Sensitivity analysis of hybrid systems with state jumps with application to trajectory tracking

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Abstract—This paper addresses the sensitivity analysis for hybrid systems with discontinuous (jumping) state trajectories. We consider state-triggered discontinuities in the state evolution, potentially accompanied by mode switching in the control vector field. For a given trajectory with state jumps, we show how to construct an approximation of the nearby perturbed trajectory corresponding to a given variation of the initial condition and input signal. A major complication in the construction of such an approximation is that, in general, the jump times corresponding to a nearby perturbed trajectory are not equal to those of the nominal one. The main contribution of this work is the development of a notion of error to clarify in which sense the approximate trajectory is, at each instant of time, a first-order approximation of the perturbed trajectory. This notion of error naturally finds application in the (local) tracking problem of a time-varying reference trajectory of a hybrid system. To illustrate the possible use of this new error definition in the context of trajectory tracking, we outline how the standard linear trajectory tracking control for nonlinear systems could be generalized for hybrid systems.

I. INTRODUCTION

Sensitivity analysis allows to predict the modification of a trajectory due to (small) changes in initial conditions and parameters. This computational technique has proven beneficial in many aspects of the analysis of dynamical systems. In this paper, we pursue such sensitivity analysis for a class of hybrid systems. A hybrid system is a dynamic system that exhibits both continuous and discrete dynamic behaviors [8]. It is this inherent nature of hybrid systems that makes sensivity analysis for this class of systems harder than for smooth nonlinear dynamics, where sensitivity analysis is well established (see, e.g., [12, Chapters 3]). The theoretical framework provided by hybrid systems with (state-triggered) jumps is suitable to model those systems which, at certain instants of time, are subjected to rapid and abrupt changes. Indeed, in the modeling of such systems, it is frequently convenient and valid to neglect the durations of these rapid changes and to assume that these changes can be represented by instantaneous state jumps. Hybrid systems with statetriggered jumps are, for example, suitable to describe dynamical models in the area of robotics and rigid body mechanics with unilateral contact constraints [14], [3], including the study of the dynamics and control of walking or juggling robots (see, e.g, [28], [23], [19]).

The sensitivity result presented in this paper draws from the application of classical sensitivity and perturbation theory of nonlinear systems (see, e.g., [12, Chapters 3 and 10]) combined with the use of the implicit function theorem to compute an estimate of the unknown switching time at which the perturbed trajectory jumps.

We are not the first authors to propose to do so. The same mathematical tools have been combined in the investigation of the sensitivity about a nominal trajectory of piecewisesmooth nonlinear systems in [15], where the concept of saltation matrix was introduced: in [15], however, discontinuities in the state evolution are not considered. Another interesting use of classical perturbation theory in combination with the implicit function theorem is provided in [11] where, although state jumps are not considered, it is recognized that part of the analysis could be carried on even in the presence of discontinuities in the state evolution. In [18, Section 6.4], one finds an interesting discussion regarding the sensitivity of hybrid systems in the context of numerical optimal control of mechanical systems. There, state jumps are specifically taken into account and the reader is referred to [27, Section 2.2] that, in turn, refers to [13] (in German) as the source of a key formula for defining the sensitivity of hybrid systems with state jumps. We will re-establish this key formula for the class of hybrid systems considered in this work. This formula allows to compute the gain - the jump gain given by (27) in Section II in this paper - associated with a discontinuous event. Continuing with our literature review on sensitivity analysis for hybrid system with state jumps, another interesting discussion is presented in [6]. There, hybrid systems with jumps obtained by combining multiple differential algebraic equations (DAEs) with switching conditions and reset maps are considered. That paper also mentions that the jump gain (27) should, in fact, be credited to the (seemingly forgotten) seminal work of Rozenvasser [24, equation (11)]. For completeness, we also mention that a formula *related* to (27) can be encountered in the context of discontinuity induced bifurcations for hybrid systems. The interested reader is referred to [4, Section 2.5.1] and the work in [20], where the concept of discontinuity map (for transversal intersections) is introduced.

We leave to historians the settling of the question on who discovered (27) first. Here, we limit ourselves to mentioning that the jump gain (27) is indeed a key result that we, as other researcher before us, rediscovered autonomously. Our goal in this paper is to rigorously define what a first-order approximation about a nominal trajectory of a hybrid system with state jumps is on the basis of the result on the jump

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gain. By doing so, a novel notion of error, which allows to locally compare a nominal trajectory with a perturbed trajectory, emerges. This error notion, in turn, leads naturally to obtain a *trajectory tracking controller to locally stabilize time-varying trajectories for hybrid systems with state jumps*. This is, in our view, the main novelty and contribution of this paper. Since the proper understanding of the jump gain (27) is a key ingredient in such developments, we also include a (re)derivation of this result in this work.

Tracking control for hybrid systems with state-triggered jumps is a recent and active field of research. Few results exist to design a controller to make a hybrid system with state-triggered jumps track a given, time-varying, reference trajectory. Recent techniques addressing this control problem both from a theoretical and an experimental viewpoint are provided in [17], [22], [21], [16], [5], and [1]. Our interest lies on the situation, commonly encounter in practice, where the jump times of plant and reference trajectories cannot be assumed to coincide.

Aiming at developing an effective trajectory tracking controllers for hybrid systems with state-triggered jumps, we propose to investigate the effects on a nominal trajectory of variations of the control input and initial conditions. In particular, we detail how to construct a linear approximation of the hybrid system about a nominal trajectory with jumps and then show how to use this approximation to construct a local trajectory tracking controller. To the best of our knowledge, the notion of linear approximation that we introduce in this paper has not be presented before.

The non-trivial aspect of the problem is how to construct the approximation of the perturbed trajectory as the sum of the nominal trajectory and a linear term when the perturbed and nominal trajectories jump at different, although close, time instants. This difference in the jump times poses also the problem of defining a proper notion of tracking error.

Mimicking what is done for dynamical systems with no jumps, the most intuitive definition of tracking error is the difference between the nominal and actual state. However, e.g., as illustrated in [1], this definition of error cannot be used to conclude stability in the sense of Lyapunov and usually leads to poor tracking performance.

For this reason, different approaches have been proposed in recent years to redefine the notion of tracking error for hybrid systems. In [7], e.g., the tracking problem has been defined in order to neglect in the analysis the times belonging to infinitesimal intervals about the jumping times. In [1], instead, a novel definition of the notion of distance between two jumping trajectories has been proposed. Moreover, for a subclass of hybrid system with state-triggered jumps corresponding to mechanical systems with fully elastic impacts, the tracking error distance has been defined as the minimum between the distance of state from the nominal trajectory and the distance of state a mirrored version of nominal trajectory. In [2], a modification of this mirroring approach has been consider to deal with dissipative impacts. The use of a mirror reference trajectory has been proposed recently also in [5] for a tracking problem in polyhedral billiards: the controller in

this case may decide to track either the real reference or the mirrored reference, mirrored through the billiard boundary.

We claim that the approximation proposed in this paper allows for a reinterpretation of the mirroring technique in terms of what we will call *extended ante- and post-event trajectories*. Furthermore, we care to emphasize the fact that the concept of extended ante- and post-event trajectories allows to cope with the problem of trajectory tracking for a hybrid-system with state-triggered jumps where the (continuous-time) dynamics before and after the jump event are qualitatively different, where simply mirroring the reference trajectory does not appear to be the best choice. This is of clear relevance for the applications we have in mind: the control of mechanical system undergoing repetitive, not necessarily periodic, impacts with rigid surfaces. A common situation for walking and jumping robots.

This paper is organized as follows. In Section II, we discuss the jump gain associated with a nominal trajectory of a hybrid system and introduce the notion of error between the nominal and perturbed trajectories of hybrid system with state jumps. This notion of error is used in Section III to propose a linear feedback control law for local trajectory tracking of time-varying reference trajectories with jumps. Conclusions are finally drawn in Section IV.

II. SENSITIVITY ANALYSIS FOR HYBRID SYSTEMS

In this section, we propose a framework for the sensitivity analysis of a jumping solution of a hybrid system with statetriggered jumps. To focus on the complexity and effect of a state jump on the sensitivity, we consider a nominal trajectory with a single jump. The treatise of sensitivity analysis of solutions with multiple jumps is left for future work.

Consider a smooth time-varying control vector field

$$\dot{x}(t) = f^a(x, u, t), \tag{1}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$. For reasons that will soon become clear, we will refer to f^a as the *ante-event* control vector field. For a given initial condition x_0 at time t_0 and a integrable signal $\mu(t) \in \mathbb{R}^m$, $t \in [t_0, t_1]$, we denote with $\alpha^a(t)$, $t \in [t_0, \tau]$, the solution of (1) with input

$$u(t) = \mu(t), \quad t \in [t_0, t_1],$$
 (2)

up to the occurrence of a triggering *event* at time $\tau \in [t_0, t_1]$. This event is defined by the satisfaction of implicit condition

$$g(\alpha^a(\tau), \tau) = 0, \tag{3}$$

with $g: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ a *smooth* real-valued function. We assume that, for all t, the level set $g(\cdot, t) = 0$ is an n-1dimensional smooth manifold embedded in \mathbb{R}^n , (a sufficient condition being $\partial g(x,t)/\partial x \neq 0$ for all x such that g(x,t) =0). At the event time τ , the state exhibits a *jump* according to a smooth single-valued *impulse* map $\Delta: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

$$\alpha^{p}(\tau) = \alpha^{a}(\tau) + \Delta(\alpha^{a}(\tau), \tau), \qquad (4)$$

and subsequently it evolves according to the following *post*event vector field

$$\dot{x}(t) = f^p(x, u, t) \tag{5}$$

with initial condition at $x(\tau) = \alpha^p(\tau)$ and input (2). We indicate with $\alpha^p(t), t \in [\tau, t_1]$, the post-event trajectory.

Ante- and post-event trajectories can be glued together by "appending" to ante-event trajectory α^a the post-trajectory α^p , obtaining the trajectory

$$\alpha(t) := \begin{cases} \alpha^a(t), & t \in [t_0, \tau) \\ \alpha^p(t), & t \in [\tau, t_1]. \end{cases}$$
(6)

The state trajectory α is (generally) not continuous, due to the state jump caused by the impulse map Δ . Without loss of generality, α is by construction right continuous. Together with the nominal input $\mu(t)$, the state trajectory $\alpha(t)$ forms what we term the *nominal state-input trajectory* $\xi(t) = (\alpha(t), \mu(t)), t \in [t_0, t_1].$

We are interested in defining and computing the sensitivity of $\xi(t)$ to small variations of the initial condition x_0 and input $\mu(t), t \in [t_0, t_1]$. To this end, we will perturb the initial condition in the direction $z_0 \in \mathbb{R}^n$ and the nominal input curve μ in the direction v, where v denotes an integrable curve $v(t) \in \mathbb{R}^m, t \in [t_0, t_1]$. Hence, the perturbed initial condition and input curve are defined, respectively, as

$$x_{\varepsilon}(t_0) = x_0 + \varepsilon z_0 \tag{7}$$

$$u_{\varepsilon}(t) = \mu(t) + \varepsilon v(t), \quad t \in [t_0, t_1],$$
(8)

with $\varepsilon \in \mathbb{R}$ typically small. The corresponding state-input trajectory will be denoted $\xi_{\varepsilon}(t) = (x_{\varepsilon}(t), u_{\varepsilon}(t)), t \in [t_0, t_1]$. We aim at defining a notion of sensitivity to predict, for small values of ε , the effect of perturbations to the initial condition and the nominal input on a jumping solution of the hybrid system.

The perturbed trajectory $\xi_{\varepsilon} = (x_{\varepsilon}, u_{\varepsilon})$ is defined, similarly to the nominal trajectory $\xi = (\alpha, \mu)$, by appending to the *perturbed* ante-event trajectory x_{ε}^a the the *perturbed* post-event trajectory x_{ε}^p . The event time of the perturbed trajectory, that we denote τ_{ε} , typically varies as ε changes. Indeed, this perturbed event time is implicitly defined by the condition

$$g(x_{\varepsilon}^{a}(\tau_{\varepsilon}), \tau_{\varepsilon}) = 0.$$
(9)

The above equation is satisfied, for $\varepsilon = 0$, by the nominal ante-event trajectory α^a at the nominal event time τ : see (3).

In order to ensure that $\varepsilon \mapsto \tau_{\varepsilon}$ is, about $\varepsilon = 0$, a continuously differentiable function the following assumption needs to be fulfilled by the nominal state-input trajectory ξ .

Assumption 2.1: The nominal state-input trajectory satisfies the following *transversality* condition

$$D_1g \cdot f^a + D_2g \cdot 1 \neq 0, \tag{10}$$

where g is evaluated at $(\alpha^a(\tau), \tau)$ and the ante-event vector field f^a at $(\alpha^a(\tau), u(\tau), \tau)$. The notation D_1 and D_2 denote, respectively, the partial differentiation with respect the first and second arguments.

The above transversality condition is a common requirement when dealing with hybrid system with state-triggered jumps (see, e.g., [19]). Its role in the sensitivity analysis will become clear in the proof of the theorem presented later on in this section. For $t = \tau_{\varepsilon}$, the value of x_{ε}^{p} is computed using the impulse map Δ , similarly as done in (4). Therefore, we can formulate the following conditions that the perturbed ante- and postevent trajectories have to satisfy:

$$x_{\varepsilon}^{a} = x_{0} + \varepsilon z_{0}, \qquad \qquad t = t_{0} \tag{11}$$

$$\dot{x}^a_{\varepsilon} = f^a(x^a_{\varepsilon}, u_{\varepsilon}, t), \qquad t \in [t_0, \tau_{\varepsilon}] \qquad (12)$$

$$x_{\varepsilon}^{p} = x_{\varepsilon}^{a} + \Delta(x_{\varepsilon}^{a}, t), \qquad t = \tau_{\varepsilon}$$
(13)

$$\dot{x}^p_{\varepsilon} = f^p(x^p_{\varepsilon}, u_{\varepsilon}, t), \qquad t \in [\tau_{\varepsilon}, t_1].$$
(14)

One important observation is that, although naturally defined in the time intervals $[t_0, \tau_{\varepsilon}]$ and $[\tau_{\varepsilon}, t_1]$, both the anteand post-event trajectories x_{ε}^a and x_{ε}^p can individually be extended over the whole time interval $[t_0, t_1]$ by, respectively, forward and backward time integration starting from τ_{ε} . We will denote those extensions as $\bar{x}_{\varepsilon}^a(t)$ and $\bar{x}_{\varepsilon}^p(t)$, $t \in [t_0, t_1]$. These extended ante- and post-event perturbed trajectories satisfy

$$\bar{x}^a_{\varepsilon} = x_0 + \varepsilon z_0 \qquad \qquad t = t_0, \tag{15}$$

$$\bar{x}_{\varepsilon}^{p} = f^{u}(\bar{x}_{\varepsilon}^{a}, u_{\varepsilon}, t), \qquad t \in [t_{0}, t_{1}], \tag{16}$$

$$\bar{x}_{\varepsilon}^{p} = \bar{x}_{\varepsilon}^{a} + \Delta(\bar{x}_{\varepsilon}^{a}, t), \qquad t = \tau_{\varepsilon}, \qquad (17)$$

$$\bar{x}_{\varepsilon}^{p} = f^{p}(\bar{x}_{\varepsilon}^{p}, u_{\varepsilon}, t), \qquad t \in [t_{0}, t_{1}], \qquad (18)$$

where the perturbed event time τ_{ε} is implicitly defined by the condition

$$g(\bar{x}^a_{\varepsilon}(\tau_{\varepsilon}), \tau_{\varepsilon}) = 0.$$
(19)

This apparently innocuous extension is the cornerstone to understand the sensitivity differential equation.

Figure 1 gives an indication on why in general we cannot expect to be able to write the perturbed trajectory x_{ε} as

$$x_{\varepsilon}(t) = \alpha(t) + \varepsilon z(t) + o(\varepsilon)$$

with z the solution to an appropriately defined time-varying linear system. The obstacle is represented by the difference in the event times for x_{ε} and α . The following theorem shows how to overcome this difficulty, defining the extended ante- and post-event linearization trajectories \bar{z}^a and \bar{z}^p and defining the expansion about the extended ante- and post-event trajectories $\bar{\alpha}^a$ and $\bar{\alpha}^p$ instead of simply about α .

Theorem 2.1: Consider a nominal state-control trajectory $\xi(t) = (\alpha, \mu)(t), t \in [t_0, t_1]$, of system (1), (3), (4), (5) with nominal event time $\tau \in [t_0, t_1]$ and associated extended ante- and post-event trajectories $\bar{\alpha}^a(t)$ and $\bar{\alpha}^p(t), t \in [t_0, t_1]$. Adopt Assumption 2.1. The perturbed state trajectory $x_{\varepsilon}(t)$, $t \in [t_0, t_1]$, corresponding to perturbations in the initial condition and input as in (7), satisfies

$$x_{\varepsilon}(t) = \begin{cases} \bar{\alpha}^{a}(t) + \varepsilon \bar{z}^{a}(t) + o(\varepsilon), & t < \tau_{\varepsilon} \\ \bar{\alpha}^{p}(t) + \varepsilon \bar{z}^{p}(t) + o(\varepsilon), & t \ge \tau_{\varepsilon}. \end{cases}$$
(20)

where the extended ante- and post-event linearization trajectories $\bar{z}^a(t)$ and $\bar{z}^p(t)$, $t \in [t_0, t_1]$, are computed as

$$\bar{z}^a = z_0, \qquad \qquad t = t_0 \qquad (21)$$

$$\dot{\bar{z}}^a = A^a(t)\bar{z}^a + B^a(t)v$$
 $t \in [t_0, t_1]$ (22)

$$\bar{z}^p = \bar{z}^a + H(\tau)\bar{z}^a \qquad \qquad t = \tau \qquad (23)$$

$$\dot{\bar{z}}^p = A^p(t)\bar{z}^p + B^p(t)v$$
 $t \in [t_0, t_1]$ (24)



Fig. 1. The ante- and post-event nominal trajectories α^a and α^p , perturbed trajectories x^a_{ε} and x^p_{ε} , and the linear approximations $\alpha^a + \varepsilon z^a$ and $\alpha^p + \varepsilon z^p$. The extended trajectories, indicated with a bar sign, are depicted using dashed lines.

where

$$A^{s}(t) = D_{1}f^{s}(\bar{\alpha}^{s}(t), \mu(t), t)$$
(25)

$$B^{s}(t) = D_{2} f^{s}(\bar{\alpha}^{s}(t), \mu(t), t)$$
(26)

with $s = \{a, p\}$ and

$$H(\tau) = \frac{f^+ - f^- - \dot{\Delta}^-}{\dot{g}^-} D_1 g^- + D_1 \Delta^-, \qquad (27)$$

where

$$\alpha^{p}(\tau) = \alpha^{a}(\tau) + \Delta(\alpha^{a}(\tau), \tau)$$
(28)

$$f^+ = f^p(\alpha^p(\tau), \mu(\tau), \tau), \tag{29}$$

$$f^{-} = f^{a}(\alpha^{a}(\tau), \mu(\tau), \tau), \qquad (30)$$

$$\Delta^{-} = D_1 \Delta^{-} \cdot f^{-} + D_2 \Delta^{-} \cdot 1 \tag{31}$$

$$\dot{g}^{-} = D_1 g^{-} \cdot f^{-} + D_2 g^{-} \cdot 1 \tag{32}$$

$$D_k \Delta^- = D_k \Delta(\alpha^a(\tau), \tau), \qquad \qquad k = \{1, 2\}$$
(33)

$$D_k g^- = D_k g(\alpha^a(\tau), \tau), \qquad \qquad k = \{1, 2\}.$$
 (34)

Proof: The extended ante- and post-event perturbed trajectories satisfy (15)-(18) where the perturbed event time τ_{ε} is implicitly defined by the condition (19). On the basis of Assumption 2.1, the implicit function theorem allows to conclude the existence of a unique event time τ_{ε} for small values of ε . Furthermore, as (19) is identically zero for each ε in a neighbourhood of zero, the derivative of (19) with respect to ε allows to obtain a linear approximation of τ_{ε} as a function of ε , namely to compute $\tau'_0 := \partial \tau_{\varepsilon} / \partial \varepsilon|_{\varepsilon=0}$ and approximate τ_{ε} as

$$\tau_{\varepsilon} = \tau + \varepsilon \, \tau_0' + o(\varepsilon). \tag{35}$$

The formula to compute τ'_0 will be obtained at the end of the proof.

The sensitivity equations for the extended ante- and postevent trajectories \bar{x}^a and \bar{x}^p are the standard (see, e.g., [12, Chapter 10]) sensitivity equations given by (22) and (24). The missing link is how to relate \bar{z}_p to \bar{z}_a , that is, to show that \bar{z}_p is indeed reinitialized as in (23) at the *nominal* event time τ using the linear map H given by (27).

Expanding $\bar{x}_{\varepsilon}^{p}(\tau_{\varepsilon})$ in series with respect to ε results in

$$\begin{split} \bar{x}_{\varepsilon}^{p}(\tau_{\varepsilon}) &= \bar{\alpha}^{p}(\tau_{\varepsilon}) + \varepsilon \bar{z}^{p}(\tau_{\varepsilon}) + o(\varepsilon) \\ &= \alpha^{p}(\tau) + \varepsilon \dot{\alpha}^{p}(\tau) \tau_{0}' + \varepsilon \bar{z}^{p}(\tau_{\varepsilon}) + o(\varepsilon) \\ &= \alpha^{p}(\tau) + \varepsilon (\dot{\alpha}^{p}(\tau) \tau_{0}' + \bar{z}^{p}(\tau)) + o(\varepsilon). \end{split}$$
(36)

To obtain the above expression, $\bar{\alpha}^p(\tau_{\varepsilon})$ has been approximated by linear extrapolation using the value and the time derivative of $\bar{\alpha}^p$ at time τ and τ_{ε} has been approximated using (35). Note that, as $\bar{\alpha}^p$ and its time derivative are then evaluated at τ , there is no need for the overline sign because α^p is always defined at τ . We have then discarded the terms of order higher than one in $\varepsilon \bar{z}^p(\tau_{\varepsilon})$ obtaining $\varepsilon \bar{z}^p(\tau)$. A similar expansion can be computed for $\bar{x}^a_{\varepsilon}(t_{\varepsilon})$ obtaining

$$\bar{x}^a_{\varepsilon}(\tau_{\varepsilon}) = \alpha^a(\tau) + \varepsilon(\dot{\alpha}^a(\tau)\tau'_0 + \bar{z}^a(\tau)) + o(\varepsilon).$$
(37)

Using (37) and (35), $\Delta(\bar{x}^a_{\varepsilon}(\tau_{\varepsilon}), \tau_{\varepsilon})$ appearing in (17) can be expanded as

$$\Delta(\bar{x}^{a}_{\varepsilon}(\tau_{\varepsilon}), \tau_{\varepsilon}) = \Delta(\alpha^{a}(\tau), \tau) + \varepsilon D_{1}\Delta(\alpha^{a}(\tau), \tau) \cdot (\dot{\alpha}^{a}(\tau) \tau_{0}' + \bar{z}^{a}(\tau)) + \varepsilon D_{2}\Delta(\alpha^{a}(\tau), \tau) \cdot \tau_{0}' + o(\varepsilon).$$
(38)

Using (37) and (35), (19) can be expanded as

$$g(\bar{x}^{a}_{\varepsilon}(\tau_{\varepsilon}), \tau_{\varepsilon}) = \varepsilon D_{1}g(\alpha^{a}(\tau), \tau) \cdot (\dot{\alpha}^{a}(\tau) \tau_{0}' + \bar{z}^{a}(\tau)) + \varepsilon D_{2}g(\alpha^{a}(\tau), \tau) \cdot \tau_{0}' + o(\varepsilon).$$
(39)

As the above expression is identically zero for every ε , we get

$$\tau_0' = -\frac{D_1 g(\alpha^a(\tau), \tau) \cdot \bar{z}^a(\tau)}{D_1 g(\alpha^a(\tau), \tau) \cdot \dot{\alpha}^a(\tau) + D_2 g(\alpha^a(\tau), \tau) \cdot 1}$$
$$= -\frac{D_1 g^- \cdot \bar{z}^a(\tau)}{\dot{g}^-} \tag{40}$$

where \dot{g}^- and D_1g^- are defined, respectively, as in (32) and (34). The expression for (40) is valid as long as \dot{g}^- is different from zero, that is, if Assumption 2.1 is satisfied.

Making use of the series expansions (36), (37), and (38), we can match the first-order terms of (17), obtaining, at the event time τ ,

$$\dot{\alpha}^{p}\tau_{0}^{\prime} + \bar{z}^{p} = \dot{\alpha}^{a}\tau_{0}^{\prime} + \bar{z}^{a} + D_{1}\Delta(\alpha^{a},\tau) \cdot (\dot{\alpha}^{a}\tau_{0}^{\prime} + \bar{z}^{a}) + D_{2}\Delta(\alpha^{a},\tau) \cdot \tau_{0}^{\prime}.$$
(41)

The above expression can be rearranged as

$$\bar{z}^p = \bar{z}^a - (f^+ - f^- - \dot{\Delta}^-) \,\tau_0' + D_1 \Delta^- \cdot \bar{z}^a \qquad (42)$$

where f^+ , f^- , $\dot{\Delta}^-$, and $D_1\Delta^-$ are defined, respectively, as in (29), (30), (31), and (33). Substituting in the above equation the expression for τ'_0 given in (40), we obtain (23) and in particular the expression for H provided in (27). This concludes the proof of the theorem.

Remark. Note that the reset map (23) is linear in \overline{z}^a and the reset occurs at the nominal event time τ .

As mentioned in the introduction, (27) is a (uncommonly) known expression in the context of numerical optimal control [27], [18], and parametric sensitivity for hybrid systems [24], [6]. It is also strictly related to equation (57) that appears in [11] and can also be interpreted as a generalization, for piecewise-smooth nonlinear systems with state jumps, of the saltation matrix introduced in [15] (in [15], Δ is identically equal to zero as there is no state jump).

Our contribution lies in the use of extended ante- and postevent trajectories to achieve the first-order approximation presented in (20). To the best of our knowledge, this approximation and the use extended anti- and post-event trajectories is new.

It is worth mentioning that [11, Appendix A] suggests "a procedure for refining the estimate of the perturbed trajectory" in a neighbourhood of the event time. Non surprisingly, the need for this refining is due to the difference between the nominal and perturbed trajectory event times (respectively, τ and τ_{ε} , according to our notation). Note, however, that the approximation in [11] differs from the one that we propose here as it is not based on using the extended ante- and postevent trajectories and therefore is not an $o(\varepsilon)$ approximation of the nominal trajectory.

The approximation (20) is key to address the problem of (local) trajectory tracking for systems with state-triggered jumps, as we discuss in the following section.

III. TRAJECTORY TRACKING OF A TIME-VARYING REFERENCE TRAJECTORY

Let $\xi = (\alpha, \mu)$ be a nominal state-input trajectory for a hybrid system characterized by ante- and post-event vector fields f^a and f^p . As done in the previous section, τ will indicate the nominal event time and $\bar{\alpha}^a$ and $\bar{\alpha}^p$ the extended ante- and post-event trajectories, respectively.

Consider the following state feedback control law

$$u = \begin{cases} \mu + K(t)(\bar{\alpha}^a - x), & \text{before event detection} \\ \mu + K(t)(\bar{\alpha}^p - x), & \text{after event detection} \end{cases}$$
(43)

where $K(t) \in \mathbb{R}^{m \times n}$ is a time-varying matrix gain to be designed. In (43), by *event detection* we mean the satisfaction of the condition g(x(t), t) = 0 for the current value of the state at time t, which is generally not equal to the nominal event time τ . En passant, as in the example of tracking control for a bouncing ball proposed in [1], we mention that event detection is not strictly needed to implement a switching feedback law as (43). Indeed, due to the discontinuity in the nominal state trajectory, an equivalent result might be obtained by simply choosing, between $\bar{\alpha}^a - x$ and $\bar{\alpha}^p - x$, the one with minimum norm.

Remark. Strictly speaking, in [1], the nominal trajectory $\alpha(t)$ and its negative version $-\alpha(t)$ are used in place of $\bar{\alpha}^{a}(t)$ and $\bar{\alpha}^{p}(t)$. For a bouncing ball impacting without energy loss on a surface (located at position zero), the use of the mirror trajectory $-\alpha(t)$ can be justified within our framework observing that $-\alpha(t)$ corresponds to choosing $\bar{\alpha}^p$ before the nominal event time and to $\bar{\alpha}^a$ after the nominal event time, so that an equivalent switching law to (43) is obtained. When the bouncing is not elastic, $-\alpha(t)$ is no longer a good representative of the extended behavior and needs to be corrected. Indeed, in [2], the case of non-elastic impact is considered and the mirrored nominal trajectory is corrected via the use of the impact restitution coefficient. Again, this can be interpreted as the need to obtain (an estimate of) the extended nominal trajectories $\bar{\alpha}^a(t)$ and $\bar{\alpha}^{p}(t)$ for properly defining the notion of tracking error to deal with the difference between the nominal and perturbed event times. A similar remark applies for the mirroring technique presented in [5]. \square

Our goal in this section is to discuss why (43) is a suitable choice to design a trajectory tracking controller (assuming ξ is of infinite extent) and suggest how the time-varying gain K could be designed. Using (43), we obtain the following closed-loop ante- and post-event vector fields

$$f^{a}_{cl}(x,t) := f^{a}(x,\mu + K(t)(\bar{\alpha}^{a}(t) - x), t), \qquad (44)$$

$$f_{cl}^{p}(x,t) := f^{p}(x,\mu + K(t)(\bar{\alpha}^{p}(t) - x), t).$$
(45)

By construction, the resulting hybrid system (with no inputs) has α as nominal trajectory and, consequently, the nominal switching time remains τ . The sensitivity analysis developed in Section II leads to the following state matrices for the anteand post-event linearization of (44)-(45):

$$A^{a}_{cl}(t) = A^{a}(t) - B^{a}(t)K(t),$$
(46)

$$A_{cl}^{p}(t) = A^{p}(t) - B^{p}(t)K(t).$$
(47)

The input matrices B_{cl}^a and B_{cl}^p are zero as (44) and (45) have no input. Finally, the gain H, computed using (27), is by construction equal to the one associated to the nominal open-loop trajectory $\xi = (\alpha, \mu)$. This concludes the derivation of the extended linearization as discussed in Theorem 2.1 for the closed-loop dynamics (44)-(45).

In virtue of Theorem 2.1 and, in particular, of the approximation (20), one can reasonably expect to be able to shape the *local* behavior of the closed-loop response of the hybrid system about the nominal trajectory ξ by choosing suitably the matrix gain K(t) in (43).

We expect that this will be related to controllabilitylike assumptions on the ante- and post-event linearizations associated to the time-varying matrices A^a , B^a , A^p , B^p , as well as the jump gain H given in (27). Furthermore, a modification of the standard linear quadratic regulator (LQR) problem for linear system with fixed-time jump-gain-induced state jumps could be used to design the gain K in an optimal manner. For the sake of brevity, we leave to future publications the burden of filling in the gaps by providing a mathematical proof that the proposed approach does work and demonstrating via numerical simulations and physical experiments that the control strategy is indeed effective.

IV. CONCLUSIONS

This paper addresses the sensitivity analysis of hybrid systems with discontinuous state trajectories. We developed a novel notion of error to obtain, at each instant of time, a first-order approximation of the change in a trajectory due to small changes in initial conditions and inputs. This notion of error naturally finds application in the local tracking problem of a time-varying reference trajectory of a hybrid system. We outlined how the standard linear trajectory tracking control for nonlinear systems can be generalized for hybrid systems, leaving for future investigation the generalization of linear quadratic regulator (LQR) theory to compute the optimal feedback gain in this context. We highlighted the connection between the switching linear feedback law that we propose with the idea of trajectory mirroring recently appeared in the literature.

The notion of error developed in this paper opens the possibility of further developing perturbation analysis in the context of hybrid systems. Our current efforts are directed toward the development of a second-order approximation to be used in the context of numerical optimal control [25], [10], [9] as well as the investigation of sensitivity analysis for mechanical systems with changing state dimension such as, e.g., the robotic hopping leg described in [26].

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