Statistical bilinearization in stochastic nonlinear dynamics

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Abstract

A response approximation method for stochastically excited, nonlinear, dynamic systems is presented. Herein, the output of the nonlinear system is approximated by a finite-order Volterra series. The original nonlinear system is replaced by a bilinear system in order to determine the kernels of this Volterra series. The parameters of the bilinear system are determined by minimizing the difference between the original system and the bilinear system in a statistical sense. Application to a piece-wise linear system illustrates the effectiveness of this approach in approximating truly nonlinear, stochastic response phenomena in both the statistical moments and the power spectral density of the response of this system in case of a white noise excitation.

1 Introduction

Stochastically excited nonlinear dynamic systems are often encountered in practice. Examples are nonlinear suspensions in vehicles on random road surfaces, high-rise buildings forced by wind or earthquakes and offshore structures excited by wave motions at sea.

The simulation of the stochastic response of such systems [Kloeden and Platen, 1992] is, in general, very time-consuming since accurate estimates of the response statistics require the simulation of long (or many) time-series. Therefore, response approximation methods are needed. For strongly nonlinear systems the statistical linearization technique [Roberts and Spanos, 1990] generally fails to provide accurate results.

To overcome these problems, here a nonlinear approximation method, called statistical bilinearization, is presented. Herein, the input-output relation of the nonlinear system is described as a Volterra series [Volterra, 1959]. Of course, the computational efficiency using this description should be significantly higher than that of simulation of the response of the original nonlinear system. Therefore, a finite-order Volterra series will be used to describe the input-output relation of the nonlinear system. Finite-order Volterra systems are systems with polynomial nonlinearities [Rugh, 1981]. Consequently, a nonlinear approximation technique using polynomial nonlinearities can be seen as a natural extension of linearization. In order to determine the Volterra kernels in the Volterra series, the original, nonlinear system is replaced by a bilinear system which has the same Volterra kernels up to a certain order [Rugh, 1981]. The parameters of this bilinear system are determined to ensure that it describes the original system optimally in a statistical sense. Besides the fact that the replacement of the original system by a finite-order Volterra model gives us the advantage of computational efficiency, the gradual extension of the replacing models from linear towards polynomial will enhance our understanding of the nonlinear response phenomena of the original system.

The method of statistical bilinearization is applied to a piece-wise linear system. The piece-wise linear system can represent many systems with one-sided stiffness phenomena. Practical examples are elastic stops in vehicle suspensions, snubbers on solar arrays connected to satellites

[Van Campen et al., 1997], suspension bridges or models used in the offshore industry [Thompson and Stewart, 1986].

The basic ideas behind the use of an input-output description for the nonlinear system based on Volterra series are briefly described in section 2. In section 3, a technique called bilinearization (or Carleman linearization) is used to construct a finite-order Volterra model. In section 4, the statistical bilinearization technique is proposed and applied to the piece-wise linear system. In section 5, some results of the application of the statistical bilinearization technique to the piece-wise linear system are discussed and compared to those of simulation and statistical linearization. Conclusions will be presented in section 6.

2 Problem definition

Consider an affine, nonlinear system with the following state equations:

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{a}(t,\underline{x}(t)) + \underline{b}(t,\underline{x}(t))\ u(t) \\ y(t) &= \underline{c}(t)\ \underline{x}(t), \qquad t \geq 0,\ \underline{x}(0) = \underline{0}, \end{aligned} \tag{1}$$

where $\underline{x}(t)$ is an n-dimensional state column-vector, while $u(t)$ is a scalar input and $y(t)$ a scalar output. It is assumed that $\underline{a}$, $\underline{b}$ and $\underline{c}$ are analytic functions in $\underline{x}$ and continuous in t.

In Rugh [1981], it is stated that, when a solution of the state equation (1) exists for $u(t) = 0$ ($t \in [0, T]$) and initial condition $\underline{x}(0) = \underline{x}_0$, there is a Volterra system representation [Volterra, 1959] for (1) (with initial condition $\underline{x}(0) = \underline{0}$):

$$\begin{aligned} y(t) = &\int_0^t h_1(t-\tau_1)\ u(\tau_1)\ d\tau_1 + \ldots + \\ &+ \int_0^t \ldots \int_0^{\tau_{k-1}} h_k(t-\tau_1,\ldots,t-\tau_k) \\ &\quad u(\tau_1)\ldots u(\tau_k)\ d\tau_1 \ldots d\tau_k + \ldots, \end{aligned} \tag{2}$$

Herein, $h_k(t-\tau_1, t-\tau_2, \ldots, t-\tau_k),\ k = 1, 2, \ldots,$ is called the k^{th}-order Volterra kernel. The first term in (2) corresponds to the well-known convolution representation of linear systems. The subsequent terms in (2), for $k = 2, 3, \ldots$, represent natural extensions of the linear system using polynomial, nonlinear terms.

Since in general solving (1) for a given arbitrary input function $u(t)$ is a difficult or even impossible task, many people have tried some kind of approximation technique to describe the input-output behaviour of this system in an approximate way. In this perspective, we will aim to determine a polynomial input-output expression for (1) up to order p by truncating the series in (2):

$$\begin{aligned} y(t) = &\int_0^t h_1(t-\tau_1)\ u(\tau_1)\ d\tau_1 + \\ &\sum_{k=2}^{p} \int_0^t \ldots \int_0^{\tau_{k-1}} h_k(t-\tau_1,\ldots,t-\tau_k) \\ &\quad u(\tau_1)\ldots u(\tau_k)\ d\tau_1 \ldots d\tau_k, \end{aligned} \tag{3}$$

which approximates (1) sufficiently close at least on a finite time interval and for small enough inputs. The main issue in the characterization of such system is the determination of $h_k(t-\tau_1, t-\tau_2, \ldots, t-\tau_k),\ k = 1, 2, \ldots, p$. Hereto, the bilinearization procedure, as described in section 3, will be used.

3 Bilinearization

In this section, a method called bilinearization or Carleman linearization [Rugh, 1981] is described. The idea is that an affine, nonlinear system with analytic nonlinearities, as in (1), can be approximated by a system with bilinear state equations of the form [Lesiak and Krener, 1978]:

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{A}(t)\underline{x}(t) + (\underline{D}(t)\underline{x}(t) + \underline{e}(t))u(t) \\ y(t) &= \underline{C}(t)\underline{x}(t), \qquad t \geq 0,\ \underline{x}(0) = \underline{0}, \end{aligned} \tag{4}$$

where $\underline{x}(t)$ is an n_b-dimensional ($n_b = \sum_{l=1}^{p} n^l$) column-vector with state variables, while $u(t)$ and $y(t)$ are scalar inputs and outputs, respectively. Moreover, it is important to note that analytic expressions for the Volterra kernels of such a bilinear system are available [Bruni et al., 1971]. One can truncate the resulting Volterra system at a specific order to obtain a finite-order Volterra system description as in (3). This system description can then be used to approximate the response statistics of the bilinear system and, thus, of the original, nonlinear system (1).

3.1 The bilinearization technique

Here, the bilinearization technique will be described briefly. We aim to determine a polynomial input-output expression for (1) up to order p, as in (3), which approximates (1) sufficiently close. To do so, first, bilinear state equations, as in (4), have to be constructed in such a way that these can be represented by the same Volterra kernels (up to order p) as the system in (1). Next, the input-output relation for that bilinear system can be determined. Then, an approximation for the input-output relation of (1) is available and can be used to approximate the response statistics of this system.

In order to find an approximate description of (1) in terms of a system with bilinear state equations, the right-hand side of (1) can be replaced by a power series representation:

$$\begin{aligned}\underline{a}(t,\underline{x}(t)) &= \sum_{k=1}^{\infty} \underline{A}_k(t)\ \underline{x}^{(k)}(t)\\ \underline{b}(t,\underline{x}(t)) &= \sum_{k=0}^{\infty} \underline{B}_k(t)\ \underline{x}^{(k)}(t),\end{aligned} \tag{5}$$

where the Kronecker product notation is used: $\underline{x}^{(2)}(t) = \underline{x}(t) \otimes \underline{x}(t)$. Using (5), (1) can be rewritten as

$$\begin{aligned}\dot{\underline{x}}(t) = &\sum_{k=1}^{p} \underline{A}_k(t)\ \underline{x}^{(k)}(t) + \\ &+ \sum_{k=0}^{p-1} \underline{B}_k(t)\ \underline{x}^{(k)}(t)\ u(t) + \dots\\ y(t) = &\underline{c}(t)\ \underline{x}(t), \quad \underline{x}^{(k)}(0) = \underline{0},\ t \geq 0.\end{aligned} \tag{6}$$

In order to determine the first p kernels corresponding to (6), differential equations are developed for $\underline{x}^{(j)}(t)$ [Rugh, 1981]:

$$\begin{aligned}\frac{d}{dt}[\underline{x}^{(j)}(t)] = &\sum_{k=1}^{p-j+1} \underline{A}_{j,k}(t)\ \underline{x}^{(k+j-1)}(t)\\ &+ \sum_{k=0}^{p-j} \underline{B}_{j,k}(t)\ \underline{x}^{(k+j-1)}(t) + \dots,\end{aligned} \tag{7}$$

with $\underline{x}^{(j)}(0) = \underline{0}$ (for $j = 1,\dots,p$), $\underline{A}_{1,k} = \underline{A}_k$ and, for $j > 1$,

$$\begin{aligned}\underline{A}_{j,k}(t) = &\underline{A}_k(t) \otimes \underline{I}_n \otimes \dots \otimes \underline{I}_n\\ &+ \underline{I}_n \otimes \underline{A}_k(t) \otimes \dots \otimes \underline{I}_n +\\ &+ \dots + \underline{I}_n \otimes \dots \otimes \underline{I}_n \otimes \underline{A}_k(t).\end{aligned} \tag{8}$$

The notation for $\underline{B}_{j,k}(t)$ is likewise. Now, by setting

$$\underline{x}^{\otimes}(t) = \left[\underline{x}^{(1)}(t)\ \underline{x}^{(2)}(t)\ \dots\ \underline{x}^{(p)}(t)\right]^T, \tag{9}$$

(7) can be written as a bilinear state equation neglecting terms of order larger than p, i.e. $\underline{x}^{(p+i)}(t),\ i > 0$:

$$\begin{aligned}\frac{d}{dt}\underline{x}^{\otimes}(t) = &\begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \dots & \underline{A}_{1p} \\ \underline{0} & \underline{A}_{21} & \dots & \underline{A}_{2,p-1} \\ \underline{0} & \underline{0} & \dots & \underline{A}_{3,p-1} \\ \vdots & \vdots & \vdots & \vdots \\ \underline{0} & \underline{0} & \dots & \underline{A}_{p1} \end{bmatrix} \underline{x}^{\otimes}(t)\\ + &\begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} & \dots & \underline{B}_{1,p-1} & \underline{0} \\ \underline{B}_{20} & \underline{B}_{21} & \dots & \underline{B}_{2,p-2} & \underline{0} \\ \underline{0} & \underline{B}_{30} & \dots & \underline{B}_{3,p-3} & \underline{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \underline{0} & \underline{0} & \dots & \underline{B}_{p0} & \underline{0} \end{bmatrix} \underline{x}^{\otimes}(t)u(t)\\ &+ \begin{bmatrix}\underline{B}_{10} & \underline{0} & \underline{0} & \dots & \underline{0}\end{bmatrix}^T u(t),\\ y(t) = &\begin{bmatrix}\underline{c}(t) & \underline{0} & \dots & \underline{0}\end{bmatrix} \underline{x}^{\otimes}(t),\end{aligned} \tag{10}$$

where $\underline{x}^{\otimes}(0) = \underline{0}$ and $\underline{x}^{\otimes}(t)$ is an n_b-dimensional column-vector of state variables ($n_b = \sum_{l=1}^{p} n^l$). This equation is called a Carleman linearization or bilinearization of the linear-analytic state equation (1).

3.2 Input-output relation for bilinear state equations

Here, the Volterra representation of (10) will be described. Since the Volterra representation of (10) coincides with that of (1) up to order p, we can now use (10) to evaluate the input-output behaviour of (1). Note that (10) is a bilinear system as in (4).

It can be derived that the input-output relation of the bilinear state equations (4) can be written as [Rugh, 1981]:

$$\begin{aligned}y(t) = &\sum_{k=1}^{\infty} \int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{k-1}} \underline{C}(t)\underline{\Phi}(t,\tau_1)\underline{D}(\tau_1)\\ &\underline{\Phi}(\tau_1,\tau_2)\underline{D}(\tau_2)\dots\underline{D}(\tau_{k-1})\underline{\Phi}(\tau_{k-1},\tau_k)\\ &\underline{e}(\tau_k)\ u(\tau_1)\dots u(\tau_k)\ d\tau_k \dots d\tau_1,\end{aligned} \tag{11}$$

where $\underline{\Phi}(t,\tau)$ is the transition matrix of $\underline{A}(t)$ defined by the Peano-Baker series [Rugh, 1981].

We once more assume that $\underline{x}(0) = \underline{0}$ in equation (11). For a stationary system $\underline{A}(t)$ is a constant matrix and thus $\underline{\Phi}(t_1, t_2) := \underline{\Phi}(t_1 - t_2) = e^{\underline{A}(t_1 - t_2)}$. Combining (11), (10), (4) and (3) gives the Volterra kernels of the bilinear system (10). The kernels up to order p also represent the kernels of (1).

4 Statistical bilinearization: application to the piecewise linear system

In this section, the bilinearization technique will be used within a technique that will be termed statistical bilinearization. The statistical bilinearization technique will be applied to a piecewise linear system that can be described by the following differential equation:

$$\ddot{x} + 2\zeta\, \dot{x} + x + \alpha\, \epsilon(x)\, x = u, \tag{12}$$

where

$$\epsilon(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}, \tag{13}$$

ζ represents a dimensionless damping parameter, α is a nonlinearity parameter and u is a stationary, random, Gaussian, zero-mean excitation process. Hereto, this system will be approximated using polynomial nonlinearities. Here, only terms up to order two ($p = 2$) will be used. The approximating system can, therefore, be written as

$$\begin{aligned} \ddot{x} + 2\zeta\, \dot{x} + \beta_1\, x_E \\ + \beta_2 \left(x_E^2 - E\{x_E^2\}\right) = u, \end{aligned} \tag{14}$$

where $x_E = x - E\{x\}$. Equivalent to the procedure followed in statistical linearization [Roberts and Spanos, 1990], in which the approximating system is described by (14) with $\beta_2 = 0$, an error has to be defined. This error embodies the difference between the piece-wise linear system and the quadratic system (14):

$$\begin{aligned} \varepsilon_{bilin} =& (x + \alpha\, \epsilon(x)\, x) - \beta_1\, x_E \\ & - \beta_2\, \left(x_E^2 - E\{x_E^2\}\right), \end{aligned} \tag{15}$$

for given β_1 and β_2. Subsequently, our goal is to minimize $E\{\varepsilon_{bilin}^2\}$ with respect to β_1 and β_2. This results in the following equations:

$$\begin{aligned} \beta_1 =& \frac{E\{x_E(x + \alpha\, \epsilon(x)\, x)\}}{\sigma_x^2}, \\ \beta_2 =& \frac{E\{x_E^2(x + \alpha\epsilon(x)x)\}}{E\{x_E^4\} - \sigma_x^4} \\ & - \frac{E\{x + \alpha\epsilon(x)x\}\sigma_x^2}{E\{x_E^4\} - \sigma_x^4}. \end{aligned} \tag{16}$$

At this point we have four unknown quantities (μ_x, σ_x, β_1 and β_2) and two equations. A third equation can be found through the averaging of equation (12):

$$E\{x + \alpha\, \epsilon(x)\, x\} = 0. \tag{17}$$

In order to find a necessary fourth equation to solve for the unknowns the bilinearization procedure will be applied. This will yield an expression for σ_x^2 for given values of β_1 and β_2.

By choosing an approximating Volterra system as in (14), the power series representation of the original, nonlinear system as required in (5) is readily defined. Consequently, the matrices in equation (10) can be determined. Since this equation is a bilinear state equation of the form of (4), the matrices of (4) are also known. Next, (11) can be used to compute the kernels of this bilinear system. However, first $\underline{\Phi}(t - \tau) = e^{\underline{A}(t-\tau)}$ is computed. This can be done using the relation $e^{\underline{A}t} = \mathcal{L}^{-1}\left\{(s\underline{I}_N - \underline{A})^{-1}\right\}$, where $\mathcal{L}$ is the Laplace operator and $s \in \mathbb{C}$. The elements of $\underline{\Phi}(t-\tau)$ will be denoted by $\Phi_{jk}(t-\tau)$, $j, k = 1, \ldots, 6$. Now, the Volterra kernels of the bilinear system can be evaluated. Firstly, the stationary first-order (linear) kernel can be determined from (11):

$$h_1(t - \tau_1) = \underline{c}\underline{\Phi}(t - \tau_1)\underline{e} = \Phi_{12}(t - \tau_1). \tag{18}$$

Secondly, the observation of (11) admits the determination of the second-order kernel:

$$\begin{aligned} h_{2tri}(t, \tau_1, \tau_2) = \theta(t - \tau_1)\, \theta(\tau_1 - \tau_2) \\ \left[\Phi_{12}(\tau_1 - \tau_2)\, (\Phi_{14}(t - \tau_1) + \Phi_{15}(t - \tau_1)) + \right. \\ \left. 2\, \Phi_{16}(t - \tau_1)\Phi_{22}(\tau_1 - \tau_2)\right], \end{aligned} \tag{19}$$

where

$$\theta(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}. \tag{20}$$

The presence of the θ terms in (19) implies that $t > \tau_1 > \tau_2$, which means that h_{2tri} is

a triangular kernel, indicated by the subscript tri. The fact that h_{2tri} is a triangular kernel follows from the integration limits in (11). Since $h_{2tri}(t, \tau_1, \tau_2) = h_{2tri}(t + \Delta t, \tau_1 + \Delta t, \tau_2 + \Delta t)$, h_{2tri} is a stationary kernel. In case of stationarity, the kernel h_{2tri} can be written as $h_{2tri}(\tau_1, \tau_2) := h_{2tri}(0, -\tau_1, -\tau_2)$. At this point, we have information on the first-order and second-order Volterra kernels of the bilinear system.

This information can be used to compute the variance of the output of the bilinear system $\sigma_y^2 (= \sigma_x^2)$ using

$$\sigma_y^2 = \int_{-\infty}^{\infty} S_{yy}(\omega)\, d\omega. \tag{21}$$

For a second-order Volterra system, such as (14), the power spectral density $S_{yy}(\omega)$ obeys

$$\begin{aligned} S_{yy}(\omega) =& \mid H_1(i\omega) \mid^2 \; S_{uu}(\omega) \\ &+ 2 \int_{-\infty}^{\infty} H_{2symm}(i(\omega - \gamma), i\gamma) S_{uu}(\gamma) \\ & H_{2symm}(i(-\omega + \gamma), -i\gamma) S_{uu}(\omega - \gamma) d\gamma, \end{aligned} \tag{22}$$

where $H_1(s)$, $s \in \mathbb{C}$ is the first-order transfer function, which can be determined by taking the one-dimensional Laplace transform of $h_1(t)$. Moreover, $H_2(s_1, s_2)$, with $s_1 \in \mathbb{C}$ and $s_2 \in \mathbb{C}$, is the second-order, symmetric transfer function, which can be found by, firstly, performing a two-dimensional Laplace transform on $h_{2tri}(\tau_1, \tau_2)$ and, secondly, performing a symmetrizing operation on the result. Now, using (22), the power spectral density of the output can be computed. Consequently, the variance of the output can be evaluated through (21) for specific values of β_1 and β_2 and, thus, we have defined the fourth equation needed in the statistical bilinearization technique.

Since $\sigma_y^2 = \sigma_x^2$ is now known, a new estimate for the mean of the response of the piecewise linear system can be determined using (17). New values for β_1 and β_2 can be computed by solving the equations in (16). In order to be able to evaluate the expected values in these equations, a functional form for the probability density function of the response has to be chosen. Here, for the sake of efficiency, a Gaussian probability density function is used. Of course, the procedure, described here, has to be applied recursively in an optimization loop.

5 Results

The statistical bilinearization technique is applied to the piece-wise linear system. Hereby, we investigate the white-noise excited case with $S_{uu}(\omega) = \frac{1}{2\pi}$. Of course, cases involving non-white excitations can be investigated as well.

In figure 1, the estimates for the standard deviation of the response of the piece-wise linear system, obtained by application of the statistical bilinearization technique, are displayed. In this

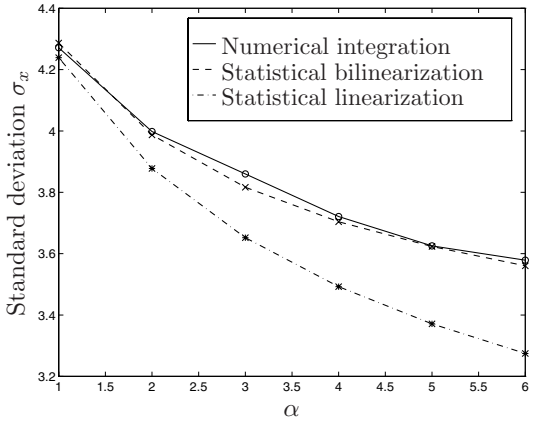

Figure 1: Estimation of the standard deviation σ_x for $\zeta = 0.01$.

figure, these results are compared to the results of the statistical linearization technique and simulation for varying nonlinearity α and $\zeta = 0.01$. It should be noted that the statistical linearization technique is equivalent to the statistical bilinearization technique when the quadratic terms in (14) are omitted. Clearly, the statistical bilinearization technique estimates the standard deviation of the response very accurately, in contradiction to the statistical linearization technique. The source of this accurate approximation can be found by observing the frequency domain information, see figure 2. This figure shows that two important nonlinear frequency domain phenomena, namely, the multiple resonance peaks (two in this case) and the high-energy, low-frequency spectral content, are modeled very well by the bilinearization procedure, whereas the statistical linearization technique fails to model these specifically nonlinear

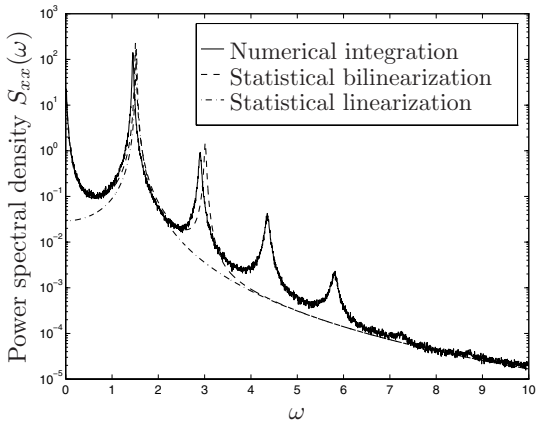

Figure 2: Estimation of the power spectral density $S_{xx}(\omega)$ for $\alpha = 6$ and $\zeta = 0.01$.

response phenomena. These phenomena represent important contributions to the energy in the response. As a consequence, the variance of the response can only be estimated accurately when these phenomena are modeled. Clearly, only the second resonance frequency appears, whereas higher resonances are absent in the output of the Volterra model. This is a consequence of the fact that we only incorporated a second-order polynomial nonlinearity in our nonlinear model. Higher resonances could be approximated by including higher-order polynomial terms in our Volterra model. It should be noted that the high-energy, low-frequency, spectral content of the response is due to the stiffness asymmetry in the piece-wise linear system. The presence of the (asymmetric) quadratic nonlinearity of the bilinear system ensures accurate approximation of the phenomenon.

In this form, the statistical bilinearization technique is computationally very efficient (computation times are comparable to those of the statistical linearization technique). Moreover, the bilinearization approach provides much more accurate results than the statistical linearization technique (in the application to the piece-wise linear system). This is a consequence of the fact that in the statistical bilinearization technique the most important, nonlinear, frequency-domain response phenomena are modeled. The statistical bilinearization procedure is so efficient because it can provide very accurate results using only a second-order Volterra system. In this respect it distincts itself from the bilinearization procedure as proposed in Lesiak and Krener [1978]. Namely, in the bilinearization procedure a bilinear system is pursued, whose output $y_b(t)$ converges to the output of the original, nonlinear system $y(t)$. The system parameters of the bilinear system are determined by minimizing the error on the output through $\mid y_b(t) - y(t) \mid$ for all t in the time interval of interest. In the statistical bilinearization procedure the parameters of the bilinear system are determined by minimizing an error that represents a weighted closeness of the original, nonlinear system and bilinear system; namely, the parameters are determined by minimizing $E\{\varepsilon_{bilin}^2\}$ with ε_{bilin} given in (15). As a consequence, accurate results can be obtained using a low-order Volterra model.

6 Conclusions

In this paper, a method called statistical bilinearization was developed. The strength of the method can be recognized in the combination of two features. Firstly, the response statistics of the bilinear model can be computed very efficiently (as long as the order of the polynomial model is low). Secondly, a truly nonlinear approximation approach is followed, which makes it possible to accurately approximate typically nonlinear phenomena in the original, nonlinear system in accordance to (the nonlinear) reality.

The statistical bilinearization technique was applied successfully to the piece-wise linear system. This application resulted in very accurate variance estimates for the response. Furthermore, typically nonlinear, frequency-domain response phenomena, such as multiple resonance peaks and high-energy, low-frequency spectral content, are modeled accurately. Moreover, it should be noted that the method is numerically far more efficient than simulation and can even compete with the statistical linearization technique in this respect, as long as the polynomial model used in the bilinearization technique is of a low order. Such a low-order Volterra model can provide accurate results because its parameters are determined according to a statistical criterion.

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