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Sampled-data extremum-seeking control for optimization of constrained dynamical systems using barrier function methods

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Abstract— **Most extremum-seeking control approaches focus solely on the problem of finding the extremum of some unknown, steady-state performance map. However, many industrial applications also have to deal with constraints on operating conditions due to, e.g., actuator limitations, limitations on design or tunable system parameters, or constraints on measurable signals. These constraints, which can be unknown a-priori, may conflict with the otherwise optimal operational condition, and should be taken into account in performance optimization. In this work, we propose a sampled-data extremum-seeking approach for optimization of constrained dynamical systems using barrier function methods, where both the objective function and the constraint function are available through measurement only. We show that, under the assumption that initialization does not violate constraints, the interconnection between a constrained dynamical system and optimization algorithms that employ barrier function methods is stable, the constraints are satisfied, and optimization is achieved. We illustrate the results by means of a numerical example.**

I. INTRODUCTION

Optimization of complex nonlinear systems is often a challenging task. Namely, most (numerical) optimization techniques usually rely on an accurate model of the process to be optimized, while such a model can be hard or impossible to obtain for complex nonlinear systems. Nevertheless, the steady-state input-output behavior of many of such systems possesses optimal performance under particular operating conditions and we often desire to find such optimal operating conditions. Based solely on output measurements, *extremum-seeking control* is able to optimize such systems in real-time by adjusting these operating conditions and drive the system into a vicinity of the optimal steady-state input-output behavior [1], [2].

Along with the pioneering work done in [1] on convergence proofs for continuous-time extremum-seeking schemes based on sinusoidal perturbations, a notable contribution to the field of extremum-seeking control was made in [2]. In [2], it was shown that under assumptions on the asymptotic stability of both the system and a discrete-time nonlinear programming method, extremum seeking can be achieved within a periodic sampled-data framework. This framework allows the use of a wide class of smooth and nonsmooth optimization algorithms for achieving optimization of general nonlinear systems. In [15], closed-loop stability of the sampled-data scheme is studied from an interconnected systems' theory point-of-view, in which stability results are obtained by imposing stronger conditions on the nonlinear programming methods than done in [2].

Notable extensions of the framework in [2] are provided in [6] and [4]. The work in [6] utilizes a trajectory-based approach to prove semi-global practical asymptotical stability of the proposed sampled-data extremum-seeking schemes as opposed to the Lyapunov-type arguments used in [2]. The former exploits the notion of multi-step consistency (see, e.g., [7]) while the latter exploits closeness of solutions of a differential inclusion over a single time step. The framework in [6] allows to use a broader class of optimization algorithms, such as those which do not admit a state-update realization and/or Lyapunov function. Subsequently in [4], the framework in [6] was extended to a more generic framework, which in addition to gradient-based optimization algorithms, also encompasses sampling-based methods capable of nonconvex optimization, enabling extremum seeking for an even wider class of problems. For example, in [5] and [8], two sampling-based algorithms are presented that are able to achieve (a weaker type of) convergence to a global optimum.

Most extremum-seeking approaches, whether it is of the continuous-time type as in [1] or the sampled-data type as in [2], [6], focus solely on the problem of finding the extremum of some unknown steady-state input-output map. However, many industrial applications also have to deal with constraints on operating conditions due to, e.g., actuator limitations, limitations on design or tunable system parameters, or constraints on measurable signals. These constraints may conflict with the otherwise optimal operational condition, and should be taken into account in the optimization procedure.

In terms of dealing with constraints in extremum-seeking schemes, existing approaches can be divided into two main categories: i) approaches that assume a-priori knowledge on constrained operating conditions in the form of explicit constraint functions, and ii) approaches that deal with *unknown* but *measurable constraint functions*. Extremum-seeking approaches that explicitly deal with known constraint functions are considered in, e.g., [10], [11], [17]. In [10] and [11], penalty/barrier functions are employed to adapt the search space so as not to violate the constraints. Another approach proposed in [10] employs an anti-windup scheme to prevent the optimizer from leaving the admissible

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search space. In [17], constraint satisfaction is achieved by employing a projection operator in the extremum-seeking scheme. Although not aimed at constrained optimization, the sampling-based algorithms in [5] and [8] operate in an a-priori defined compact set, i.e., these allow incorporation of known (input) constraints to adjust the search space.

In [12] and [13], extremum-seeking approaches for (strictly) convex optimization problems with unknown but measurable constraint functions are considered, albeit in the continuous-time extremum-seeking setting. In [12], a combined barrier/penalty function approach is employed to transform the constrained optimization problem into an unconstrained problem using an augmented cost. This method does not enforce strict satisfaction of the constraints to avoid difficulties associated with small violations of constraints and to relax the choice of initial, possibly inadmissible inputs. In [13], a combination of the classical extremum-seeking approach as in [1] and so-called saddle point algorithms as in [9] are used to find the constrained global minimizer.

In this work, we focus on sampled-data extremum-seeking schemes as studied in [2] and [15], as opposed to the continuous-time extremum-seeking schemes employed in, e.g., [12] and [13]. Sampled-data schemes are compelling given the potential of including diverse types of optimization algorithms [2], [4]. We adopt the class of optimizers from [15], where the search vector is more explicit. As argued in [15], it allows Lyapunov-type arguments to be imposed on the search vector to study closed-loop stability, which turns out to be especially suitable for optimization problems that employ barrier functions.

The main contribution of this work can be summarized as follows. First, we propose a sampled-data extremum-seeking approach for optimization of constrained dynamical systems using barrier function methods, where both the objective function and the constraint function are available through measurement only. Second, under the assumption that initialization does not violate constraints, we prove closed-loop stability of the interconnection between constrained dynamical systems and optimization algorithms that employ barrier function methods, and show strict constraint satisfaction. Third, we illustrate the working principle by means of a numerical example.

The paper is organized as follows. Section II presents the problem formulation. Section III presents a barrier function method for static, constrained optimization problems. The sampled-data extremum-seeking framework and class of algorithms are stated in Section IV. In Section V, a closed-loop stability analysis is provided. Section VI presents a numerical example. Section VII closes with conclusions.

II. PROBLEM FORMULATION FOR CONSTRAINED DYNAMICAL SYSTEMS

We consider multi-input multi-output dynamical systems of the following form:

$$\Sigma_p : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ y(t) = h(\mathbf{x}(t)) \\ \mathbf{z}(t) = \mathbf{g}(\mathbf{x}(t)), \end{cases} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$ denotes the state of system, $\mathbf{u} \in \Omega \subseteq \mathbb{R}^{n_\mathbf{u}}$ denotes the input of the system, $y \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{R}^{n_\mathbf{z}}$ are outputs of the system, and $t \in \mathbb{R}$ is time. In the context of extremum-seeking control, Σ_p represents the system to be optimized, where the input $\mathbf{u}$ can be regarded as a vector of tunable system parameters. We consider multiple outputs, separated into two output channels. In the context of optimization of constrained dynamical systems, we relate the output y to an unknown, but measurable, cost to be minimized, and the output $\mathbf{z}$ to unknown, but measurable, constraint functions. We adopt the following assumption on the system in (1).

Assumption 1. *There exists a continuous map denoted by $\bar{\mathbf{x}} : \Omega \to \mathbb{R}^{n_\mathbf{x}}$ such that $\mathbf{f}(\bar{\mathbf{x}}(\mathbf{u}), \mathbf{u}) = 0$ for all constant $\mathbf{u} \in \Omega$. Moreover, there exists $L_{\bar{\mathbf{x}}} \in \mathbb{R}_{>0}$ such that $\|\bar{\mathbf{x}}(\mathbf{u}_1) - \bar{\mathbf{x}}(\mathbf{u}_2)\| \leq L_{\bar{\mathbf{x}}} \|\mathbf{u}_1 - \mathbf{u}_2\|$ for any $\mathbf{u}_1, \mathbf{u}_2 \in \Omega$. Let us define $\tilde{\mathbf{x}} = \mathbf{x} - \bar{\mathbf{x}}(\mathbf{u})$. There exists a continuously differentiable function $V_{\Sigma_p} : \mathbb{R}^{n_\mathbf{x}} \to \mathbb{R}_{\geq 0}$ which is radially unbounded, such that:*

i) $V_{\Sigma_p}(\tilde{\mathbf{x}}) > 0 \ \forall \tilde{\mathbf{x}} \in \mathbb{R}^{n_\mathbf{x}} \backslash \{0\}$, $V_{\Sigma_p}(0) = 0$.
ii) $\exists \ \gamma > 0$ *such that* $\dot{V}_{\Sigma_p}(\tilde{\mathbf{x}}) = \nabla V_{\Sigma_p}(\tilde{\mathbf{x}}) \mathbf{f}(\tilde{\mathbf{x}} + \bar{\mathbf{x}}(\mathbf{u}), \mathbf{u}) \leq -\gamma V_{\Sigma_p}(\tilde{\mathbf{x}})$ *for all constant* $\mathbf{u} \in \Omega$ *and* $\tilde{\mathbf{x}} \in \mathbb{R}^{n_\mathbf{x}}$.

Remark 2. *Assumption 1 is a common assumption in the ESC literature on the system exhibiting globally exponentially stable equilibria for constant inputs, see, e.g., [1], [18].*

From Assumption 1, it follows that for any constant $\mathbf{u} \in \Omega$ we have the following steady-state input-output maps $Q : \Omega \to \mathbb{R}$ and $G : \Omega \to \mathbb{R}^{n_\mathbf{z}}$:

$$Q(\mathbf{u}) := h(\bar{\mathbf{x}}(\mathbf{u})) = \lim_{t \to \infty} h(\mathbf{x}(t)), \tag{2}$$

and

$$\mathbf{G}(\mathbf{u}) := \mathbf{g}(\bar{\mathbf{x}}(\mathbf{u})) = \lim_{t \to \infty} \mathbf{g}(\mathbf{x}(t)). \tag{3}$$

$Q(\mathbf{u})$ and $\mathbf{G}(\mathbf{u})$ are referred to as the objective function and the constraint function, respectively. The goal is to find the inputs $\mathbf{u}$ that minimize the objective function $Q(\mathbf{u})$, while satisfying the constraint $\mathbf{G}(\mathbf{u}) \leq \mathbf{0}$. In the extremum-seeking control context, we assume that *both* the information of the objective function *and* the constraint function are only accessible through measurement of the system outputs. We adopt the following assumption on these functions.

Assumption 3. *There exists a nonempty, compact, possibly disconnected,* unknown *set $\mathcal{T} \subseteq \Omega$ defined as $\mathcal{T} := \{\mathbf{u} \in \Omega \mid \mathbf{G}(\mathbf{u}) \leq \mathbf{0}\}$. We call this set the admissible set. If the admissible set $\mathcal{T}$ is a disconnected set, it can be decomposed into p nonempty, compact, and connected sets $\mathcal{T}_i$ such that $\bigcup_{i=1}^{p} \mathcal{T}_i = \mathcal{T}$. On each set $\mathcal{T}_i$, Q takes its (global) constrained minimum value in a nonempty, compact set $\mathcal{C}_{\mathcal{T}_i} \subset \mathcal{T}_i$, i.e., for each $i = 1, \ldots, p$, there exists a $\mathbf{u}^{*,i} \in \mathcal{T}_i$ such that for all $\mathbf{u} \in \mathcal{T}_i$, $Q(\mathbf{u}) \geq Q(\mathbf{u}^{*,i})$. Moreover, there exists an admissible initialization set $\mathcal{V}$ which is a nonempty,* known *compact (and possibly disconnected) subset $\mathcal{V} \subset \mathcal{T}$.*

Remark 4. *Assumption 3 states that we have some limited knowledge about the admissible set $\mathcal{T}$, such that we can initialize within $\mathcal{V} \subset \mathcal{T}$. This is a reasonable assumption* *since we usually know where we can initialize our system without violating the constraints immediately.*

In the next section, we present a barrier function method to solve static constrained optimization problems. Section IV presents a sampled-data extremum-seeking approach that employs the barrier function method to minimize $\mathbf{Q}(\mathbf{u})$ while satisfying $\mathbf{G}(\mathbf{u}) \leq \mathbf{0}$, solely based on output measurements.

III. BARRIER FUNCTION METHODS FOR CONSTRAINED OPTIMIZATION PROBLEMS

In this section, we will present the classical barrier function method to solve *static* constrained optimization problems, see, e.g., [14] and [16]. Consider the following constrained optimization problem:

$$\min_{\mathbf{u}\in\Omega} \quad Q(\mathbf{u}) \quad \text{subject to } \mathbf{G}(\mathbf{u}) \leq \mathbf{0}, \tag{4}$$

where $\mathbf{G}(\mathbf{u}) = [G_1(\mathbf{u}), \ldots, G_{n_\mathbf{z}}(\mathbf{u})]^\top$. The admissible set is defined as $\mathcal{T} := \{\mathbf{u} \in \Omega \subseteq \mathbb{R}^{n_\mathbf{u}} \mid G_i(\mathbf{u}) \leq 0,\ i = 1, 2, \ldots, n_\mathbf{z}\}$, and is a nonempty, compact (and possibly disconnected) set, see Assumption 3. The barrier function method is a well-known approach to solve (4); it approximates the constrained optimization problem by an unconstrained problem. It works by establishing a barrier on the boundary of the admissible set that prevents a properly tuned optimization algorithm from leaving that region, as long as it starts well inside the admissible set. We employ the following definition.

Definition 5. *We define the strict admissible set as* $\mathcal{T}^o := \{\mathbf{u} \in \mathcal{T} \mid G_i(\mathbf{u}) < 0,\ i = 1, 2, \ldots, n_\mathbf{z}\}$*, which is the interior of the admissible set* $\mathcal{T}$*. The boundary of the admissible set* $\mathcal{T}$*, denoted by* $\partial\mathcal{T}$*, is defined as* $\partial\mathcal{T} = \mathcal{T} \setminus \mathcal{T}^o$*. For each (sufficiently small) barrier parameter* $\mu > 0$*, a barrier function* $B(\mathbf{u}, \mu) : \mathcal{T}^o \times \mathbb{R}_{>0} \to \mathbb{R}$ *is defined on the interior of* $\mathcal{T}$ *such that*

(i) $B(\mathbf{u}, \mu)$ *is continuous,*
(ii) $B(\mathbf{u}, \mu) \approx 0$ *for all* $\mathbf{u} \in \mathcal{T}^o$*,*
(iii) $B(\mathbf{u}, \mu) \to \infty$ *as* $\mathbf{u}$ *approaches* $\partial\mathcal{T}$*.*

By exploiting barrier functions as defined in Definition 5, we can approximate the constrained optimization problem in (4) by the following unconstrained problem:

$$\min_{\mathbf{u}\in\mathcal{T}^o} \tilde{Q}(\mathbf{u}, \mu), \tag{5}$$

with the *modified* objective function $\tilde{Q}(\mathbf{u}, \mu) := Q(\mathbf{u}) + B(\mathbf{u}, \mu)$. In case the admissible set $\mathcal{T}$ is disconnected, the strict admissible set $\mathcal{T}^o$ is disconnected as well, and we can decompose it into p nonempty, connected sets $\mathcal{T}_i^o$ for which hold that $\bigcup_{i=i}^{p} \mathcal{T}_i^o = \mathcal{T}^o$, see Assumption 3. Minimizers of the unconstrained problem in (5), denoted by $\mathbf{u}^{*,i}(\mu)$, are called *approximate constrained minimizers* of the constrained problem in (4). Dependent on where the optimization procedure starts, i.e., in which connected set $\mathcal{T}_i^o$ the initial point is chosen, the optimization algorithm should find an approximate constrained minimizer belonging to the set of approximate minimizers denoted by $\tilde{\mathcal{C}}_{\mathcal{T}_i}$, where $i = 1, \ldots, p$.

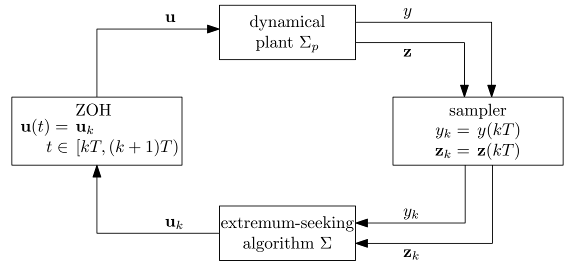

Fig. 1. Sampled-data extremum-seeking control with multiple output channels.

IV. CONSTRAINED SAMPLED-DATA EXTREMUM-SEEKING

In this work, we focus on finding the optimal inputs $\mathbf{u}^*$ for the problem described in Section II by means of a sampled-data extremum-seeking control approach, based on the work on *unconstrained* sampled-data extremum seeking in [2], [4], and [15]. In Section IV-A, we discuss the sampled-data extremum-seeking framework. In Section IV-B, we elaborate on a class of optimization algorithms employed within the sampled-data extremum-seeking framework. In Section V, a stability analysis of the resulting constrained sampled-data extremum-seeking scheme is provided.

A. Extremum-seeking framework

Figure 1 depicts the sampled-data extremum-seeking control framework, i.e., the interconnection of a dynamical system Σ_p, a T-periodic sampler, a discrete-time extremum-seeking algorithm Σ, and a zero-order-hold (ZOH) element. Let $\{\mathbf{u}_k\}_{k=0}^{\infty}$ be a sequence of vectors in Ω, generated by the extremum-seeking algorithm Σ based on collected measurements, and define the ZOH operation as follows:

$$\mathbf{u}(t) := \mathbf{u}_k \quad \text{for all } t \in [kT,\ (k+1)T), \tag{6}$$

with $k = 0, 1, 2, \ldots$, and sampling period $T > 0$. Let us define the ideal periodic sampling operations: $\mathbf{x}_k := \mathbf{x}(kT)$:

$$y_k := y(kT) \quad \forall k = 1, 2, \ldots, \tag{7}$$

and

$$\mathbf{z}_k := \mathbf{z}(kT) \quad \forall k = 1, 2, \ldots., \tag{8}$$

where y_k and $\mathbf{z}_k$ are the collected measurements as used by the extremum-seeking algorithm Σ.

B. Class of algorithms

We consider algorithms for finding a minimizer $\mathbf{u}^{*,i}(\mu) \in \tilde{\mathcal{C}}_{\mathcal{T}_i}$ of $\tilde{Q}(\mathbf{u}, \mu)$ for some $i = 1, \ldots, p$ and $\mu > 0$. In the remainder of this work, we omit the superscript i for notational clarity. In particular, we consider a class of algorithms that can be described as follows:

$$\Sigma : \mathbf{u}_{k+1} = \mathbf{u}_k + \mathbf{s}(\mathbf{u}_k), \quad \forall\ k \in \mathbb{N}, \tag{9}$$

where $\mathbf{s}(\mathbf{u}_k)$ denotes the search vector. The structure in (9) is adopted from [15], and is common to many numerical optimization methods, such as, e.g., gradient-descent, Newton's method, etc., see, e.g., [16]. We adopt the following assumption on the algorithms in (9).

Assumption 6. *For the class of algorithms in* (9)*, there exist a twice continuously differentiable function* $V_\Sigma : \mathcal{T}^o \to \mathbb{R}_{\geq 0}$ *such that for any* $\mu > 0$ *we have the following:*

i) $V_\Sigma(\mathbf{u}) > 0\ \forall \mathbf{u} \in \mathcal{T}^o \backslash \{\mathbf{u}^*(\mu)\}$, $V_\Sigma(\mathbf{u}^*(\mu)) = 0$, *and* $V_\Sigma(\mathbf{u})$ *is radially unbounded on a compact set, i.e.,* $V_\Sigma \to \infty$ *if* $\|\mathbf{u}\|_{\partial\mathcal{T}} := \inf_{\mathbf{t}\in\partial\mathcal{T}} \|\mathbf{u} - \mathbf{t}\| \to 0$.
ii) *there exists* $L_\Sigma \in \mathbb{R}_{>0}$ *such that* $\|\nabla^2 V_\Sigma(\mathbf{u})\| \leq L_\Sigma$ *for all* $\mathbf{u} \in \mathcal{T}^o$.
iii) *there exists* $\kappa_{V_\Sigma} > 0$ *such that* $\nabla V_\Sigma(\mathbf{u})^\top \mathbf{s}(\mathbf{u}) \leq -\kappa_{V_\Sigma} \|\nabla V_\Sigma(\mathbf{u})\|^2 \quad \forall \mathbf{u} \in \mathcal{T}^o \backslash \{\mathbf{u}^*(\mu)\}$, *and* $\nabla V_\Sigma(\mathbf{u}^*(\mu))^\top \mathbf{s}(\mathbf{u}^*(\mu)) = 0$.
iv) *there exist a function* $\gamma(\cdot) \in \mathcal{K}_\infty$[1] *such that* $\|\nabla V_\Sigma(\mathbf{u})\|^2 \geq \gamma(V_\Sigma(\mathbf{u}))\ \forall \mathbf{u} \in \mathcal{T}^o$.
v) *there exists* $\kappa_{\mathbf{s}} > 0$ *such that* $\|\mathbf{s}(\mathbf{u})\|^2 \leq -\kappa_{\mathbf{s}} \nabla V_\Sigma(\mathbf{u})^\top \mathbf{s}(\mathbf{u})$, $\forall \mathbf{u} \in \mathcal{T}^o$.

Remark 7. *Similar to the work in [15], the assumption on the algorithms of the type in* (9) *is motivated by Lyapunov-type arguments used to prove convergence of optimization schemes in the literature. In particular, shared conditions (iii) and (v) on the function* V_Σ *and the search vector* $s(\cdot)$ *of the sort in Assumption 6 naturally arise in the scope of optimization (see, e.g., [16, Chapter 9], in which similar conditions are used to show convergence for decent methods in combination with (strongly) convex functions). Similar conditions as in Assumption 6 can be exploited in the case of modified cost functions with a barrier function approach in combination with descent methods. For example, the conditions on* $V_\Sigma(\mathbf{u})$*, in particular* $V_\Sigma(\mathbf{u}^*(\mu)) = 0$ *and* $V_\Sigma(\mathbf{u}) \to \infty$ *if* $\|\mathbf{u}\|_{\partial\mathcal{T}} \to 0$*, are such that* $\tilde{Q}(\mathbf{u}, \mu) \geq \tilde{Q}(\mathbf{u}^*(\mu), \mu)$ *for all* $\mathbf{u} \in \mathcal{T}^o$ *and* $\tilde{Q}(\mathbf{u}, \mu) \to \infty$ *if* $\|\mathbf{u}\|_{\partial\mathcal{T}} \to 0$.

V. STABILITY ANALYSIS

The class of algorithms in (9) and its properties as assumed in Assumption 6 are geared towards the minimization of the modified objective function $\tilde{Q}(\mathbf{u}, \mu)$. However, in the extremum-seeking context, we do not have direct access to the modified objective function $\tilde{Q}(\mathbf{u}, \mu)$, and we can only approximate the search vector in (9) based on the measurable outputs y and $\mathbf{z}$. Note that, due to the dynamics in (1), the measured cost y and measured constraint functions $\mathbf{z}$ differ from the (steady-state) objective function Q and the (steady-state) constraint function $\mathbf{G}$, respectively.

To study the closed-loop behavior of the interconnection of the system in (1) and a discrete-time algorithm of the form in (9) via a ZOH element and a T-periodic sampler, see Figure 1, we consider a perturbed version of the algorithm in (9) as follows:

$$\hat{\Sigma} : \mathbf{u}_{k+1} = \mathbf{u}_k + \hat{\mathbf{s}}(\mathbf{u}_k), \qquad \forall k \in \mathbb{N} \tag{10}$$

where $\hat{\mathbf{s}}(\mathbf{u}_k) := \mathbf{s}(\mathbf{u}_k) + \boldsymbol{\delta}(\tilde{\mathbf{x}}_k) + \boldsymbol{\delta}_0$, and with $\tilde{\mathbf{x}}_k = \mathbf{x}_k - \bar{\mathbf{x}}(\mathbf{u}_k)$, $\boldsymbol{\delta}(\cdot) : \mathbb{R}^{n_\mathbf{x}} \to \mathbb{R}^{n_\mathbf{u}}$ a state dependent perturbation term, and $\boldsymbol{\delta}_0 \in \mathbb{R}_{\geq 0}$ a non-vanishing perturbation term. We adopt the following assumption on the perturbation $\boldsymbol{\delta}(\tilde{\mathbf{x}})$.

[1] A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\gamma(0) = 0$. If γ is also unbounded, then $\gamma \in \mathcal{K}_\infty$

Assumption 8. *There exists a* $\kappa_{\Sigma_p} > 0$ *such that* $\|\boldsymbol{\delta}(\tilde{\mathbf{x}})\|^2 \leq \kappa_{\Sigma_p} V_{\Sigma_p}(\tilde{\mathbf{x}})$ *for all* $\tilde{\mathbf{x}} \in \mathbb{R}^{n_\mathbf{x}}$*, with* V_{Σ_p} *satisfying Assumption 1.*

Remark 9. $\mathbf{s}(\mathbf{u}_k)$ *is the ideal search vector (available when the steady-state cost and its gradient would be available) and* $\hat{s}(\mathbf{u}_k)$ *is a perturbed version of that search vector. We can view the perturbation* $\boldsymbol{\delta}(\tilde{\mathbf{x}})$ *as a perturbation caused by the system dynamics in* (1)*, i.e., the measurement (*$y(t)$ *and* $\mathbf{z}(t)$ *which involve transients) only provide perturbed measurements of the steady-state maps* $Q(\mathbf{u})$ *and* $\mathbf{G}(\mathbf{u})$*, respectively. This perturbation and its effect on the extremum-seeking controlled system can be made small by taking the sampling time* T *long enough such that transients are sufficiently decayed. Considering, e.g., gradient-based algorithms, the perturbation* $\boldsymbol{\delta}_0$ *can be viewed as a mismatch between the approximation of the gradient, based on, e.g., finite differences, and the actual gradient, which is typically non-zero even when* $\tilde{\mathbf{x}} = 0$*. The perturbation* $\boldsymbol{\delta}_0$ *can be made small by taking a small step size for the gradient estimation.*

For the purpose of stability analysis let us now define the following function $V(\tilde{\mathbf{x}}_k, \mathbf{u}_k) := V_{\Sigma_p}(\tilde{\mathbf{x}}_k) + V_\Sigma(\mathbf{u}_k)$, which will be used as a Lyapunov-like function in Theorem 10 below. Moreover, we define $\Delta V_{\Sigma_p}(\tilde{\mathbf{x}}_k) := V_{\Sigma_p}(\tilde{\mathbf{x}}_{k+1}) - V_{\Sigma_p}(\tilde{\mathbf{x}}_k)$, and $\Delta V_\Sigma(\mathbf{u}_k) := V_\Sigma(\mathbf{u}_{k+1}) - V_\Sigma(\mathbf{u}_k)$. The next result states conditions on initial conditions and parameters such that the trajectories $(\tilde{\mathbf{x}}_k, \mathbf{u}_k)$ converge to a neighborhood of $(0, \mathbf{u}^*)$, while steady-state constraint satisfaction is guaranteed.

Theorem 10. *Let the admissible initialization set be* $\mathcal{V} = \{\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}} \mid \|V_\Sigma(\mathbf{u})\| \leq \beta_\Sigma\} \subset \mathcal{T}^o$*, for some* $\beta_\Sigma \in \mathbb{R}_{>0}$*. Under Assumptions 1, 6, and 8, there exist* $\kappa_s^*, \kappa_{\Sigma_p}^*, T^*, \beta_{\Sigma_p} \in \mathbb{R}_{>0}$*, and sufficiently small* $\boldsymbol{\delta}_0 \in \mathbb{R}_{\geq 0}$*, such that for any* $\kappa_s < \kappa_s^*$*,* $\kappa_{\Sigma_p} < \kappa_{\Sigma_p}^*$*, waiting time* $T > T^*$*, and initial input* $\mathbf{u}_0 \in \mathcal{V}$*, there exist a set of initial conditions* $\mathcal{X}_0 = \{\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}} | V_{\Sigma_p}(\tilde{\mathbf{x}}) \leq \beta_{\Sigma_p}\}$*, such that for any* $\mathbf{x}_0 \in \mathcal{X}_0$ *we have that* $\mathbf{u}_k \in \mathcal{T}^o$ *for all* $k \in \mathbb{N}$*, implying constraint satisfaction. In addition, there exist* $\beta \in \mathbb{R}_{>0}$*,* $\alpha_3(\|\boldsymbol{\delta}_0\|) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$*, and a function* $\tilde{\gamma}(\cdot) \in \mathcal{K}_\infty$*, such that the solutions* $\mathbf{u}_k$ *and* $\tilde{\mathbf{x}}_k$ *of the closed-loop system consisting of the plant in* (1)*, a* T*-periodic sampler in* (7)*,* (8)*, the discrete-time extremum-seeking algorithm in* (10)*, and the ZOH operator in* (6) *converge to a neighborhood of the optimum characterized by the set* $\mathcal{Y}_\mathbf{u} = \{\mathbf{u} \in \mathcal{T}^o \mid V_\Sigma(\mathbf{u}) \leq \tilde{\gamma}^{-1}\left(\frac{\alpha_3(\|\boldsymbol{\delta}_0\|)}{\beta}\right)\}$*, and a neighborhood of the steady-state equilibria of the system characterized by the set* $\mathcal{Y}_{\tilde{\mathbf{x}}} = \{\tilde{\mathbf{x}} \in \mathcal{X} \mid V_{\Sigma_p}(\tilde{\mathbf{x}}) \leq \tilde{\gamma}^{-1}\left(\frac{\alpha_3(\|\boldsymbol{\delta}_0\|)}{\beta}\right)\}$*, respectively.*

Proof. Consider the function $V(\tilde{\mathbf{x}}_k, \mathbf{u}_k) = V_{\Sigma_p}(\tilde{\mathbf{x}}_k) + V_\Sigma(\mathbf{u}_k)$.

Let us first derive a bound on $\Delta V_{\Sigma_p}(\tilde{\mathbf{x}}_k) := V_{\Sigma_p}(\tilde{\mathbf{x}}_{k+1}) - V_{\Sigma_p}(\tilde{\mathbf{x}}_k)$. From Assumption 1 we have that $V_{\Sigma_p}(\tilde{\mathbf{x}}(t)) \leq e^{-\gamma(t-t_0)} V_{\Sigma_p}(\tilde{\mathbf{x}}(t_0))\ \forall\ t \geq t_0$ and fixed $\mathbf{u} \in \Omega$. Given the T-periodic sampling $\mathbf{x}_k := \mathbf{x}(kT)$, we obtain the following inequality:

$$\begin{aligned} V_{\Sigma_p}(\mathbf{x}_{k+1} - \bar{\mathbf{x}}(\mathbf{u}_{k+1})) &\leq e^{-\gamma T} V_{\Sigma_p}(\mathbf{x}_k - \bar{\mathbf{x}}(\mathbf{u}_{k+1})) \\ &\leq e^{-\gamma T} V_{\Sigma_p}(\mathbf{x}_k - \bar{\mathbf{x}}(\mathbf{u}_k) + \bar{\mathbf{x}}(\mathbf{u}_k) - \bar{\mathbf{x}}(\mathbf{u}_{k+1})). \end{aligned} \tag{11}$$

Let us consider an arbitrarily large compact set $\mathcal{X} \subset \mathbb{R}^{n_{\mathbf{x}}}$. Since V_{Σ_p} is continuously differentiable by Assumption 1, there exists a $L_{V_{\Sigma_p}} \in \mathbb{R}_{>0}$ such that $\|V_{\Sigma_p}(\mathbf{x}_1) - V_{\Sigma_p}(\mathbf{x}_2)\| \leq L_{V_{\Sigma_p}}\|\mathbf{x}_1 - \mathbf{x}_2\|$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$. Using this property, Assumption 1, and (10) we obtain the following inequality:

$$\begin{aligned} V_{\Sigma_p}(\tilde{\mathbf{x}}_{k+1}) &\leq e^{-\gamma T} V_{\Sigma_p}(\tilde{\mathbf{x}}_k + \bar{\mathbf{x}}(\mathbf{u}_k) - \bar{\mathbf{x}}(\mathbf{u}_{k+1})) \\ &\leq e^{-\gamma T}\left(V_{\Sigma_p}(\tilde{\mathbf{x}}_k) + L_{V_{\Sigma_p}} L_{\bar{\mathbf{x}}}\|\mathbf{s}(\mathbf{u}_k) + \boldsymbol{\delta}(\tilde{\mathbf{x}}_k) + \boldsymbol{\delta}_0\|\right). \end{aligned} \tag{12}$$

From Young's inequality, i.e., $ab \leq \frac{a^2}{2\epsilon_1} + \frac{\epsilon_1 b^2}{2}$ for some $\epsilon_1 > 0$, it follows that

$$\begin{aligned} \Delta V_{\Sigma_p}(\tilde{\mathbf{x}}_k) \leq& -(1 - e^{-\gamma T}) V_{\Sigma_p}(\tilde{\mathbf{x}}_k) + \frac{1}{2\epsilon_1} e^{-\gamma T}(L_{V_{\Sigma_p}} L_{\bar{\mathbf{x}}})^2 \\ &+ \frac{\epsilon_1}{2} e^{-\gamma T}\|\mathbf{s}(\mathbf{u}_k) + \boldsymbol{\delta}(\tilde{\mathbf{x}}_k) + \boldsymbol{\delta}_0\|^2. \end{aligned} \tag{13}$$

From the Cauchy-Schwarz inequality, i.e., $\|\mathbf{a} + \mathbf{b}\|^2 \leq 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$, Assumption 6(v), and Assumption 8, we obtain the following inequality:

$$\begin{aligned} \|\mathbf{s}(\mathbf{u}_k) + \boldsymbol{\delta}(\tilde{\mathbf{x}}_k) + \boldsymbol{\delta}_0\|^2 &\leq 2\|\mathbf{s}(\mathbf{u}_k)\|^2 \\ &\quad + 4\|\boldsymbol{\delta}(\tilde{\mathbf{x}}_k)\|^2 + 4\|\boldsymbol{\delta}_0\|^2 \\ &\leq -2\kappa_s \nabla V_\Sigma^\top(\mathbf{u}_k)\mathbf{s}(\mathbf{u}_k) + 4\kappa_{\Sigma_p} V_{\Sigma_p}(\tilde{\mathbf{x}}_k) + 4\|\boldsymbol{\delta}_0\|^2. \end{aligned} \tag{14}$$

In combination with (13) and $\epsilon_1 = 2$, this yields

$$\begin{aligned} \Delta V_{\Sigma_p}(\tilde{\mathbf{x}}_k) \leq& -(1 - e^{-\gamma T}(1 + 4\kappa_{\Sigma_p})) V_{\Sigma_p}(\tilde{\mathbf{x}}_k) \\ &+ \frac{1}{4} e^{-\gamma T}(L_{V_{\Sigma_p}} L_{\bar{\mathbf{x}}})^2 + 4e^{-\gamma T}\|\boldsymbol{\delta}_0\|^2 \\ &- 2\kappa_s e^{-\gamma T} \nabla V_\Sigma^\top(\mathbf{u}_k)\mathbf{s}(\mathbf{u}_k). \end{aligned} \tag{15}$$

Next, let us derive a bound on $\Delta V_\Sigma(\mathbf{u}_k) := V_\Sigma(\mathbf{u}_{k+1}) - V_\Sigma(\mathbf{u}_k)$. Since $V_\Sigma(\cdot)$ is twice continuously differentiable on $\mathcal{T}^o$ (Assumption 6), it follows from Taylor's Theorem that

$$\begin{aligned} V_\Sigma(\mathbf{u}_1 + \mathbf{u}_2) =& V_\Sigma(\mathbf{u}_1) + \nabla V_\Sigma^\top(\mathbf{u}_1)\mathbf{u}_2 \\ &+ \mathbf{u}_2^\top \int_0^1 (1 - \sigma)\nabla^2 V_\Sigma^\top(\mathbf{u}_1 + \sigma \mathbf{u}_2) d\sigma \mathbf{u}_2. \end{aligned} \tag{16}$$

From Assumption 6(ii), (10), and (16), we obtain the following inequality:

$$\begin{aligned} \Delta V_\Sigma(\mathbf{u}_k) \leq& \nabla V_\Sigma^\top(\mathbf{u}_k)(\mathbf{s}(\mathbf{u}_k) + \boldsymbol{\delta}(\tilde{\mathbf{x}}_k) + \boldsymbol{\delta}_0) \\ &+ \frac{L_{V_\Sigma}}{2}\|\mathbf{s}(\mathbf{u}_k) + \boldsymbol{\delta}(\tilde{\mathbf{x}}_k) + \boldsymbol{\delta}_0\|^2. \end{aligned} \tag{17}$$

Using (14), applying Young's inequality again with $\epsilon_1 = \frac{1}{\kappa_{V_\Sigma}}$, and Assumption 8, we can write the following inequality:

$$\begin{aligned} \Delta V_\Sigma(\mathbf{u}_k) \leq& (1 - L_{V_\Sigma}\kappa_s)\nabla V_\Sigma^\top(\mathbf{u}_k)\mathbf{s}(\mathbf{u}_k) \\ &+ \frac{\kappa_{V_\Sigma}}{2}\|\nabla V_\Sigma(\mathbf{u}_k)\|^2 + (2L_{V_\Sigma} + \frac{1}{2\kappa_{V_\Sigma}})\|\boldsymbol{\delta}_0\|^2 \\ &+ \kappa_{\Sigma_p}(2L_{V_\Sigma} + \frac{1}{2\kappa_{V_\Sigma}}) V_{\Sigma_p}(\tilde{\mathbf{x}}_k). \end{aligned} \tag{18}$$

Let us now analyze the sum of the two increments: $\Delta V(\tilde{\mathbf{x}}_k, \mathbf{u}_k) = \Delta V_{\Sigma_p}(\tilde{\mathbf{x}}_k) + \Delta V_\Sigma(\mathbf{u}_k)$. Using (15) and (18), for any $\kappa_s < \frac{1}{L_{V_\Sigma} + 2e^{-\gamma T}}$, we obtain from Assumption 6(iii) the following inequality:

$$\begin{aligned} \Delta V(\tilde{\mathbf{x}}_k, \mathbf{u}_k) \leq& -\beta_1\|\nabla V_\Sigma(\mathbf{u}_k)\|^2 - \beta_2 V_{\Sigma_p}(\tilde{\mathbf{x}}_k) \\ &+ \alpha_3(\|\boldsymbol{\delta}_0\|), \end{aligned} \tag{19}$$

with $\alpha_3(\|\boldsymbol{\delta}_0\|) := \beta_3\|\boldsymbol{\delta}_0\|^2 + \beta_4$, and

$$\begin{aligned} \beta_1 &:= \kappa_{V_\Sigma}(\frac{1}{2} - \kappa_s(L_{V_\Sigma} + 2e^{-\gamma T})), \\ \beta_2 &:= 1 - \kappa_{\Sigma_p}(2L_{V_\Sigma} + \frac{1}{2\kappa_{V_\Sigma}}) - e^{-\gamma T}(1 + 4\kappa_{\Sigma_p}), \\ \beta_3 &:= 2L_{V_\Sigma} + 4e^{-\gamma T} + \frac{1}{2\kappa_{V_\Sigma}}, \beta_4 := \frac{1}{4}e^{-\gamma T}(L_{V_{\Sigma_p}} L_{\bar{\mathbf{x}}})^2. \end{aligned}$$

For any $\kappa_s < \kappa_s^* := \frac{1}{2L_{V_\Sigma} + 4}$, it follows that $\beta_1 > 0$. To show that $\beta_2 > 0$, let $\epsilon_3 := 1 - \kappa_{\Sigma_p}(2L_{V_\Sigma} + \frac{1}{2\kappa_{V_\Sigma}})$. For any $0 < \kappa_{\Sigma_p} < \kappa_{\Sigma_p}^* := \frac{1}{2L_{V_\Sigma} + \frac{1}{2\kappa_{V_\Sigma}}}$, it follows that $\epsilon_3 \in (0, 1)$. Then, if $T > T^* := \frac{1}{\gamma}\ln\left(\frac{1 + 4\kappa_{\Sigma_p}}{\epsilon_3}\right)$, then $\beta_2 > 0$. Note that from the positive constants defined in Assumptions 1, 6, 8, for any $T > 0$ it follows that $\beta_3, \beta_4 > 0$.

From Assumption 6(iv) and (19), we obtain the following inequality:

$$\Delta V(\tilde{\mathbf{x}}_k, \mathbf{u}_k) \leq -\beta_1\gamma(V_\Sigma(\mathbf{u}_k)) - \beta_2 V_{\Sigma_p}(\tilde{\mathbf{x}}_k) + \alpha_3(\|\boldsymbol{\delta}_0\|). \tag{20}$$

Define a function $\bar{\gamma}(V) := \min\{\gamma(V), V\}$, with $\bar{\gamma}(\cdot) \in \mathcal{K}_\infty$. This implies that $\bar{\gamma}(V) \leq \gamma(V)$, and $\bar{\gamma}(V) \leq V$. Moreover, define $\beta := \min\{\beta_1, \beta_2\}$. From this, we obtain the following inequality:

$$\Delta V(\tilde{\mathbf{x}}_k, \mathbf{u}_k) \leq -\beta\left(\bar{\gamma}(V_\Sigma(\mathbf{u}_k)) + \bar{\gamma}(V_{\Sigma_p}(\tilde{\mathbf{x}}_k))\right) + \alpha_3(\|\boldsymbol{\delta}_0\|). \tag{21}$$

Moreover, given the fact that $\bar{\gamma} \in \mathcal{K}_\infty$, we define a function $\tilde{\gamma}(\cdot) \in \mathcal{K}_\infty$ such that $\tilde{\gamma}(V_\Sigma + V_{\Sigma_p}) := \bar{\gamma}(\frac{1}{2}(V_\Sigma + V_{\Sigma_p})) \leq \bar{\gamma}(V_\Sigma) + \bar{\gamma}(V_{\Sigma_p})$ for all $V_\Sigma, V_{\Sigma_p} \geq 0$ [19]. This yields the following inequality:

$$\begin{aligned} \Delta V(\tilde{\mathbf{x}}_k, \mathbf{u}_k) &\leq -\beta\tilde{\gamma}(V_\Sigma(\mathbf{u}_k) + V_{\Sigma_p}(\tilde{\mathbf{x}}_k)) + \alpha_3(\|\boldsymbol{\delta}_0\|) \\ &= -\beta\tilde{\gamma}(V(\tilde{\mathbf{x}}_k, \mathbf{u}_k)) + \alpha_3(\|\boldsymbol{\delta}_0\|). \end{aligned} \tag{22}$$

Finally, from (22) it follows that $\Delta V(\tilde{\mathbf{x}}_k, \mathbf{u}_k) \leq 0$ for all $V(\tilde{\mathbf{x}}_k, \mathbf{u}_k) \geq \tilde{\gamma}^{-1}\left(\frac{\alpha_3(\|\boldsymbol{\delta}_0\|)}{\beta}\right)$. Let $\mathcal{V}_\Omega := \{(\tilde{\mathbf{x}}, \mathbf{u}) \in \mathcal{X} \times \mathcal{T}^o \mid V(\tilde{\mathbf{x}}, \mathbf{u}) \leq \Omega_V\}$ with $\Omega_V \in \mathbb{R}_{>0}$ be the largest sublevel set of $V(\tilde{\mathbf{x}}, \mathbf{u})$ contained in $\mathcal{X} \times \mathcal{T}^o$. Define the set $\mathcal{Y} = \{(\tilde{\mathbf{x}}, \mathbf{u}) \in \mathcal{X} \times \mathcal{T}^o \mid V(\tilde{\mathbf{x}}, \mathbf{u}) \leq \tilde{\gamma}^{-1}\left(\frac{\alpha_3(\|\boldsymbol{\delta}_0\|)}{\beta}\right)\}$ with sufficiently small $\boldsymbol{\delta}_0$ and choose $\mathcal{X}$ sufficiently large such that $\mathcal{Y} \subset \mathcal{V}_\Omega$. The set $\mathcal{Y}$ is a positively invariant set to which all solutions starting at initial conditions in $\mathcal{V}_\Omega$ converge. Moreover, as $V(\tilde{\mathbf{x}}, \mathbf{u})$ is always positive and remains bounded in $\mathcal{X} \times \mathcal{T}^o$, $V_\Sigma(\mathbf{u})$ is bounded as well in $\mathcal{X} \times \mathcal{T}^o$. In addition, $V_\Sigma(\mathbf{u}) \leq V(\tilde{\mathbf{x}}, \mathbf{u})$ for any $(\tilde{\mathbf{x}}, \mathbf{u})$ in $\mathcal{V}_\Omega$. This implies boundedness of V_Σ and hence constraint satisfaction is guaranteed. The same holds for $V_{\Sigma_p}(\tilde{\mathbf{x}}_k)$, i.e., $V_{\Sigma_p}(\tilde{\mathbf{x}}) \leq V(\tilde{\mathbf{x}}, \mathbf{u})$ for any $(\tilde{\mathbf{x}}, \mathbf{u})$ in $\mathcal{V}_\Omega$. Let $\mathcal{V} = \{\mathbf{u} \in \mathcal{T}^o \mid V_\Sigma(\mathbf{u}) \leq \beta_\Sigma\}$ be the admissible initialization set with some $\beta_\Sigma \in (0, \Omega_V)$. Then, for any $\mathbf{u}_0 \in \mathcal{V}$ and initial conditions $\mathbf{x}_0 \in \mathcal{X}_0$ with $\mathcal{X}_0 = \{\mathbf{x} \in \mathcal{X} \mid V_{\Sigma_p}(\tilde{\mathbf{x}}) \leq \beta_{\Sigma_p}\}$ and $\beta_{\Sigma_p} \leq \Omega_V - \beta_\Sigma > 0$, such that for $k \to \infty$ all solutions $\mathbf{u}_k$ converge to the set $\mathcal{Y}_{\mathbf{u}} = \{\mathbf{u} \in \mathcal{T}^o \mid V_\Sigma(\mathbf{u}_k) \leq \tilde{\gamma}^{-1}(\frac{\alpha_3(\|\boldsymbol{\delta}_0\|)}{\beta})\}$, and all solutions $\tilde{\mathbf{x}}_k$ converge to the set $\mathcal{Y}_{\tilde{\mathbf{x}}} = \{\tilde{\mathbf{x}} \in \mathcal{X} \mid V_{\Sigma_p}(\tilde{\mathbf{x}}_k) \leq \tilde{\gamma}^{-1}(\frac{\alpha_3(\|\boldsymbol{\delta}_0\|)}{\beta})\}$. □

VI. NUMERICAL EXAMPLE

A. Constrained system with equilibria solutions

Consider the following dynamical system:

$$\begin{aligned} \dot{x}_1 &= -3x_2 + u_1^2 + u_2^2 \\ \dot{x}_2 &= x_1 - 6x_2 - \frac{1}{2} + 6u_1^2 - u_1^4 - u_1u_2 - 2u_2^2, \end{aligned} \tag{23}$$

with performance output $y = 3x_2$, and constrained output $z = x_1$, and inputs u_1 and u_2. For any constant $\mathbf{u}^\top = [u_1\ u_2]$, the equilibrium solutions $\bar{\mathbf{x}}^\top = [\bar{x}_1\ \bar{x}_2]$ are globally exponentially stable (GES) (the system is LTI and the system matrix is Hurwitz). The steady-state input-output maps are as follows:

$$Q(u_1, u_2) = u_1^2 + u_2^2, \tag{24}$$

and

$$G(u_1, u_2) = \frac{1}{2} - 4u_1^2 + u_1^4 + u_1u_2 + 4u_2^2. \tag{25}$$

Figure 2 depicts contour plots of $Q(\mathbf{u})$ (bottom) and $G(\mathbf{u})$ (top), and shows that the unconstrained minimizer of $Q(\mathbf{u})$ is not a minimizer in the constrained case $G(\mathbf{u}) \leq 0$. Figure 3 depicts contour plots of the modified cost function $\tilde{Q}(\mathbf{u}, \mu) := Q(\mathbf{u}) + B(\mathbf{u}, \mu)$ for $\mu = 0.25$ (bottom) and $\mu = 0.05$ (top), where $B(\mathbf{u}, \mu) = -\mu \log(-G(\mathbf{u}))$. Moreover, the approximate constrained minimizers in both cases are shown by a blue dot.

B. Gradient-descent extremum-seeking algorithm

Here we show that we can find an approximate constrained minimizer based on the extremum-seeking method presented in Section IV. In particular, we assume no knowledge of $Q(\mathbf{u})$ and $G(\mathbf{u})$, and we obtain information on performance and constraint satisfaction only through measurements of y and z. In particular, the approximation of the modified cost function is obtained by $\tilde{Q}(\mathbf{u}_k, \mu) := y_k - \mu \log(-z_k)$ with $k = 1, 2, \ldots$. We employ the following gradient descent algorithm to optimize the vector $\mathbf{u}$:

$$\mathbf{u}_k = \mathbf{u}_{k-1} - \lambda \nabla \tilde{Q}(\mathbf{u}_{k-1}, \mu), \tag{26}$$

with λ the optimizer gain. Since $\tilde{Q}$ is unknown, its gradient $\nabla \tilde{Q}$ is unknown. As such, the gradient of $\tilde{Q}$ will be estimated based on finite differences. In particular, the gradient descent algorithm in (26) is implemented through the following extremum-seeking algorithm:

$$\mathbf{u}_k = \begin{cases} \mathbf{u}_{k-n} + \tau \mathbf{d}_n & \text{if } n \neq 0 \\ \mathbf{u}_{k-(p+1)} - \lambda \nabla \tilde{Q}(\mathbf{u}_{k-(p+1)}, \mu) & \text{if } n = 0 \end{cases}, \tag{27}$$

for all $k = 1, 2, \ldots$, with τ the step size of the gradient estimator, $\mathbf{d}_j$ with $j = 1, \ldots, p$ are dither signals, i.e., vectors

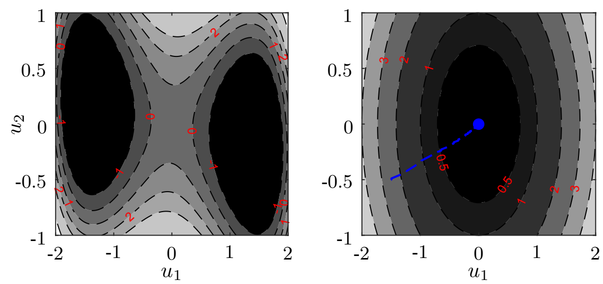

Fig. 2. Contour plots of the objective function $Q(\mathbf{u})$ (bottom) and the constraint function $G(\mathbf{u})$ (top). From the figures it is evident that the unconstrained minimizer of $Q(\mathbf{u})$ is not a minimizer in the constrained case. (**– –**) shows the convergence of the ESC scheme to the unconstrained minimizer of $Q(\mathbf{u})$ with initial input $\mathbf{u}_0 = [-1\ -\frac{1}{2}]^\top$.

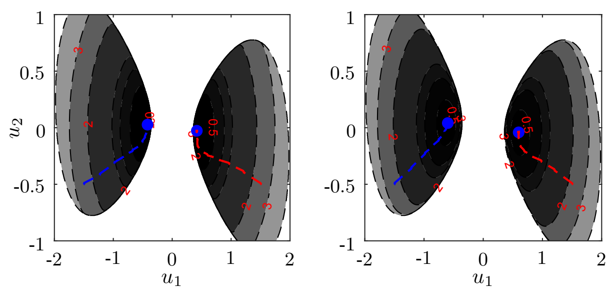

Fig. 3. Contour plots of the modified objective function $\tilde{Q}(\mathbf{u}, \mu)$ with $\mu = 0.05$ (top) and $\mu = 0.25$ (bottom). In both plots, (**– –**) and (**– –**) depict the convergence of the ESC scheme to the approximate constrained minimizers of $Q(\mathbf{u})$ with initial input $\mathbf{u}_0 = [-1\ -\frac{1}{2}]^\top$ and $\mathbf{u}_0 = [-1\ \frac{1}{2}]^\top$, respectively.

where the jth element is equal to one, and all other elements are zero, $n = \text{mod}(k, p+1) \in \mathbb{N}$, initial input u_0, and

$$\nabla \tilde{Q}(\mathbf{u}_{k-(p+1)}, \mu) = \frac{1}{\tau} \begin{bmatrix} \tilde{Q}(\mathbf{u}_{k-p}, \mu) - \tilde{Q}(\mathbf{u}_{k-(p+1)}, \mu) \\ \vdots \\ \tilde{Q}(\mathbf{u}_{k-1}, \mu) - \tilde{Q}(\mathbf{u}_{k-(p+1)}, \mu) \end{bmatrix}. \tag{28}$$

Note that the case $n = 0$ in (27) implements an update of the control signal $\mathbf{u}$.

C. Simulation results

We have performed three simulations with different settings of the barrier parameter: i) $\mu = 0.25$, ii) $\mu = 0.05$, and iii) $\mu = 0$. The case when $\mu = 0$ boils down to minimizing the cost $y \approx Q(u)$ without taking into account any constraints. For all simulations, we have used a step size $\tau = 1 \cdot 10^{-2}$ and an optimizer gain $\lambda = 10$ for the extremum-seeking algorithm in (27), and we employed a waiting time $T = 10$. We choose the initial input vector $\mathbf{u}_0^\top = [-\frac{3}{2}\ -\frac{1}{2}]$, and to simulate the dynamics in (23) we used the initial state as $\mathbf{x}_0^\top := \mathbf{x}(0)^\top = [-3\ 1]$.

Figure 2 (bottom) shows the trajectory of $\mathbf{u}$ in a 2D plane for $\mu = 0$. In the absence of constraints, the optimizer state converges towards a neighborhood of the input vector that minimizes $Q(\mathbf{u})$, i.e., $\mathbf{u}^* = [0\ 0]^\top$. Figure 3 shows the trajectories of $\mathbf{u}$ in a 2D plane in case $\mu = 0.05$ (top) and $\mu = 0.25$ (bottom). In both cases, convergence towards a

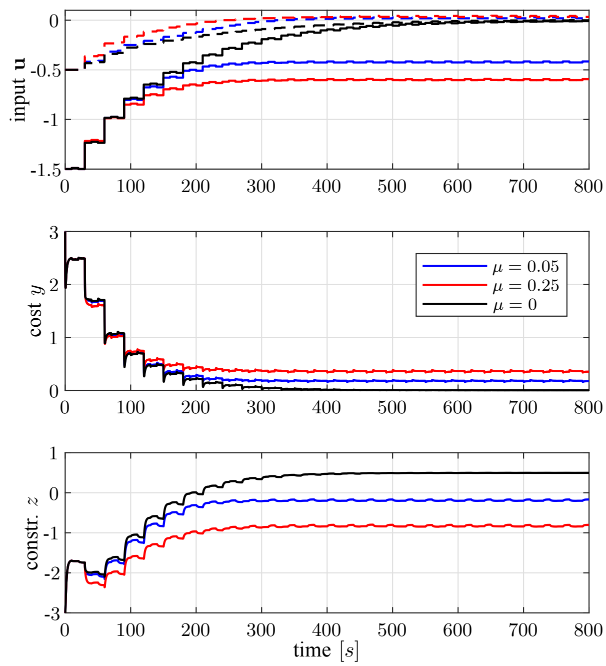

Fig. 4. Time-domain convergence of the inputs u_1 (solid) and u_2 (dashed) towards a neighborhood of the approximate constrained minimizers of $\tilde{Q}(\mathbf{u}, \mu)$ for that particular μ, and the corresponding cost and constraint values.

neighborhood of inputs $\mathbf{u}$ that minimize $\tilde{Q}(\mathbf{u}, \mu)$ for that particular μ is achieved.

Similar conclusions can be drawn from Figure 4, which shows the evolution of u_1 and u_2, and the corresponding measurements of y and z, in time, for the cases $\mu = 0.25$, $\mu = 0.05$, and $\mu = 0$. In case $\mu = 0$, the optimizer state converge towards a neighborhood of the unconstrained optimum $\mathbf{u}^* = [0\ 0]^\top$. This however yields a violation of the constraint (see bottom figure). In case of $\mu = 0.25$ and $\mu = 0.05$, we satisfy the constraint and the optimizer states converge towards a neighborhood of inputs $\mathbf{u}$ that minimize $\tilde{Q}(\mathbf{u}, \mu)$ for that particular μ.

Remark 11. *For even smaller values of μ, the optimizer state will tend more closely to the actual constrained minimizer. As mentioned in Section III, we can also employ a decreasing sequence of μ such that the approximate constrained minimizers, i.e., minimizers of the problem in* (5)*, approach the minimizers of the actual problem in* (4)*. This may require a different tuning of the parameters of the extremum-seeking algorithm.*

Remark 12. *If we would have initialized differently, for example, $\mathbf{u}_0^\top = [\frac{3}{2}\ \text{-}\frac{1}{2}]$, then the state converges towards the optimum in the other admissible region, as we can only find the minimum in a subset of the admissible set in which we initialize, see, e.g., the trajectories depicted by the red lines in Figure 3.*

VII. CONCLUSION

We have proposed a sampled-data extremum-seeking approach for optimization of constrained dynamical systems using barrier function methods, where both the objective function and the constraint function are available for measurement only. We have shown that, under appropriate conditions, the interconnection between a constrained dynamical system and a class of optimization algorithms that employ barrier function methods is stable, strict constraint satisfaction is guaranteed, and optimization is achieved. A numerical example is provided that illustrates the working principle of the sampled-data extremum-seeking approach using barrier functions.

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