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An extended model order reduction technique for linear delay systems*

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Abstract— **This paper presents a model reduction technique for linear delay differential equations that, first, preserves the infinite-dimensional nature of the system, and, second, enables the preservation of additional properties such as physical interconnection structures or uncertainties. This structured/robust reduction of delay systems is achieved by allowing additional degrees of freedom in the characterization of (bounds on) controllability and observability energy functionals, leading to a so-called extended balancing procedure. In addition, the proposed technique preserves stability properties and provides an a priori error bound. The relevance of the method in controller reduction is discussed and illustrative numerical examples are presented.**

I. Introduction

Engineering systems such as drilling and electric/electronic systems [1], [2], and also phenomena in economics and biology, can be described by models in terms of delay differential equations [3]. However, such models are often complex as those are described by a large number of equations, which complicates or even prohibits simulation, analysis, or controller synthesis. To address this issue of complexity, we present a model order reduction technique for such delay models in this paper.

For the model order reduction of linear delay-free systems, many techniques, such as balanced truncation [4], have been proposed over the past four decades (see [5] for an overview). For time delay systems, some model reduction techniques have also been proposed. Most of these aim at approximating a time delay system by a low-order finite-dimensional model [6], [7], [8]. This is because analysis and design based on a finite-dimensional model enables the use of well-developed classical systems and control theory. However, today, powerful analysis and controller design techniques are available for time delay system. Moreover, for a particular order of the reduced model, a reduced model in terms of delay differential equations has in general the potential to be more accurate than a finite-dimensional approximation of the same order [9]. Therefore, delay-structure preserving methods, i.e., methods that preserve the infinite-dimensional nature of the time delay system during model reduction, have gained much attention [9], [10], [11], [12], [13]. In many cases, however, there is a need to preserve additional properties during model reduction. Important examples are physical interconnection structures (e.g., the interconnection of a system and a controller) or the presence of uncertainties. In this paper, we develop such structured/robust model reduction techniques for linear time delay systems.

Specifically, the main contribution of this paper is the introduction of a so-called *extended* balanced truncation procedure for time delay systems. Following [12], [13], bounds on controllability and observability energy functionals are computed and used as a basis for model reduction. However, compared to the results in [13], the current paper introduces additional degrees of freedom in the computation of (bounds on) these functionals through the use of slack variables. This procedure is motivated by the technique of extended balanced truncation for finite-dimensional systems in [14], [15], which is known to enable structured/robust model reduction [16].

We will show that extended balanced truncation for delay differential equations enjoys similar properties. In particular, we will show that the proposed technique is useful for the structured model reduction of closed-loop time delay systems and also for delay systems with polytopic uncertainties. We will also prove that this technique preserves both asymptotic stability and the infinite-dimensional nature of the time delay system. Moreover, we show that it provides an a priori computable and guaranteed delay-dependent error bound.

Outline. After introducing notation, a detailed problem statement is given in Section II. Section III introduces and gives a characterization of the observability and controllability energy functionals. Section IV is devoted to the description of the proposed delay-dependent model order reduction procedure and prospective applications of the proposed technique are given in Section V. Numerical examples are provided in Section VI and conclusions are presented in Section VII.

Notation. The notation $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$) refers to the field of (non-negative) real numbers. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x| = \sqrt{x^T x}$. The space of all functions $x : [a, b] \to \mathbb{R}^n$ with bounded norm $\|x\|_2^2 = \int_a^b |x(t)|^2 \, dt$ is denoted by $\mathcal{L}_2([a, b], \mathbb{R}^n)$; $\mathcal{L}_\infty([a, b], \mathbb{R}^n)$ indicates the space of all bounded piecewise continuous functions mapping $[a, b]$ into $\mathbb{R}^n$. The notation $\mathcal{C}_n = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ refers to the Banach space of absolutely continuous functions that map the interval $[-\tau, 0]$ into $\mathbb{R}^n$. Moreover, $\mathcal{W}_n = \mathcal{W}([-\tau, 0], \mathbb{R}^n)$ refers to the space of functions $\varphi \in \mathcal{C}_n$ with square-integrable derivative, i.e., $\dot{\varphi} \in \mathcal{L}_2([-\tau, 0], \mathbb{R}^n)$ for $\varphi \in \mathcal{W}_n$ [17]. A block-diagonal matrix with A_1, ..., A_m on the diagonal is represented as $\mathrm{blkdiag}\{A_1, \cdots, A_m\}$, and I_m is the $m \times m$ identity matrix. Superscripts T and H denote matrix trans-

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position and conjugate transposition, respectively, whereas a star $*$ in a symmetric matrix represents a symmetric term.

II. Problem statement

Consider a time delay system Ω with a point-wise delay in the state variables as

$$\Omega : \begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bu(t), \\ y(t) = Cx(t) + C_d x(t-\tau) + Du(t), \\ \quad x_0 = \varphi, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ denotes the external input, $y(t) \in \mathbb{R}^p$ is the output, and τ denotes a constant time delay. We assume that there exists a constant $\bar{\tau} > 0$ such that for each $\tau \in [0, \bar{\tau}]$, the system is asymptotically stable for zero input. For $t \in \mathbb{R}$, the function segment $x_t : [-\tau, 0] \to \mathbb{R}^n$ denotes the state of Ω at the time instance t with $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0]$. The initial condition is given by φ, such that $x(t) = \varphi(t)$, $t \in [-\tau, 0]$. The objective is to find a reduced-order model $\hat{\Omega}$ that has the same stability properties and delay structure as Ω and that closely approximates the input-output behaviour of Ω. Moreover, the model reduction procedure itself should be applicable to time delay systems with polytopic uncertainties and it should facilitate structured model order reduction (i.e., model order reduction with the preservation of physical interconnection structures in a system) for time delay systems.

We should emphasize that due to the fact that the state belongs to $\mathcal{C}_n$, the system Ω has an infinite-dimensional nature despite the, possibly large, finite number of dynamical equations describing it. In this paper, model order reduction is accomplished with respect to only the latter aspect, i.e., by reducing the number of dynamical equations of Ω.

III. Observability and controllability matrix inequalities

In this section, we define observability and controllability functionals for the time delay system Ω. These functionals are, however, in general challenging to compute exactly. Therefore, we then define computable Lyapunov-Krasovskii functionals that upper/lower bound these energy functionals. We also provide matrix inequalities that characterise these bounds. The solution to these inequalities is later used as a basis for an extended model order reduction procedure.

A. Observability functional

The observability energy functional of a system characterizes the output energy of that system for a non-zero initial condition and zero input. We give the following definition.

Definition 1 ([12]): The observability functional of the system (1) is the functional $L_o : \mathcal{C}_n \to \mathbb{R}_{\geq 0}$ defined as

$$L_o(\varphi) = \int_0^\infty |y(t)|^2 dt, \tag{2}$$

where $y(\cdot)$ is the output of the system (1) for the initial condition $x_0 = \varphi$ and zero input.

We note that the existence of the observability functional in (2) is guaranteed by the asymptotic stability of the system Ω for $u = 0$. As mentioned before, it is in general difficult to compute the observability functional (2). Therefore, in the next lemma, we define a computable functional that can upper-bound this observability functional.

Lemma 1: Consider the asymptotically stable system (1). If there exist symmetric matrices $Q > 0$, $Q_d \geq 0$, $\bar{Q} \geq 0$ and $S > 0$, and a scalar α_o such that

$$M_o = \begin{bmatrix} N_{11} & \bar{Q} + SA_d & Q - S + \alpha_o A^T S & C^T \\ * & -Q_d - \bar{Q} & \alpha_o A_d^T S & C_d^T \\ * & * & -2\alpha_o S + \tau^2 \bar{Q} & 0 \\ * & * & * & -I_p \end{bmatrix} < 0, \tag{3}$$

with $N_{11} = SA + A^T S + Q_d - \bar{Q}$, then the functional $E_o : \mathcal{W}_n \times \mathcal{L}_2([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ defined as

$$\begin{aligned} E_o(\varphi, \dot{\varphi}) = \varphi^T(0) Q \varphi(0) + \int_{-\tau}^0 \varphi^T(s) Q_d \varphi(s)\, ds \\ + \tau \int_{-\tau}^0 \int_\theta^0 \dot{\varphi}^T(s) \bar{Q} \dot{\varphi}(s)\, ds d\theta, \end{aligned} \tag{4}$$

satisfies

$$E_o(\varphi, \dot{\varphi}) \geq L_o(\varphi), \tag{5}$$

for all $\varphi \in \mathcal{W}_n$ and with L_o as in Definition 1.

Proof: For the time derivative of $E_o(x_t, \dot{x}_t)$ along the solutions of (1) for zero input, we obtain

$$\begin{aligned} \dot{E}_o(x_t, \dot{x}_t) \leq\ & 2\dot{x}^T(t) Q x(t) + x^T(t) Q_d x(t) \\ & - x^T(t-\tau) Q_d x(t-\tau) + \tau^2 \dot{x}^T(t) \bar{Q} \dot{x}(t) \\ & - (x(t) - x(t-\tau))^T \bar{Q} (x(t) - x(t-\tau)), \end{aligned} \tag{6}$$

where Jensen's inequality [18] and the Newton-Leibniz formula have been used. Next, we incorporate the slack variables S and α_o. To this end, we consider the term $2(x^T(t) + \alpha_o \dot{x}^T(t)) S (Ax(t) + A_d x(t-\tau) - \dot{x}(t))$. Given (1), this term is always zero for $u = 0$ and $t \geq 0$. Therefore, adding it to the right-hand side of (6), we obtain

$$\begin{aligned} \dot{E}_o(x_t, \dot{x}_t) \leq\ & 2\dot{x}^T(t) Q x(t) + x^T(t) Q_d x(t) \\ & - x^T(t-\tau) Q_d x(t-\tau) + \tau^2 \dot{x}^T(t) \bar{Q} \dot{x}(t) \\ & - (x(t) - x(t-\tau))^T \bar{Q} (x(t) - x(t-\tau)) \\ & + 2(x(t) + \alpha_o \dot{x}(t))^T S (Ax(t) + A_d x(t-\tau) - \dot{x}(t)). \end{aligned} \tag{7}$$

Next, by adding and subtracting the term $|y(t)|^2$, with $y(t)$ from (1) for $u = 0$, to the right hand-side of (7) and writing the resulting inequality in a compact form, we obtain $\dot{E}_o(x_t, \dot{x}_t) \leq \xi_o^T(t) \bar{M}_o \xi_o(t) - |y(t)|^2$, where $\bar{M}_o$ results in M_o as in (3) after applying a Schur complement with respect to the output matrices C and C_d, and $\xi_o^T(t) := [x^T(t), x^T(t-\tau), \dot{x}^T(t)]$. This result implies that if $\bar{M}_o < 0$, and equivalently $M_o < 0$ due to the Schur complement, then $\dot{E}_o(x_t, \dot{x}_t) \leq -|y(t)|^2$; integration of both sides of which on the interval $[0, T]$ yields

$$E_o(x_T, \dot{x}_T) - E_o(x_0, \dot{x}_0) \leq -\int_0^T |y(t)|^2 dt. \tag{8}$$

Now, given the facts that $E_o(x_T, \dot{x}_T) \to 0$ for $T \to \infty$, due to the asymptotic stability of the system for zero input, and that $x_0 = \varphi$, (8) implies that $E_o(\varphi, \dot{\varphi}) \geq L_o(\varphi)$, which follows from Definition 1. This completes the proof. ■

B. Controllability functional

A controllability functional characterizes the minimum input energy required by a system of the form (1) to reach from the zero-state to a final state φ. In what follows, we provide a precise definition for this functional [12].

Definition 2: The controllability functional of the system (1) is the functional $L_c : \mathcal{D}_n \to \mathbb{R}_{\geq 0}$ defined as

$$L_c(\varphi)=\inf\left\{\int_{-\infty}^{0}|u(t)|^2 dt \;\middle|\; u\in \mathcal{L}_2\cap\mathcal{L}_\infty\big((-\infty,0],\mathbb{R}^m\big),\right.$$
$$\left.\lim_{T\to\infty} x_{-T}=0,\, x_0=\varphi\right\}, \tag{9}$$

where x_t is the solution of (1) for u that satisfies the above and $\mathcal{D}_n \subset \mathcal{C}_n$ is the domain of L_c, that is the space of function segments φ for which $L_c(\varphi)$ is well-defined.

The following lemma provides a characterization of a computable lower-bound on the controllability energy functional.

Lemma 2: Consider the system in (1). If there exists symmetric matrices $P > 0$, $P_d \geq 0$, $\bar{P} \geq 0$ and $R > 0$, and a scalar α_c such that

$$M_c = \begin{bmatrix} M_{11} & \bar{P}+A_dR & P-R+\alpha_c RA^T & B \\ * & -P_d-\bar{P} & \alpha_c RA_d^T & 0 \\ * & * & -2\alpha_c R+\tau^2\bar{P} & \alpha_c B \\ * & * & * & -I_m \end{bmatrix} < 0, \tag{10}$$

with $M_{11} = AR + RA^T + P_d - \bar{P}$, then the functional $E_c : \mathcal{W}_n \times \mathcal{L}_2([-\tau,0],\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ defined as

$$E_c(\varphi,\dot\varphi) = \varphi^T(0)U\varphi(0) + \int_{-\tau}^{0}\varphi^T(s)U_d\varphi(s)\,ds + \tau\int_{-\tau}^{0}\int_{\theta}^{0}\dot\varphi^T(s)\bar{U}\dot\varphi(s)\,dsd\theta, \tag{11}$$

with $U = R^{-1}PR^{-1}, U_d = R^{-1}P_dR^{-1}, \bar{U} = R^{-1}\bar{P}R^{-1}$, satisfies

$$E_c(\varphi,\dot\varphi) \leq L_c(\varphi), \tag{12}$$

for all $\varphi \in \mathcal{D}_n \cap \mathcal{W}_n$ and L_c as in Definition 2.

Proof: The proof is similar to that of Lemma 1. ■

Remark 1: The variables S, α_o in (2), and R, α_c in (10) are referred to as the slack variables. On the contrary, the variables $Q, Q_d, \bar{Q}$ and $U, U_d, \bar{U}$ (also $P, P_d, \bar{P}$) that characterise the energy functionals (4) and (11) are called the main decision variables in this context.

Remark 2: The introduction of the slack variables into the matrix inequalities (3) and (2) follows an idea in [3, Chapter 3], where a free term with slack variables is added to the derivative of an energy functional to alleviate the conservatism of stability criteria of time delay systems. Inspired by extended model reduction for delay-free systems [14], we use the idea of slack variables here to enhance a model reduction procedure for time delay systems such that the quality of model approximation is improved and structured/robust reduction of these systems is facilitated. In particular, we can freely set $S = Q$, $\alpha_o = \tau^2\bar{\alpha}_o$, and $R = P$ and $\alpha_c = \tau^2\bar{\alpha}_c$, with $\bar{\alpha}_o$ and $\bar{\alpha}_c$ some positive scalar variables. Substituting these into (3) and (10), while imposing the constraints $\bar{Q} = \bar{\alpha}_o Q$ and $\bar{P} = \bar{\alpha}_c P$, one can recover the matrix inequalities introduced in [13] for $\tau \neq 0$. The structured/robust reduction is yet to be discussed.

IV. Model order reduction by truncation

Next, we explain how a general model of the form (1) can be reduced through a truncation procedure. To this end, consider a partitioned form of $x(t)$ and x_t (and φ) as

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\; x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix},\; \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \tag{13}$$

where $x_1(t) \in \mathbb{R}^k$ and $\varphi_1 \in \mathcal{W}_k$, with $1 \leq k < n$. The corresponding partitioning of the system matrices is

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, A_d = \begin{bmatrix} A_{d,11} & A_{d,12} \\ A_{d,21} & A_{d,22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, C_d = \begin{bmatrix} C_{d,1} & C_{d,2} \end{bmatrix}. \tag{14}$$

Using this partitioning, a reduced-order approximation of (1), denoted by $\hat\Omega$, is obtained by truncation of the dynamics corresponding to x_2. Such an approximate model reads

$$\hat\Omega : \begin{cases} \dot\zeta(t) = A_{11}\zeta(t) + A_{d,11}\zeta(t-\tau) + B_1u(t), \\ \hat{y}(t) = C_1\zeta(t) + C_{d,1}\zeta(t-\tau) + Du(t), \\ \zeta_0 = \hat\varphi, \end{cases} \tag{15}$$

where $\zeta(t) \in \mathbb{R}^k$ and $\hat{y}(t) \in \mathbb{R}^p$ is an approximate of $y(t)$, and $\hat\varphi \in \mathcal{W}_k$ is the initial condition of the reduced model.

The system $\hat\Omega$ approximates x_1 in the partitioned coordinate. As can be clearly seen from (15), this model approximation preserves the delay structure. Moreover, as shown in the sequel, it guarantees the preservation of stability properties and also a computable a priori error bound, under certain conditions. To present these central properties, we first show that the observability and controllability energy functionals of the reduced-order system can be characterized in terms of those of the original system Ω.

Lemma 3: Let (3) hold for a scalar α_o and symmetric matrices $Q > 0$, $Q_d \geq$, $\bar{Q} \geq 0$ and $S > 0$ of the form

$$S = \mathrm{blkdiag}\{S_1, S_2\}, \quad S_1 \in \mathbb{R}^{k\times k}, \tag{16}$$

Then, the observability functional $\hat{L}_o : \mathcal{W}_k \to \mathbb{R}_{\geq 0}$ of the reduced-order system (15) exists, and the functional $\hat{E}_o : \mathcal{W}_k \times \mathcal{L}_2([-\tau,0],\mathbb{R}^k) \to \mathbb{R}_{\geq 0}$ given as

$$\hat{E}_o(\hat\varphi,\dot{\hat\varphi}) = \hat\varphi^T(0)Q_{11}\hat\varphi(0) + \int_{-\tau}^{0}\hat\varphi^T(s)Q_{d,11}\hat\varphi(s)\,ds + \tau\int_{-\tau}^{0}\int_{\theta}^{0}\dot{\hat\varphi}^T(s)\bar{Q}_{11}\dot{\hat\varphi}(s)\,dsd\theta, \tag{17}$$

with Q_{11}, $Q_{d,11}$ and $\bar{Q}_{11}$ respectively the upper left $k \times k$ subblocks of Q, Q_d and $\bar{Q}$, satisfies $\hat{E}_o(\hat\varphi,\dot{\hat\varphi}) \geq \hat{L}_o(\hat\varphi)$ for all $\hat\varphi \in \mathcal{W}_k$.

Proof: The proof uses a similar reasoning as the proof of [13, Lemma 4]. ■

The next lemma gives a similar characterization for the controllability functional of the reduced-order model.

Lemma 4: Let (10) hold for a scalar α_c and symmetric matrices $U > 0$, $U_d \geq 0$, $\bar{U} \geq 0$ and $R > 0$ of the form

$$R = \text{blkdiag}\{R_1, R_2\}, \quad R_1 \in \mathbb{R}^{k \times k}. \tag{18}$$

Moreover, let $\hat{L}_c : \mathcal{D}_k \to \mathbb{R}_{\geq 0}$ be the controllability functional of the reduced-order system (15). Then, the functional $\hat{E}_c : \mathcal{W}_k \times \mathcal{L}_2([-\tau, 0], \mathbb{R}^k) \to \mathbb{R}_{\geq 0}$ given as

$$\begin{aligned}\hat{E}_c(\hat{\varphi}, \dot{\hat{\varphi}}) = \hat{\varphi}^T(0)U_{11}\hat{\varphi}(0) + \int_{-\tau}^{0} \hat{\varphi}^T(s)U_{d,11}\hat{\varphi}(s)\, ds \\ + \tau \int_{-\tau}^{0} \int_{\theta}^{0} \dot{\hat{\varphi}}^T(s)\bar{U}_{11}\dot{\hat{\varphi}}(s)\, ds d\theta,\end{aligned} \tag{19}$$

with $U_{11}, U_{d,11}$ and $\bar{U}_{11}$ respectively the upper-left $k \times k$ subblocks of U, U_d and $\bar{U}$, satisfies $\hat{E}_c(\hat{\varphi}, \dot{\hat{\varphi}}) \leq \hat{L}_c(\hat{\varphi})$ for all $\hat{\varphi} \in \mathcal{D}_k \cap \mathcal{W}_k$.

Proof: The proof is similar to that of Lemma 3. ■

Given R as in (18) and the satisfaction of (10), it can be shown that an inequality of the same form and in terms of P_{11}, $P_{d,11}$, $\bar{P}_{11}$ and R_1 holds for the reduced-order model $\hat{\Omega}$ in (15). Likewise, there is also an equality of the form (3) holding for the reduced-order model. As a result, Lemmas 3 and 4 imply that the observability and controllability functionals of the reduced-order system can be obtained by relevant parts of the energy functionals of the original system (1) when S in (3) and R in (10) are block-diagonal as in (16) and (18), respectively. We stress that there is no structure required in the energy functionals themselves, but only in the slack variables. These structures on the slack variables are crucial as without those the properties of the reduced-order system (15) cannot be guaranteed.

Next, we define an extended-balanced realization of Ω.

Definition 3: A realization as in (1) is said to be extended balanced if there exists symmetric matrices $S > 0$, $Q > 0$, $Q_d \geq 0$, $\bar{Q} \geq 0$, and a scalar α_o satisfying (3), symmetric matrices $R > 0$, $P > 0$, $P_d \geq 0$, $\bar{P} \geq 0$, and a scalar α_c satisfying (10), and, additionally, S and R are such that

$$S = R = \Sigma = \text{blkdiag}\{\sigma_1 I_{m_1}, \sigma_2 I_{m_2}, \cdots, \sigma_q I_{m_q}\}. \tag{20}$$

Here, the constants $\sigma_i > 0$, satisfying $\sigma_i > \sigma_{i+1}$, $i \in \{1, ..., q-1\}$, are extended singular values and $\Sigma_{i=1}^{q} m_i = n$.

As S and R are symmetric positive definite matrices, there exists a coordinate transformation T which transforms (1) into an extended balanced form, as stated in the next lemma, the proof of which follows from standard results in, e.g., [19].

Lemma 5: Let there exists symmetric matrices $S > 0$, $Q > 0$, $Q_d \geq 0$ and $\bar{Q} \geq 0$, and a scalar α_o satisfying (3), and symmetric matrices $R > 0$, $P > 0$, $P_d \geq 0$ and $\bar{P} \geq 0$, and a scalar α_c satisfying (10). Then, there exists a coordinate transformation $x(t) = Tz(t)$, with $T \in \mathbb{R}^{n \times n}$, such that the realization in the new coordinates is extended balanced, i.e., the nonsingular matrix T can be chosen such that $T^T S T = T^{-1} R T^{-T} = \Sigma$, with Σ, as in (20), being the solution (for S and R simultaneously) of (3) and (10).

Remark 3: In the literature on finite-dimensional systems, a realization is said to be balanced if 1) the states that are easy to observe are those which are also easy to control, and vice versa, and, 2) the state components are ordered in terms of their contribution to the input-output behaviour of the system [20]. However, the transformed system due to Lemma 5 does not fully possess these properties. This is because the balancing procedure is based on the slack variables S and R, which do not explicitly contribute to the energy functionals (4) and (11). Moreover, the balancing procedure is performed in a finite-dimensional Euclidean space with respect to $x(t)$, while the state of a time delay system is a function segment.

The next theorem states that the described extended model order reduction technique preserves stability properties.

Theorem 1: Let the system (1), which is asymptotically stable for zero input, be in an extended-balanced realization and consider the reduced-order system (15) obtained by truncation for k such that $k = \Sigma_{i=1}^{r} m_i$ for some $r > 0$ and m_i as in Definition 3. Then, the reduced-order system $\hat{\Omega}$ is asymptotically stable for zero input.

Proof: The satisfaction of (3) and (10) guarantees for the reduced system the satisfaction of inequalities of the same form which are sufficient for asymptotic stability. ■

As stated in the following theorem, an interesting property of the proposed model reduction technique is the availability of an a priori bound on the $\mathcal{H}_\infty$-norm of error system $\Omega - \hat{\Omega}$.

Theorem 2: Let the asymptotically stable system Ω as in (1) be in an extended balanced realization, as defined in Definition 3, and consider the reduced-order system $\hat{\Omega}$, as in (15), obtained by truncation for $k = \Sigma_{i=1}^{r} m_i$ for some $r > 0$. Moreover, let $\alpha_o = \alpha_c = \alpha$. Then, for any common input function $u \in \mathcal{L}_2 \cap \mathcal{L}_\infty([0, T], \mathbb{R}^m)$ and initial conditions $\varphi = 0$ and $\hat{\varphi} = 0$ for (1) and (15), respectively,

$$\int_0^T |y(t) - \hat{y}(t)|^2\, dt \leq \varepsilon^2 \int_0^T |u(t)|^2\, dt,$$

for all $T \geq 0$ and where the error bound ε is given as $\varepsilon = 2\sum_{i=r+1}^{q} \sigma_i$, with σ_i as in (20).

Proof: This theorem is proved by extending the proof of [12, Theorem 7] and involving the slack variables. ■

Remark 4: Given Remark 2, the proposed extended model reduction technique in this paper recovers the model reduction technique proposed in [13] as a special case. Thus, the technique of this paper can outperform that of [13] in terms of error bound, at the cost of additional complexity in the characterization of the bounds on the energy functions.

V. Prospective applications

An extended model reduction is particularly suited for applications such as structured and robust model reduction.

Structured model order reduction: If the states of a system have a particular physical interpretation that should be preserved in the reduced model, a structured reduction technique should be used (see [21] for a detailed discussion). An example is given by the reduction of a plant and/or controller in a closed-loop setting. One approach for this problem in the case of delay-free systems is to impose a proper block-diagonal structure to the gramians in conventional techniques. This approach, however, compromises

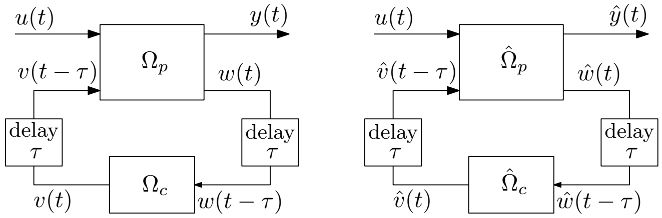

Fig. 1. A closed-loop system with measurement and actuation delays: (right) original system, (left) after structured model order reduction.

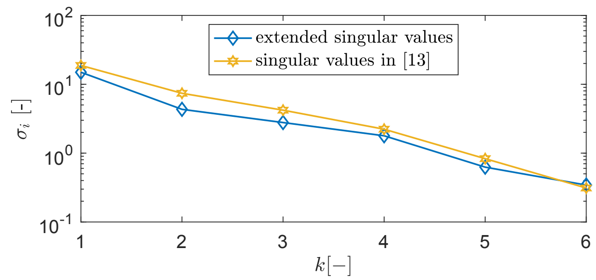

Fig. 2. Singular values of Example 1.

the flexibility and feasibility of equations involved and can, moreover, deteriorate the accuracy of the approximation. Alternatively, in extended structured model reduction, these structures are imposed to the slack variables [14] to avoid enforced structures on the main decision variables. With the proposed technique in this paper, we can take one step further and speak of extended structured reduction for time delay systems. The structured model reduction of closed-loop systems with measurement and actuation delays, as in the left-side of Fig. 1, is one of the applications of this technique. More precisely, assume that the plant Ω_p is described by

$$\Omega_p : \begin{cases} \dot{x}_p(t) = A_p x(t) + B_{pu} u(t) + B_{pv} v(t), \\ y(t) = C_{py} x(t), \\ w(t) = C_{pw} x(t), \end{cases} \tag{21}$$

with $x_p(t) \in \mathbb{R}^{n_p}$, and the controller Ω_c is described by

$$\Omega_c : \begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c w(t), \\ v(t) = C_c x_c(t), \end{cases} \tag{22}$$

with $x_c(t) \in \mathbb{R}^{n_c}$. Then, the closed-loop system in the presence of delays in the feedback channel can be described by a model of the form (1) with $x^T(t) = [x_p^T(t), x_c^T(t)]$ and

$$A = \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \; A_d = \begin{bmatrix} 0 & B_{pw} C_c \\ B_c C_{pw} & 0 \end{bmatrix}, \\ B^T = [B_{pu}^T, 0], \; C = [C_{py}, 0], \; C_d = 0, \; D = 0. \tag{23}$$

The objective is to perform model reduction that preserves the feedback structure of the closed-loop system, i.e., the state vector of the reduced system $\zeta^T(t) = [\zeta_p^T(t), \zeta_c^T(t)]$ is such that $\zeta_i(t) \in \mathbb{R}^{k_i}, 1 \le k_i < n_i$, in (15) is only a function of $x_i(t)$, $i \in \{p, c\}$. For this desirable structure to be preserved during model reduction, the balancing transformation T should have a proper block-diagonal form, obtained by enforcing a block-diagonal structure on S and R as follows:

$$S = \mathrm{blkdiag}\{S_p, S_c\}, \; R = \mathrm{blkdiag}\{R_p, R_c\}. \tag{24}$$

where $S_p, R_p \in \mathbb{R}^{n_p \times n_p}$. Summarizing, we propose the following corollary.

Corollary 1: Consider a closed-loop system as in Fig. 1 with stable Ω_p and Ω_c described, respectively, by (21) and (22) and a measurement and actuation time delay τ. Let us write this system in the form (1) with the realization (23). Moreover, let the inequalities (3) and (10) admit solutions of the form (24). Then, there exists an extended balancing transformation of the form $T = \mathrm{blkdiag}\{T_p, T_c\}$ that results in $\Sigma_s = \mathrm{blkdiag}\{\Sigma_p, \Sigma_c\}$, with T_p, Σ_p and T_c, Σ_c corresponding to the plant and controller, respectively, where both Σ_p and Σ_c individually have the properties of Σ in (20). Moreover, T preserves the feedback structure of original system and the system obtained by the truncation of any $k < n_p + n_c$ state components of the extended-balanced system is asymptotically stable and the corresponding error bound is twice the summation of all distinct truncated singular values.

Robust model order reduction: A second application of our extended balanced truncation procedure is in robust model reduction. Namely, a large class of uncertain (time delay) systems can be written in the form of (time delay) systems with polytopic uncertainties. Although the methods in [13] and [12] can be used for the reduction of this type of systems, those can result in low quality model approximations and conservative error bounds, if not infeasible. Conversely, the extended model reduction in this paper improves both the feasibility and the accuracy of model approximation for this type of systems. This is due to the fact that in an extended model reduction method we can assign a polytopic structure to the main decision variables and, thereby, increase the degrees of freedom in the model reduction procedure. However, due to page limitations, we do not present the details of this adaptation, but care to mention that it is similar to the case of finite-dimensional systems [16].

VI. Illustrative Examples

This section presents numerical examples. The involved matrix inequalities are solved using the software CVX [22].

Example 1: Here, we use the proposed technique for the reduction of the model in Example 1 in [13]. In this example, the original model with $\tau = 1.6$ s is of order $n = 6$ and it is approximated by a model of order $k = 2$. As can be seen in Fig. 2, the extended singular values obtained from the proposed technique are generally smaller than the singular values in [13]. As a result, the error bound obtained with the the proposed technique $\varepsilon = 11.06$ is significantly smaller than the error bound $\varepsilon = 15.26$ obtained in [13]. A comparison between the frequency response functions of the original model $G(j\omega)$ and the reduced-order model $\hat{G}(j\omega)$ is provided in Fig. 3. We observe, however, that in terms of the $\mathcal{H}_\infty$-norm of the error system $E(j\omega) = G(j\omega) - \hat{G}(j\omega)$, the technique of [13] outperforms the proposed one.

Example 2: we consider the structured model reduction of

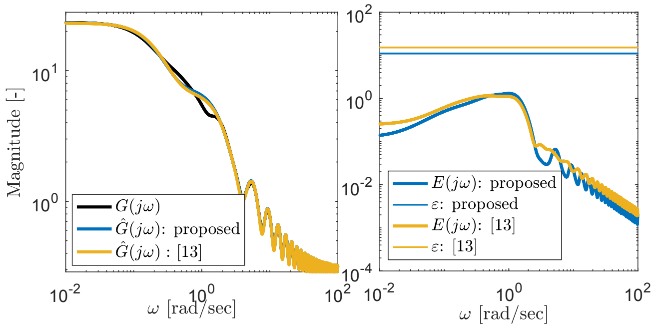

Fig. 3. Frequency response functions and error bounds in Example 1.

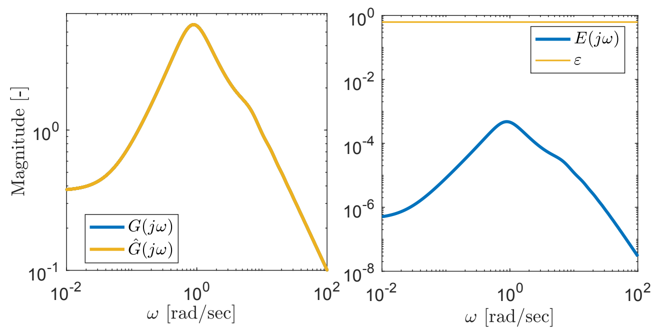

Fig. 4. Frequency response functions and error bound in Example 2.

the feedback system (23) with $\tau = 0.45$ s,

$$\begin{aligned} A_p &= \begin{bmatrix} 0 & 1 \\ -0.1 & -3 \end{bmatrix}, A_c = \begin{bmatrix} 0 & 1 \\ -1 & -20 \end{bmatrix}, B_{pu} = \begin{bmatrix} 0.05 \\ 1 \end{bmatrix}, \\ B_c &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_{pv} = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}, C_{pw}^T = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, C_c^T = -\begin{bmatrix} 2 \\ 0.1 \end{bmatrix}, \end{aligned} \tag{25}$$

and $C_{py} = C_{pw}$. The objective is to perform a structured model order reduction that preserves the feedback structure of the system using the technique proposed in Section V, as it is not feasible by using the method in [13]. The structured singular values obtained from the proposed method are $\Sigma_p = \mathrm{diag}\{19.12, 18.61\}$ and $\Sigma_c = \mathrm{diag}\{17.97, 0.31\}$. It is clear from Σ_p that truncation of any of the plant states results in a large error bound and it can cause a poor model approximation. Conversely, there is a large difference between the singular values of the controller. Therefore, we do not reduce the plant, but only truncate the state of the transformed controller that corresponds to the smallest singular value in Σ_c. In this way, we can still expect a good closed-loop model approximation. To have a comparison between the reduced-order and the original closed-loop systems, the frequency response function (from input u to output y) of the original closed-loop system $G(j\omega)$ is compared to that of the reduced-order system, indicated by $\hat{G}(j\omega)$, in Fig. 4. Clearly, the approximation is highly accurate in terms of the $\mathcal{H}_\infty$-norm of the error system $E(j\omega) = G(j\omega) - \hat{G}(j\omega)$.

VII. Conclusions

In this paper, by introducing slack variables in the computation of bounds on the energy functionals, we have obtained an extended model reduction technique for linear delay systems. This technique exhibits more flexibility compared to existing counterparts, making it interesting for purposes such as robust and structured model reduction. Moreover, the proposed technique preserves stability properties and also provides a computable error bound. The performance of this method has been illustrated through numerical examples.

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