of the i'th vehicle is given by [7], [14]

$$\begin{pmatrix} \dot{s}_i \\ \dot{v}_i \\ \dot{a}_i \end{pmatrix} = \begin{pmatrix} v_i \\ a_i \\ -\tau^{-1}a_i + \tau^{-1}u_i \end{pmatrix}, \tag{7}$$

and in which s_i, v_i, and a_i are the vehicle position, speed and acceleration, respectively, and u_i is the external input. τ is a time constant representing the engine dynamics. However, as the inter-vehicle distances are more relevant than the absolute vehicle positions, the vehicle model (7) can be rewritten with relative distance (between two subsequent vehicles) d_i as a state, instead of the absolute position s_i, as follows:

$$\begin{pmatrix} \dot{d}_i \\ \dot{v}_i \\ \dot{a}_i \end{pmatrix} = \begin{pmatrix} v_{i-1} - v_i \\ a_i \\ -\tau^{-1}a_i + \tau^{-1}u_i \end{pmatrix}. \tag{8}$$

The equilibrium state for $u_i = 0$ is obtained by setting the time derivatives in (8) to zero. The solution of the resulting algebraic set of equations is

$$\begin{pmatrix} d_{i,eq} \\ v_{i,eq} \\ a_{i,eq} \end{pmatrix} = \begin{pmatrix} C_i \\ v_{i-1,eq} \\ 0 \end{pmatrix} = \begin{pmatrix} C_i \\ v_{eq} \\ 0 \end{pmatrix}, \tag{9}$$

where C_i is a constant that can be different for each vehicle. So, in the equilibrium, all of the vehicles are moving with constant velocity $v_{eq} = v_{i+1} = v_i$ for all i and vehicle i attains a constant distance C_i to the preceding vehicle $i-1$. We employ another state transformation as follows:

$$x_i = \begin{pmatrix} d_i \\ v_i \\ a_i \end{pmatrix} - \begin{pmatrix} C_i \\ v_{eq} \\ 0 \end{pmatrix}.$$

Hence x_i represents the error between the state $\left(\begin{smallmatrix} d_i \\ v_i \\ a_i \end{smallmatrix}\right)$ of (8) and the equilibrium state in (9). In terms of this error state, we obtain the following model:

$$\begin{aligned} \dot{x}_i(t) &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\tau^{-1} \end{pmatrix} x_i(t) + \\ &\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_{i-1}(t) + \begin{pmatrix} 0 \\ 0 \\ \tau^{-1} \end{pmatrix} u_i(t). \end{aligned} \tag{10}$$

After applying the z-transform, the system (10) becomes

$$\begin{aligned}\frac{d}{dt}\check{x}(z,t) &= \begin{pmatrix} 0 & z^{-1}-1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\tau^{-1} \end{pmatrix}\check{x}(z,t) + \\ &\quad \begin{pmatrix} 0 \\ 0 \\ \tau^{-1} \end{pmatrix}\check{u}(z,t) \\ &=: \check{A}(z)\check{x}(z,t) + \check{B}(z)\check{u}(z,t). \end{aligned} \tag{11}$$

Since the finite-dimensional system $\Sigma(\check{A}(1), \check{B}(1), -)$ is not (exponentially) stabilizable, we obtain by Theorem 1.1 that the infinite-dimensional system $\Sigma(\check{A}, \check{B}, -)$ is not exponentially stabilizable. Still, from a practical point of view it is essential to stabilize these dynamics (in particular to enforce accurate vehicle following and string stability).

C. Problem statement

In this paper we study the infinite horizon, linear quadratic (LQ) optimal control problem for the infinite-dimensional system (4)–(5), or equivalently (1)–(2). We will study this problem under the challenging condition that the system is not exponentially stabilizable.

II. Main result

In this section, we study the optimal control problem for the spatial invariant system (1)–(2) (or equivalently system (4)–(5)). That is, we consider the system (4)–(5), and want to find for every initial condition $\check{x}_0$ an input $\check{u}$ which minimizes the cost function

$$J(\check{x}_0, \check{u}) = \int_0^\infty \|\check{y}(z,t)\|^2 + \|\check{u}(z,t)\|^2 dt, \tag{12}$$

where the norms on $\check{y}$ and $\check{u}$ are the $L^2(\partial\mathbb{D})$-norms. Note that for the sake of simplicity we have assumed that the weighting on $\check{u}$ is one. The optimal control problem for spatial invariant systems has been studied before, and has let some discussion, [8], [11], [12]

We assume that $\check{A}, \check{B}$, and $\check{C}$ in (4)–(5) are continuous functions on the unit circle.

Next we state a result which links the *optimizability* of the system (4)–(5) to the (point-wise) solution of an Algebraic Riccati Equation, obtained by fixing z in the z-domain.

Theorem 2.1: For the system (4)–(5) with cost-function (12) the following statements are equivalent:

1) The system is optimizable; that is, for every initial condition $\check{x}_0$ there exists an input $\check{u}$ such that $J(\check{x}_0, \check{u}) < \infty$;
2) For almost all $z \in \partial\mathbb{D}$, the Algebraic Riccati Equation (ARE)

$$\begin{aligned}&\check{A}^*(z)\check{P}(z) + \check{P}(z)\check{A}(z) - \\ &\qquad \check{P}(z)\check{B}(z)\check{B}^*(z)\check{P}(z) + \check{C}^*(z)\check{C}(z) = 0\end{aligned} \tag{13}$$

possesses a non-negative solution $P(z)$ and $\operatorname{ess\,sup}_{z\in\partial\mathbb{D}} \|P(z)\| < \infty$.

If the system is optimizable, then the optimal control is given by $\check{u}(z,t) = -\check{B}^*(z)\check{P}(z)\check{x}(z,t)$.

Proof: The proof follows from standard LQ theory for infinite-dimensional systems, see e.g. [13, Section 6.2]. ■

Now we would like to find conditions such that item 1) or 2) of Theorem 2.1 can be directly checked on the finite-dimensional systems obtained by fixing z on the unit circle. If every such finite-dimensional system is exponentially stabilizable, then 1) holds, see Theorem 1.1. However, as we have seen from the motivating example in Section I-B, the system under study may not be exponentially stabilizable. The following example shows that the condition: "all but finitely-many finite-dimensional systems (obtained from (4)–(5) by fixing z on the unit circle) are exponentially stabilizable" is not sufficient for item 2) in Theorem 2.1 to hold.

Example 2.2: We consider the following system

$$\check{A}(z) = 1 - z, \quad \check{B}(z) = -1 + z, \quad \check{C}(z) = -1 + z. \tag{14}$$

Since for $z \in \partial\mathbb{D}$, and $z \neq 1$, the real part of $\check{B}(z)$ is non-zero, for such z the (finite-dimensional) system is exponentially stabilizable. Solving the ARE (13) gives

$$P(z) = \frac{Re(\check{A}(z)) + \sqrt{Re(\check{A}(z))^2 + |\check{B}(z)|^2|\check{C}(z)|^2}}{|\check{B}(z)|^2}. \tag{15}$$

Writing $z = e^{i\theta}$ gives

$$\begin{aligned} P(z) &= \frac{1-\cos(\theta) + \sqrt{(1-\cos(\theta))^2 + 4(1-\cos(\theta))^2}}{2 - 2\cos(\theta)} \\ &= \frac{1+\sqrt{1+4}}{2} \end{aligned}$$

and hence $P(\cdot)$ is strictly positive and uniformly bounded as required under point 2) in Theorem 2.1. Clearly, this example may tempt us to believe that exponential stabilizability for almost all z on the unit circle will be sufficient for statement 2 of Theorem 2.1 to hold. However, this is by no means the case as we show with the next example.

We choose the same $\check{A}$ and $\check{C}$, but change $\check{B}$ to $\check{B}(z) = (-1+z)^2$. Using (15) once more, gives with $z = e^{i\theta}$,

$$\begin{aligned} P(z) &= \frac{1-\cos(\theta) + \sqrt{(1-\cos(\theta))^2 + 8(1-\cos(\theta))^3}}{(2-2\cos(\theta))^2} \\ &= \frac{1+\sqrt{1+8(1-\cos(\theta))}}{4(1-\cos(\theta))}. \end{aligned}$$

Since now $P(z)$ is unbounded on the unit circle, the LQR-problem is not solvable for this system.

Remark 2.3: We note that even a simpler counter example could have been obtained by choosing $\check{C} = 1$. However, then it is obvious, since the output equals the state, and thus the system is optimizable if and only if it is exponentially stabilizable, see [13, Exercise 6.5].

What we did in the above examples is not to observe those states which are not stabilizable, i.e., we made sure that all finite-dimensional systems are optimizable, or equivalently that every finite-dimensional system is output stabilizable. However, even under that condition a spatially invariant system, such as e.g. the platoon model in Section I-B, need

not to be optimizable as the second example in Example 2.2 shows. Nevertheless, the optimizability of all finite-dimensional systems is a necessary condition for the optimizability of the spatially invariant system as made explicit in the following lemma.

Lemma 2.4: Consider the system $\Sigma(\check{A}, \check{B}, \check{C})$ with cost functional (12). Furthermore, we assume that $\check{A}, \check{B}$, and $\check{C}$ are continuously depending on $z \in \partial\mathbb{D}$. If the system is optimizable, then for every $z \in \partial\mathbb{D}$, the finite-dimensional system $\Sigma(\check{A}(z), \check{B}(z), \check{C}(z))$ is optimizable.

Proof: The proof follows directly from the fact that the ARE (13) is a point-wise equation in z, and the continuity of $\check{A}, \check{B}$, and $\check{C}$. ■

A finite-dimensional system is optimizable if and only if

$$\mathbb{C}^n = V_{\text{stab}} + V_{\text{unobs}}, \tag{16}$$

where V_{stab} is the stabilizable subspace, and V_{unobs} is the non-observable subspace, see e.g. [2, Theorem 10.13].

For our example in Section I-B we have that the stabilizable subspace of $\Sigma(\check{A}(1), \check{B}(1))$ is

$$V_{\text{stab}} = \{x \in \mathbb{C}^3 \mid x_1 = 0\}. \tag{17}$$

Thus $\check{C}(1)$ must be such that its V_{unobs} contains a vector with the first component unequal to zero. A possible choice for the output matrix of the spatially invariant system in the z-domain is

$$\check{C}(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In Section III, we will revisit the vehicular platooning model of the example in Section I-B. Before doing so, we provide the following sufficient condition for optimizability of the spatially invariant system (4)–(5) (which we denote by $\Sigma(\check{A}, \check{B}, \check{C})$).

Theorem 2.5: If there exists a $P_0 \in \mathbb{C}^{n\times n} \geq 0$ such that for all $z \in \partial\mathbb{D}$

$$\begin{aligned} &\check{A}^*(z)P_0 + P_0\check{A}(z) \\ &\quad - P_0\check{B}(z)\check{B}^*(z)P_0 + \check{C}^*(z)\check{C}(z) \leq 0 \end{aligned} \tag{18}$$

then the system $\Sigma(\check{A}, \check{B}, \check{C})$ with cost functional (12) is optimizable.

Proof: We can rewrite equation (18) as the Lyapunov inequality

$$\begin{aligned} &(\check{A}(z) - \check{B}(z)\check{B}^*(z)P_0)^*P_0 + P_0\left(\check{A}(z) - \check{B}(z)\check{B}^*(z)P_0\right) \\ &\quad \leq -P_0\check{B}(z)\check{B}^*(z)P_0 - \check{C}^*(z)\check{C}(z). \end{aligned}$$

This inequality implies that the state of the closed-loop system

$$\frac{\partial}{\partial t}\check{x}(z,t) = \left(\check{A}(z) - \check{B}(z)\check{B}^*(z)P_0\right)\check{x}(z,t),$$

with $\check{x}(z,0) = \check{x}_0(z)$ satisfies for all $t_e > 0$

$$\int_0^{t_e} \|\check{C}\check{x}(\cdot,t)\|^2 + \|-\check{B}^*P_0\check{x}(\cdot,t)\|^2 dt \leq \langle \check{x}_0(\cdot), P_0\check{x}_0(\cdot)\rangle.$$

Since in the input leading to the closed-loop system equals $\check{u}(z,t) = -\check{B}(z)^*P_0\check{x}(z,t)$ we see that the system is optimizable. ■

We remark that it is not necessary to find one P_0 which satisfies (18) for all $z \in \partial\mathbb{D}$. If $z_0 \in \partial\mathbb{D}$ is such that $\Sigma(\check{A}(z_0), \check{B}(z_0), \check{C}(z_0))$ is not exponentially stabilizable, then it suffices to find a constant P_0 satisfying (18) in a neighborhood of z_0. Hence for every such a z_0 on the unit circle, we could find a different P_0.

We will exploit Theorem 2.5 in the next section to solve the optimal control problem for the vehicular platooning model.

III. Application to vehicular platooning model

We apply the above condition to our vehicle model of the example in Section I-B. However, we begin by obtaining an exact solution of the ARE for $\check{C}$ constant and diagonal.

Since P is symmetric, we choose $\check{P}(z)$ of the form

$$\check{P}(z) = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ \overline{p}_{12} & p_{22} & p_{23} \\ \overline{p}_{13} & \overline{p}_{23} & p_{33} \end{pmatrix}, \quad p_{ii} \in \mathbb{R}, \quad p_{ij} \in \mathbb{C}, \quad i \neq j, \tag{19}$$

where $\overline{p}_{**}$ is the complex conjugate of p_{**}. For the sake of simplicity, the $\check{C}$ matrix is chosen diagonal and thus $\check{Q}(z) := \check{C}(z)^*\check{C}(z)$ is given by

$$\check{Q}(z) = \begin{pmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{pmatrix}. \tag{20}$$

Substituting all the matrices into the Algebraic Riccati Equation (13) gives the following six scalar equations:

$$-\frac{1}{\tau^2}p_{13}\overline{p}_{13} + q_{11} = 0 \tag{21a}$$

$$(-1 + z^{-1})p_{11} - \frac{1}{\tau^2}p_{13}\overline{p}_{23} = 0 \tag{21b}$$

$$p_{12} - \frac{1}{\tau}p_{13} - \frac{1}{\tau^2}p_{13}p_{33} = 0 \tag{21c}$$

$$(-1+z^{-1})\overline{p}_{12} + (-1+\overline{z^{-1}})p_{12} - \frac{1}{\tau^2}p_{23}\overline{p}_{23} + q_{22} = 0 \tag{21d}$$

$$(-1 + z^{-1})\overline{p}_{13} + p_{22} - \frac{1}{\tau}\overline{p}_{23} - \frac{1}{\tau^2}p_{33}\overline{p}_{23} = 0 \tag{21e}$$

$$\overline{p}_{23} - \frac{1}{\tau}p_{33} + p_{23} - \frac{1}{\tau}p_{33} - \frac{1}{\tau^2}p_{33}^2 + q_{33} = 0. \tag{21f}$$

By (16) and (17) we decide first to choose $q_{11} = 0$. With this assumption, we can solve the above equations as follows:

$$(21a), q_{11} = 0 \Rightarrow p_{13} = 0 \tag{22a}$$

$$(21b), (22a) \Rightarrow p_{11} = 0 \tag{22b}$$

$$(21c), (22a) \Rightarrow p_{12} = 0 \tag{22c}$$

$$(21d), (22c) \Rightarrow |p_{23}| = \tau\sqrt{rq_{22}} \tag{22d}$$

$$(21f), (22d) \Rightarrow p_{33} = \tau[-1 \pm \sqrt{1 + (2p_{23} + q_{33})}] \tag{22e}$$

$$(21e), (22d), (22e) \Rightarrow p_{22} = \frac{1}{\tau}\overline{p}_{23} + \frac{1}{r\tau^2}p_{33}\overline{p}_{23}. \tag{22f}$$

Observe that in (22) none of the elements of $\check{P}(z)$ depends on z; so, all the solutions must be real and, therefore, from (22d), it follows that $p_{23} = \pm\tau\sqrt{rq_{22}}$. Also, $\check{P} = \check{P}^*$ must be positive semi-definite ($\check{P} \geq 0$) which requires that all the diagonal elements should be non-negative ($p_{22}, p_{33} \geq 0$). So, equation (22f) indicates that p_{23} is positive ($p_{23} = \tau\sqrt{q_{22}}$) and from equation (22e) it is obvious that $p_{33} = \tau[-1 + \sqrt{1 + (2\tau\sqrt{q_{22}} + q_{33})}]$. Summarizing, the solution to the Algebraic Riccati Equation (13) is

$$\check{P} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{q_{22}[1+(2\tau\sqrt{q_{22}}+q_{33})]} & \tau\sqrt{q_{22}} \\ 0 & \tau\sqrt{q_{22}} & \tau[-1+\sqrt{1+(2\tau\sqrt{q_{22}}+q_{33})}] \end{bmatrix}, \tag{23}$$

which is positive semi-definite.

Hence for $q_{22} = q_{33} = 1$, and $\tau = 1$ we find

$$\check{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We now assume that $\tau = 1$ and we choose P_0 as

$$P_0 = \begin{pmatrix} 3\nu^2 & 2\nu & \nu \\ 2\nu & 2 & 1 \\ \nu & 1 & 1 \end{pmatrix},$$

It is easy to see that $P_0 \geq 0$ for all $\nu \in \mathbb{R}$. Furthermore,

$$\begin{aligned} &\check{A}^*(z)P_0 + P_0\check{A}(z) - P_0\check{B}(z)\check{B}^*(z)P_0 \\ &= \begin{pmatrix} 0 & 3\nu^2(z^{-1}-1) & \nu \\ 3\nu^2(z-1) & 2\nu(z+z^{-1}-2) & \nu(z-1)+1 \\ \nu & \nu(z^{-1}-1)+1 & 0 \end{pmatrix} \\ &- \begin{pmatrix} \nu^2 & \nu & \nu \\ \nu & 1 & 1 \\ \nu & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\nu^2 & -\nu+3\nu^2(z^{-1}-1) & 0 \\ -\nu+3\nu^2(z-1) & 2\nu(z+z^{-1}-2)-1 & \nu(z-1) \\ 0 & \nu(z^{-1}-1) & -1 \end{pmatrix}. \end{aligned} \tag{24}$$

Since the two by two lower block of this matrix is negative-definite for $z = 1$, we see that the whole matrix is negative definite when its determinant is non-positive. The determinant of the matrix in equation (24) equals

$$-\nu^3(10\nu+1)(z+z^{-1}-2)$$

Since $z + z^{-1} - 2 \leq 0$ for z on the unit circle we conclude that the matrix of equation (24) is non-positive if and only if $-0.1 \leq \nu \leq 0$.

Hence by now adding $\check{C}(z)^*\check{C}(z)$, it is easy to see whether the sufficient condition of Theorem 2.5 is satisfied. For instance this is the case for $\check{C}(z) = \begin{pmatrix} z^{-1}-1 & 0 & 0 \end{pmatrix}$.

Resuming, we can say that despite the fact that the vehicular platooning model is not exponentially stabilizable, the LQR problem is solvable (i.e. an optimal controller can be designed). In order to enable this design, the first state (concerning the inter-vehicle distance) should not be included in the objective function for optimization at least for $z = 1$.

IV. Conclusion

In this paper, we have considered the optimal control problem for spatially invariant systems that are not exponentially stabilizable. We have shown that a common model for an infinite string of vehicles does not satisfy the exponential stabilizability condition, indicating the practical significance of the problem studied. The lack of exponential stabilizability implies that the LQR problem will not be solvable if we consider full state measurements in the objective function. For non-full state measurements the LQR can be solvable. We showed that a necessary condition for the solvability of the LQR problem is that the "point-wise" system must be optimizable for every z on the unit circle. However, this condition is by no means sufficient. A sufficient condition for optimizability is given in terms of a Riccati inequality and subsequently applied to the vehicular platooning problem.

References

[1] H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.

[2] H.L. Trentelman, A.A. Stoorvogel and M. Hautus. *Control Theory for Linear Systems*, Springer-Verlag, London, 2001.

[3] J.L. Lions. *Contrôle Optimal de systèmes gouvernés par des équations and derivées partielles*, Dunod, Paris, 1968. (English Translation, Springer Verlag, Berlin, New York, 1971).

[4] R.F. Curtain and A.J. Pritchard. *Infinite-Dimensional Linear Systems Theory*, Lecture Notes in Control and Information Sciences. Vol. **8**, Springer Verlag, Berlin, 1978.

[5] A. Al Alam, A. Gattami, and K. Johansson. An experimental study on the fuel reduction potential of heavy duty vehicle platooning. In *13th International IEEE Conference on Intelligent Transportation Systems*, pp. 306–311, Madeira Island, Portugal, 2010.

[6] G. Naus, R. Vugts, J. Ploeg, M. van de Molengraft, and M. Steinbuch. String-stable CACC design and experimental validation: A frequency-domain approach. *IEEE Transactions on Vehicular Technology*, 59(9):4268–4279, November 2010.

[7] J. Ploeg, N. van de Wouw, and H. Nijmeijer. Lp string stability of cascaded systems: Application to vehicle platooning, *IEEE Transactions on Control Systems Technology*, Accepted, 2013.

[8] B. Bamieh, F. Paganini, and M. Dahleh. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, 47(7):1091–1107, July 2002.

[9] R. Curtain, O.V. Iftime, and H. Zwart. System theoretic properties of a class of spatially invariant systems. *Automatica*, 45(7):1619–1627, 2009.

[10] M.R. Jovanović and B. Bamieh. On the ill-posedness of certain Vehicular platoon control problems. *IEEE Transactions on Automatic Control*, 50(9): 1307–1321, September 2005.

[11] R.F. Curtain, On some aspects of platoon control problems, *Proceedings of the 48th IEEE Conference on Decision and Control, 2009*, pp. 2357–2362, 2009.

[12] R.F. Curtain, Comments on "Distributed control of spatially invariant systems", *IEEE Trans. Automat. Control*, 56(3): 707–710, 2011.

[13] R. Curtain and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[14] J. Ploeg, B. Scheepers, E. van Nuenen, N. van de Wouw, and H. Nijmeijer. Design and experimental evaluation of cooperative adaptive cruise control. *14th International IEEE Conference on Intelligent Transportation Systems*, Washington, October 2011.