2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Controller synthesis for incremental stability: Application to symbolic controller synthesis

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Abstract— **Incremental stability is a property of dynamical and control systems, requiring the stability and convergence of trajectories with respect to each other, rather than with respect to an equilibrium point or a particular trajectory. Most design techniques providing controllers rendering control systems incrementally stable have two main drawbacks: they can only be applied to control systems in parametric-strict-feedback or strict-feedback form, and they require the control systems to be smooth. In this paper, we propose a controller design technique that is applicable to larger classes of control systems, including a class of non-smooth control systems. Moreover, we propose a recursive way of constructing incremental Lyapunov functions which have been identified as a key tool enabling the construction of finite abstractions of nonlinear control systems. The effectiveness of the proposed results in this paper is illustrated by synthesizing a controller rendering a non-smooth control system incrementally stable as well as constructing its finite abstraction, using the computed incremental Lyapunov function. Finally, using the constructed finite abstraction, we synthesize another controller for the incrementally stable closed-loop system enforcing the satisfaction of logic specifications, difficult (or even impossible) to enforce using conventional techniques.**

I. Introduction

This paper proposes a synthesis strategy rendering (non-smooth) control systems incrementally stable. In incremental stability, focus is on the stability and convergence of all trajectories with respect to each other rather than merely with respect to an equilibrium point or a specific trajectory. Note that there exist dynamical and control systems that are stable (with respect to a particular solution) but not incrementally stable [1]. Examples of applications of incremental stability include building explicit bounds on the region of attraction in phase-locking in the Kuramoto system [3], global synchronization in networks of cyclic feedback systems [5], control reconfiguration of piecewise affine systems with actuator and sensor faults [16], construction of symbolic models for nonlinear control systems [15], [4], [12], and synchronization of complex networks [18]. Unfortunately there are very few results available in the literature regarding the design of controllers enforcing incremental stability of the resulting closed-loop systems. Therefore, there is a growing need to develop design methods rendering control systems incrementally stable. One of the design approaches, which received much more attention, is the backstepping method.

Related works include controller design for convergence of Lur'e-type systems [14] and a class of piecewise affine systems [23] through the solution of linear matrix inequalities (LMIs). In contrast, the results presented in this paper do not require the solution of the LMIs and existence of controllers is always guaranteed. Existing backstepping design approaches either render parametric-strict-feedback[1] form systems incrementally globally asymptotically stable[2] using the notion of contraction metrics in [6], [20], [19], or render strict-feedback[1] form systems incrementally input-to-state stable[3] using the notion of contraction metrics and incremental Lyapunov functions in [25] and [24], respectively. The results in [14] offer a backstepping design approach rendering a larger class of control systems than those in strict-feedback form input-to-state convergent, rather than incrementally input-to-state stable. The notion of input-to-state convergence requires existence of a trajectory which is bounded on the whole time axis which is not necessarily the case in incremental (input-to-state) stability. Moreover, we note that the notion of (input-to-state) convergence can not be applied in the scope of the results in [15], [4], [12], which require uniform stability and convergence of trajectories with respect to each other rather than the uniform asymptotic stability of a particular trajectory. See [25], [17] for a comparison between the notions of convergent system and incremental stability.

The results in this paper improve upon existing backstepping techniques for incremental stability by addressing the following three aspects in unison:

1) the controller design enforces not only incremental global asymptotic stability but also incremental input-to-state stability;
2) the results are applicable to larger classes of control systems including a class of non-smooth control systems;
3) a recursive way of constructing incremental Lyapunov functions is provided.

In the first direction, our technique extends the results in [6], [20], [19], where only controllers enforcing incremental global asymptotic stability are designed. In the second direction, our technique improves the results in [6], [20], [19], which are only applicable to smooth parametric-strict-feedback form systems, and the results in [25], [24], which are only applicable to smooth strict-feedback form systems. In the third direction, our technique extends the results in [6], [20], [19], [25], where the authors only provide a recursive way of constructing contraction metrics, and the results in [14], where the authors do not provide a way to construct Lyapunov functions characterizing the input-to-state convergence property induced by the controller. Note that having incremental Lyapunov functions explicitly is necessary in many applications. Examples include construction of symbolic models for nonlinear control systems [4],

This research was sponsored in part by the grant NSF-CNS-0953994, a contract from Toyota Corp, and by the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement no. 257462 HYCON2 Network of excellence.

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[1] See [9] for a definition.
[2] Understood in the sense of Definition 2.2.
[3] Understood in the sense of Definition 2.4.

approximation of stochastic hybrid systems [7], and source-code model checking for nonlinear dynamical systems [8]. Note that incremental Lyapunov functions can be used as bisimulation functions, which are recognized as a key tool for the analysis provided in [7], [8].

Our technical results are illustrated by designing an incrementally input-to-state stabilizing controller for an unstable non-smooth control system that does not satisfy the assumptions required in [6], [20], [19], [25], [24]. Moreover, we construct a finite bisimilar abstraction for the resulting incrementally stable closed-loop system using the results in [4], which can only be applied to incrementally stable systems with explicitly available incremental Lyapunov functions. When a finite abstraction is available, the synthesis of the controllers satisfying logic specifications expressed in linear temporal logic or automata on infinite strings can be easily reduced to a fixed-point computation over the finite-state abstraction [22]. Note that satisfying those specifications is difficult or impossible to enforce with conventional control design methods. We synthesize another controller for the incrementally stable closed-loop system satisfying some logic specification explained in detail in the example section.

II. Control Systems and Stability Notions

A. Notation

The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ denote the set of natural, real, positive, and nonnegative real numbers, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, by $|x_i|$ the absolute value of x_i, and by $\|x\|$ the Euclidean norm of x; we recall that $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty :=$ (ess)sup$\{\|f(t)\|, t \geq 0\}$ and $\|f\|_{[0,\tau)} :=$ (ess)sup$\{\|f(t)\|, t \in [0,\tau)\}$. Function f is essentially bounded if $\|f\|_\infty < \infty$. For a given time $\tau \in \mathbb{R}^+$, define f_τ so that $f_\tau(t) = f(t)$, for any $t \in [0,\tau)$, and $f_\tau(t) = 0$ elsewhere; f is said to be locally essentially bounded if for any $\tau \in \mathbb{R}^+$, f_τ is essentially bounded. A function $f : \mathbb{R}^n \to \mathbb{R}_0^+$ is called radially unbounded if $f(x) \to \infty$ as $\|x\| \to \infty$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r,s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed nonzero r, the map $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$. A function $\mathbf{d} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a metric on $\mathbb{R}^n$ if for any $x, y, z \in \mathbb{R}^n$, the following three conditions are satisfied: i) $\mathbf{d}(x,y) = 0$ if and only if $x = y$; ii) $\mathbf{d}(x,y) = \mathbf{d}(y,x)$; and iii) (triangle inequality) $\mathbf{d}(x,z) \leq \mathbf{d}(x,y) + \mathbf{d}(y,z)$. Given measurable functions $f : \mathbb{R}_0^+ \to \mathbb{R}^n$ and $g : \mathbb{R}_0^+ \to \mathbb{R}^n$, we define $\mathbf{d}(f,g)_\infty :=$ (ess)sup$\{\mathbf{d}(f(t),g(t)), t \geq 0\}$.

B. Control systems

The class of control systems that we consider in this paper is formalized in the following definition.

Definition 2.1: A *control system* is a quadruple $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$, where:

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is the input set;
- $\mathcal{U}$ is the set of all measurable and locally essentially bounded functions of time from intervals of the form $]a,b[\subseteq \mathbb{R}$ to U with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $Q \subset \mathbb{R}^n$, there exists a constant $Z \in \mathbb{R}^+$ such that $\|f(x,u) - f(y,u)\| \leq Z\|x - y\|$ for all $x, y \in Q$ and all $u \in \mathsf{U}$.

A curve $\xi :]a,b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $\upsilon \in \mathcal{U}$ satisfying:

$$\dot{\xi}(t) = f\left(\xi(t), \upsilon(t)\right), \tag{II.1}$$

for almost all $t \in]a,b[$. We also write $\xi_{x\upsilon}(t)$ to denote the point reached at time t under the input υ from initial condition $x = \xi_{x\upsilon}(0)$; the point $\xi_{x\upsilon}(t)$ is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories [21].

A control system Σ is said to be forward complete if every trajectory is defined on an interval of the form $]a,\infty[$. Sufficient and necessary conditions for a system to be forward complete can be found in [2]. A control system Σ is said to be smooth if f is an infinitely differentiable function of its arguments.

C. Stability notions

Here, we recall the notions of incremental global asymptotic stability ($\delta_\exists$-GAS) and incremental input-to-state stability ($\delta_\exists$-ISS), presented in [25].

Definition 2.2: A control system Σ is incrementally globally asymptotically stable ($\delta_\exists$-GAS) if it is forward complete and there exist a metric $\mathbf{d}$ and a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$ and any $\upsilon \in \mathcal{U}$ the following condition is satisfied:

$$\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon}(t)\right) \leq \beta\left(\mathbf{d}\left(x, x'\right), t\right). \tag{II.2}$$

The notion of incremental global asymptotic stability (δ-GAS), defined in [1], requires the metric $\mathbf{d}$ to be the Euclidean metric. However, Definition 2.2 only requires the existence of a metric. We note that while δ-GAS is not generally invariant under changes of coordinates, $\delta_\exists$-GAS is. When the origin is an equilibrium point for Σ, with $\upsilon(t) = 0$ for any $t \in \mathbb{R}_0^+$, and the map $\psi : \mathbb{R}^n \to \mathbb{R}_0^+$, defined by $\psi(\cdot) = \mathbf{d}(\cdot, 0)$, is continuous[4] and radially unbounded, both $\delta_\exists$-GAS and δ-GAS imply global asymptotic stability.

Remark 2.3: Note that any smooth control system Σ admitting a contraction metric G, with respect to states, in the sense of Definition 2.4 in [25] or Definition 1 in [11] satisfies the property (II.2) with the Riemannian distance function, defined in [10], provided by the Riemannian metric G.

Definition 2.4: A control system Σ is incrementally input-to-state stable ($\delta_\exists$-ISS) if it is forward complete and there exist a metric $\mathbf{d}$, a $\mathcal{KL}$ function β, and a $\mathcal{K}_\infty$ function γ such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any υ, $\upsilon' \in \mathcal{U}$ the following condition is satisfied:

$$\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) \leq \beta\left(\mathbf{d}\left(x, x'\right), t\right) + \gamma\left(\|\upsilon - \upsilon'\|_\infty\right). \tag{II.3}$$

By observing (II.2) and (II.3), it is readily seen that $\delta_\exists$-ISS implies $\delta_\exists$-GAS while the converse is not true in general. Moreover, whenever the metric $\mathbf{d}$ is the Euclidean metric, $\delta_\exists$-ISS becomes δ-ISS as defined in [1]. We note that while δ-ISS is not generally invariant under changes of coordinates, $\delta_\exists$-ISS is. When the origin is an equilibrium point for Σ, with $\upsilon(t) = 0$ for any $t \in \mathbb{R}_0^+$, and the map $\psi : \mathbb{R}^n \to \mathbb{R}_0^+$, defined by $\psi(\cdot) = \mathbf{d}(\cdot, 0)$, is continuous[4] and radially unbounded, both $\delta_\exists$-ISS and δ-ISS imply input-to-state stability.

Remark 2.5: Note that any smooth control system Σ admitting a contraction metric G, with respect to states and inputs, in the sense of Definition 2.6 in [25] satisfies the property (II.3) with the Riemannian distance function, defined in [10], provided by the Riemannian metric G.

D. Characterizations of incremental stability

This section contains characterizations of $\delta_\exists$-GAS and $\delta_\exists$-ISS in terms of existence of incremental Lyapunov functions. We start by recalling the notions of incremental global asymptotic stability ($\delta_\exists$-GAS) Lyapunov function and incremental input-to-state stability ($\delta_\exists$-ISS) Lyapunov function, presented in [24].

Definition 2.6: Consider a control system Σ and a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ which is smooth on $\{\mathbb{R}^n \times \mathbb{R}^n\}\backslash\Delta$, where $\Delta = \{(x, x) \mid x \in \mathbb{R}^n\}$. Function V is called a $\delta_\exists$-GAS Lyapunov function for Σ, if there exist a metric $\mathbf{d}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and $\kappa \in \mathbb{R}^+$ such that:

(i) for any $x, x' \in \mathbb{R}^n$
$\underline{\alpha}(\mathbf{d}(x, x')) \le V(x, x') \le \overline{\alpha}(\mathbf{d}(x, x'))$;

(ii) for any $x, x' \in \mathbb{R}^n$ and any $u \in \mathsf{U}$
$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u) \le -\kappa V(x, x')$.

Function V is called a $\delta_\exists$-ISS Lyapunov function for Σ, if there exist a metric $\mathbf{d}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, σ, and $\kappa \in \mathbb{R}^+$ satisfying conditions (i) and:

(iii) for any $x, x' \in \mathbb{R}^n$ and for any $u, u' \in \mathsf{U}$
$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \le -\kappa V(x, x') + \sigma(\|u - u'\|)$.

Remark 2.7: It can be readily verified that for any smooth control system Σ admitting a contraction metric G, with respect to states and inputs (resp. with respect to states), in the sense of Definition 2.6 (resp. 2.4 or 1) in [25] (resp. [25] or [11]), the Riemannian distance function, defined in [10], provided by the Riemannian metric G, is a $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) Lyapunov function.

The following theorem, see [26], provides characterizations of $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) in terms of existence of $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) Lyapunov functions.

Theorem 2.8: Consider a control system Σ. If U is compact and $\mathbf{d}$ is a metric such that the function $\psi(\cdot) = \mathbf{d}(\cdot, y)$ is continuous[4] for any $y \in \mathbb{R}^n$, then the following statements are equivalent:

(1) Σ is forward complete[5] and there exists a $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) Lyapunov function, equipped with the metric $\mathbf{d}$.

(2) Σ is $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS), equipped with the metric $\mathbf{d}$.

In the next section, we propose a design approach, providing controllers rendering control systems incrementally input-to-state stable as well as providing incremental Lyapunov functions.

III. Controller Design Approach

The controller design approach proposed here is inspired by the backstepping method described in [14]. Consider the following subclass of control systems:

$$\Sigma : \begin{cases} \dot{\eta} = f(\eta, \zeta), \\ \dot{\zeta} = \upsilon, \end{cases} \tag{III.1}$$

where $x = \left[y^T, z^T\right] \in \mathbb{R}^{n_\eta + n_\zeta}$ is the state of Σ, y and z are the states η, ζ-subsystems, respectively, and υ is the control input.

[4]Here, continuity is understood with respect to the Euclidean metric.

[5]Here, forward completeness is understood with respect to the Euclidean metric, as in [2].

In order to show the main theorem, we need the following technical result, see [26] for a detailed proof.

Lemma 3.1: Consider the following interconnected control system:

$$\Sigma : \begin{cases} \dot{\eta} = f(\eta, \zeta, \upsilon), \\ \dot{\zeta} = g(\zeta, \upsilon). \end{cases} \tag{III.2}$$

Let the η-subsystem be $\delta_\exists$-ISS with respect to ζ, υ and let the ζ-subsystem be $\delta_\exists$-ISS with respect to υ for some metrics $\mathbf{d}_\eta$ and $\mathbf{d}_\zeta$, respectively, such that the solutions $\eta_{y\zeta\upsilon}$[6] and $\zeta_{z\upsilon}$ satisfy the following inequalities:

$$\begin{aligned} \mathbf{d}_\eta\left(\eta_{y\zeta\upsilon}(t), \eta_{y'\zeta'\upsilon'}(t)\right) &\le \beta_\eta\left(\mathbf{d}_\eta\left(y, y'\right), t\right) \\ &\quad + \gamma_\zeta\left(\mathbf{d}_\zeta(\zeta, \zeta')_\infty\right) + \gamma_\upsilon\left(\|\upsilon - \upsilon'\|_\infty\right), \\ \mathbf{d}_\zeta\left(\zeta_{z\upsilon}(t), \zeta_{z'\upsilon'}(t)\right) &\le \beta_\zeta\left(\mathbf{d}_\zeta\left(z, z'\right), t\right) + \widetilde{\gamma}_\upsilon\left(\|\upsilon - \upsilon'\|_\infty\right), \end{aligned}$$

where y, y' and z, z' are the initial conditions for the η, ζ-subsystems, respectively. Then, the overall system Σ in (III.2) is $\delta_\exists$-ISS with respect to υ.

Inspired by the work in [14], we can now state the main result on a backstepping controller design approach rendering the control system Σ in (III.1) $\delta_\exists$-ISS.

Theorem 3.2: Consider the control system Σ of the form (III.1). Suppose there exists a continuously differentiable function $\psi : \mathbb{R}^{n_\eta} \to \mathbb{R}^{n_\zeta}$ such that the control system

$$\Sigma_\eta : \dot{\eta} = f(\eta, \psi(\eta) + \widetilde{\upsilon}) \tag{III.3}$$

is $\delta_\exists$-ISS with respect to the input $\widetilde{\upsilon}$. Then for any $\lambda \in \mathbb{R}^+$, the state feedback control law:

$$\upsilon = k(\eta, \zeta, \widehat{\upsilon}) = -\lambda(\zeta - \psi(\eta)) + \frac{\partial \psi}{\partial y}(\eta) f(\eta, \zeta) + \widehat{\upsilon} \tag{III.4}$$

renders the control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$.

Proof: Consider the following coordinate transformation:

$$\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \phi(\xi) = \begin{bmatrix} \eta \\ \zeta - \psi(\eta) \end{bmatrix}, \tag{III.5}$$

where $\xi = \left[\eta^T, \zeta^T\right]^T$. In the new coordinate χ, we obtain the following dynamics:

$$\widehat{\Sigma} : \begin{cases} \dot{\chi_1} = f\left(\chi_1, \psi(\chi_1) + \chi_2\right), \\ \dot{\chi_2} = \upsilon - \frac{\partial \psi}{\partial y}(\chi_1) f\left(\chi_1, \psi(\chi_1) + \chi_2\right). \end{cases}$$

The proposed control law (III.4), given in the new coordinates χ by

$$\begin{aligned} \upsilon &= k(\chi_1, \chi_2 + \psi(\chi_1), \widehat{\upsilon}) \\ &= -\lambda\chi_2 + \frac{\partial \psi}{\partial y}(\chi_1) f\left(\chi_1, \psi(\chi_1) + \chi_2\right) + \widehat{\upsilon}, \end{aligned} \tag{III.6}$$

transforms the control system $\widehat{\Sigma}$ into:

$$\widetilde{\Sigma} : \begin{cases} \dot{\chi_1} = f(\chi_1, \psi(\chi_1) + \chi_2), \\ \dot{\chi_2} = -\lambda\chi_2 + \widehat{\upsilon}. \end{cases} \tag{III.7}$$

Due to the choice of ψ, the χ_1-subsystem of $\widetilde{\Sigma}$ is $\delta_\exists$-ISS with respect to χ_2. It can be easily verified that the χ_2-subsystem is input-to-state stable with respect to the input $\widehat{\upsilon}$. Since any ISS linear control system is also δ-ISS [1], χ_2-subsystem

[6]Notation $\eta_{y\zeta\upsilon}$ denotes a trajectory of η-subsystem under the inputs ζ and υ from initial condition $y \in \mathbb{R}^{n_\eta}$.

is also δ-ISS[7] with respect to $\widehat{\upsilon}$. Therefore, using Lemma 3.1, we conclude that the control system $\widetilde{\Sigma}$ is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. Since $\delta_\exists$-ISS property is coordinate invariant [25], we conclude that the original control system Σ in (III.1) equipped with the state feedback control law in (III.4) is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$ which completes the proof. ■

Remark 3.3: The $\delta_\exists$-ISS property of system Σ_η in (III.3) can be established, for example, using the controller synthesis approaches provided in [14], [23] for some classes of control systems such as e.g. piece-wise affine systems or Lur'e-type systems.

Remark 3.4: The result of Theorem 3.2 can be extended to the case in which we have an arbitrary number of integrators:

$$\Sigma : \begin{cases} \dot{\eta} = f(\eta, \zeta_1), \\ \dot{\zeta}_1 = \zeta_2, \\ \quad \vdots \\ \dot{\zeta}_k = \upsilon. \end{cases}$$

Note that in this case, the functions f and ψ must be differentiable sufficiently many times.

Although the proposed approach in Theorem 3.2 provides a controller rendering the control system Σ of the form (III.1) $\delta_\exists$-ISS, it does not provide a way of constructing $\delta_\exists$-ISS Lyapunov functions. In the next theorem, we show how to recursively construct incremental Lyapunov functions for the overall system. We note that the availability of incremental Lyapunov functions are essential e.g. in the context of the analysis in [4], [8]. In particular, these analyses exploit the fact that the incremental Lyapunov function provides an equivalent relation between the control system and its finite abstraction [22].

Theorem 3.5: Consider the control system Σ of the form (III.1). Suppose there exists a continuously differentiable function $\psi : \mathbb{R}^{n_\eta} \to \mathbb{R}^{n_\varsigma}$ such that the smooth function $\widehat{V} : \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\eta} \to \mathbb{R}_0^+$ is a $\delta_\exists$-ISS Lyapunov function for the control system

$$\Sigma_\eta : \dot{\eta} = f(\eta, \psi(\eta) + \widetilde{\upsilon}), \qquad \text{(III.8)}$$

with respect to the control input $\widetilde{\upsilon}$. Assume that $\widehat{V}$ satisfies condition (iii) in Definition 2.6 for some $\kappa \in \mathbb{R}^+$ and some $\sigma \in \mathcal{K}_\infty$, satisfying $\sigma(r) \le \widehat{\kappa} r^2$ for some $\widehat{\kappa} \in \mathbb{R}^+$ and any $r \in \mathbb{R}_0^+$. Then the function $\widetilde{V} : \mathbb{R}^{n_\eta+n_\varsigma} \times \mathbb{R}^{n_\eta+n_\varsigma} \to \mathbb{R}_0^+$, defined as

$$\widetilde{V}(x, x') = \widehat{V}(y, y') + \|(z - \psi(y)) - (z' - \psi(y'))\|^2,$$

where $x = \left[y^T, z^T\right]^T$ and $x' = \left[y'^T, z'^T\right]^T$, is a $\delta_\exists$-ISS Lyapunov function for Σ, as in (III.1), equipped with the state feedback control law (III.4) for all $\lambda \ge \frac{\kappa+\widehat{\kappa}+1}{2}$.

Proof: As explained in the proof of Theorem 3.2, using the proposed state feedback control law (III.4) and the coordinate transformation ϕ in (III.5), the control system Σ of the form (III.1) is transformed to the control system $\widetilde{\Sigma}$ in (III.7). Now we show that

$$V(\widehat{x}, \widehat{x}') = \widehat{V}(\widehat{x}_1, \widehat{x}_1') + (\widehat{x}_2 - \widehat{x}_2')^T(\widehat{x}_2 - \widehat{x}_2'),$$

is a $\delta_\exists$-ISS Lyapunov function for $\widetilde{\Sigma}$, where $\widehat{x} = \left[\widehat{x}_1^T, \widehat{x}_2^T\right]^T$ and $\widehat{x}' = \left[\widehat{x}_1'^T, \widehat{x}_2'^T\right]^T$ are the states of $\widetilde{\Sigma}$ and $\widehat{x}_1, \widehat{x}_1'$ and $\widehat{x}_2, \widehat{x}_2'$ are the states of χ_1, χ_2-subsystems, respectively. Since $\widehat{V}$ is a $\delta_\exists$-ISS Lyapunov function for χ_1-subsystem when χ_2 is the input, it satisfies condition (i) in Definition 2.6 using a metric $\mathbf{d}$ as follows:

$$\underline{\alpha}(\mathbf{d}(\widehat{x}_1, \widehat{x}_1')) \le \widehat{V}(\widehat{x}_1, \widehat{x}_1') \le \overline{\alpha}(\mathbf{d}(\widehat{x}_1, \widehat{x}_1')),$$

for some $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$. Now we define a new metric $\widehat{\mathbf{d}} : \mathbb{R}^{n_\eta+n_\varsigma} \times \mathbb{R}^{n_\eta+n_\varsigma} \to \mathbb{R}_0^+$ by

$$\widehat{\mathbf{d}}(\widehat{x}, \widehat{x}') = \mathbf{d}(\widehat{x}_1, \widehat{x}_1') + \|\widehat{x}_2 - \widehat{x}_2'\|.$$

It can be readily checked that $\widehat{\mathbf{d}}$ satisfies all three conditions of a metric. Using metric $\widehat{\mathbf{d}}$, function V satisfies condition (i) in Definition 2.6 as follows:

$$\underline{\mu}\left(\widehat{\mathbf{d}}(\widehat{x}, \widehat{x}')\right) \le V(\widehat{x}, \widehat{x}') \le \overline{\mu}\left(\widehat{\mathbf{d}}(\widehat{x}, \widehat{x}')\right),$$

where $\underline{\mu}, \overline{\mu} \in \mathcal{K}_\infty$, $\underline{\mu}\left(\widehat{\mathbf{d}}(\widehat{x}, \widehat{x}')\right) = \underline{\alpha}(\mathbf{d}(\widehat{x}_1, \widehat{x}_1')) + \|\widehat{x}_2 - \widehat{x}_2'\|^2$, and $\overline{\mu}\left(\widehat{\mathbf{d}}(\widehat{x}, \widehat{x}')\right) = \overline{\alpha}(\mathbf{d}(\widehat{x}_1, \widehat{x}_1')) + \|\widehat{x}_2 - \widehat{x}_2'\|^2$. Now we show that V satisfies condition (iii) in Definition 2.6 for $\widetilde{\Sigma}$. Since $\widehat{V}$ is a $\delta_\exists$-ISS Lyapunov function for χ_1-subsystem when χ_2 is the input, $\lambda \ge \frac{\kappa+\widehat{\kappa}+1}{2}$, $\sigma(r) \le \widehat{\kappa} r^2$, and using the Cauchy Schwarz inequality, we have:

$$\begin{aligned}
&\frac{\partial V}{\partial \widehat{x}} \left[f(\widehat{x}_1, \psi(\widehat{x}_1) + \widehat{x}_2)^T, -\lambda \widehat{x}_2^T + \widehat{u}^T\right]^T \\
&+ \frac{\partial V}{\partial \widehat{x}'} \left[f(\widehat{x}_1', \psi(\widehat{x}_1') + \widehat{x}_2')^T, -\lambda \widehat{x}_2'^T + \widehat{u}'^T\right]^T \\
&\le \frac{\partial \widehat{V}}{\partial \widehat{x}_1} f(\widehat{x}_1, \psi(\widehat{x}_1) + \widehat{x}_2) + \frac{\partial \widehat{V}}{\partial \widehat{x}_1'} f(\widehat{x}_1', \psi(\widehat{x}_1') + \widehat{x}_2') \\
&+ 2(\widehat{x}_2 - \widehat{x}_2')^T(-\lambda \widehat{x}_2 + \widehat{u}) - 2(\widehat{x}_2 - \widehat{x}_2')^T(-\lambda \widehat{x}_2' + \widehat{u}') \\
&\le -\kappa \widehat{V}(\widehat{x}_1, \widehat{x}_1') + \sigma(\|\widehat{x}_2 - \widehat{x}_2'\|) - 2\lambda \|\widehat{x}_2 - \widehat{x}_2'\|^2 \\
&+ 2(\widehat{x}_2 - \widehat{x}_2')^T(\widehat{u} - \widehat{u}') \\
&\le -\kappa \widehat{V}(\widehat{x}_1, \widehat{x}_1') + \widehat{\kappa} \|\widehat{x}_2 - \widehat{x}_2'\|^2 - 2\lambda \|\widehat{x}_2 - \widehat{x}_2'\|^2 \\
&+ 2\|\widehat{x}_2 - \widehat{x}_2'\| \|\widehat{u} - \widehat{u}'\| \\
&\le -\kappa \widehat{V}(\widehat{x}_1, \widehat{x}_1') + \widehat{\kappa} \|\widehat{x}_2 - \widehat{x}_2'\|^2 - 2\lambda \|\widehat{x}_2 - \widehat{x}_2'\|^2 \\
&+ \|\widehat{x}_2 - \widehat{x}_2'\|^2 + \|\widehat{u} - \widehat{u}'\|^2 \le -\kappa V(\widehat{x}, \widehat{x}') + \|\widehat{u} - \widehat{u}'\|^2.
\end{aligned}$$

The latter inequality implies that V is a $\delta_\exists$-ISS Lyapunov function for $\widetilde{\Sigma}$. Since $\delta_\exists$-ISS Lyapunov functions are coordinate-invariant [24], we conclude that the function $\widetilde{V} : \mathbb{R}^{n_\eta+n_\varsigma} \times \mathbb{R}^{n_\eta+n_\varsigma} \to \mathbb{R}_0^+$, defined by

$$\begin{aligned}
\widetilde{V}(x, x') &= V(\phi(x), \phi(x')) \\
&= \widehat{V}(y, y') + \|(z - \psi(y)) - (z' - \psi(y'))\|^2,
\end{aligned}$$

is a $\delta_\exists$-ISS Lyapunov function for Σ, as in (III.1), equipped with the state feedback control law in (III.4). ■

Remark 3.6: Note that the results in Theorem 3.5 differ from the classical ones [9] in the following ways: first, it is a result for incremental stability rather than for mere asymptotic stability; second, the resulting overall incremental Lyapunov function is equipped with the product topology induced by the metric $\mathbf{d}$ on $\mathbb{R}^{n_\eta}$ and the Euclidean metric on $\mathbb{R}^{n_\varsigma}$, rather than the Euclidean metric on $\mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\varsigma}$.

Remark 3.7: One can, for example, use the LMI-based results in [14], [23] to find a quadratic $\delta_\exists$-ISS Lyapunov function for system Σ_η in (III.8).

Remark 3.8: It can be verified that the backstepping design approach for strict-feedback form control systems,

[7] We recall that δ-ISS property is equivalent to $\delta_\exists$-ISS property whenever the metric is the Euclidean one.

proposed in [24], is a special case of the results in Theorem 3.5. The results in [24] can be easily obtained by recursively applying the results proposed in Theorem 3.5.

IV. Example

Here, we study a non-smooth control system and use the results in this paper to explicitly construct a $\delta_\exists$-ISS Lyapunov function, which, in turn, is employed to construct a finite equivalent abstraction using the results in [4, Theorem 4.1]. Consider the following non-smooth control system:

$$\Sigma : \begin{cases} \dot{\eta}_1 = \mathsf{sat}(\eta_1) + \eta_1 + 5\zeta_1, \\ \dot{\zeta}_1 = \zeta_1^2 + \eta_1^2 + \upsilon, \end{cases} \tag{IV.1}$$

where $\mathsf{sat} : \mathbb{R} \to \mathbb{R}$ is the saturation function, defined by:

$$\mathsf{sat}(x) = \begin{cases} -1 & \text{if } x < -1, \\ x & \text{if } |x| \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Note that here we require $\delta_\exists$-ISS and a $\delta_\exists$-ISS Lyapunov function in order to construct a finite equivalent abstraction using the results in [4, Theorem 4.1]. By introducing the feedback transformation $\widehat{\upsilon} = \zeta_1^2 + \eta_1^2 + \upsilon$, the control system Σ is transformed into the following form:

$$\widehat{\Sigma} : \begin{cases} \dot{\eta}_1 = \mathsf{sat}(\eta_1) + \eta_1 + 5\zeta_1, \\ \dot{\zeta}_1 = \widehat{\upsilon}. \end{cases}$$

Now by choosing $\psi(\eta_1) = -\eta_1$ and substituting $\psi(\eta_1) + \widetilde{\upsilon}$ instead of ζ_1, we obtain the following η-subsystem:

$$\widehat{\Sigma}_\eta : \{ \ \dot{\eta}_1 = \mathsf{sat}(\eta_1) - 4\eta_1 + 5\widetilde{\upsilon}.$$

It remains to show that $\widehat{\Sigma}_\eta$ is $\delta_\exists$-ISS with respect to $\widetilde{\upsilon}$. By choosing the function $V_1(y_1, y_1') = (y_1 - y_1')^2$, where y_1 and y_1' are states of $\widehat{\Sigma}_\eta$, and using the Cauchy Schwarz inequality, we have that:

$$\begin{aligned}
\frac{\partial V_1}{\partial y_1}\left(\mathsf{sat}(y_1) - 4y_1 + 5\widetilde{u}\right) + \frac{\partial V_1}{\partial y_1'}\left(\mathsf{sat}\left(y_1'\right) - 4y_1' + 5\widetilde{u}'\right) \leq \\
-8(y_1 - y_1')^2 + 2|y_1 - y_1'||\mathsf{sat}(y_1) - \mathsf{sat}(y_1')| \\
+10(y_1 - y_1')(\widetilde{u} - \widetilde{u}') \leq \\
-8(y_1 - y_1')^2 + 2(y_1 - y_1')^2 + 10(y_1 - y_1')(\widetilde{u} - \widetilde{u}') \leq \\
-5(y_1 - y_1')^2 + 25(\widetilde{u} - \widetilde{u}')^2,
\end{aligned}$$

showing that V_1 is a $\delta_\exists$-ISS Lyapunov function for $\widehat{\Sigma}_\eta$ and, hence, $\widehat{\Sigma}_\eta$ is $\delta_\exists$-ISS with respect to $\widetilde{\upsilon}$. By using the results in Theorem 3.2 for the control system $\widehat{\Sigma}$, we conclude that the state feedback control law:

$$\begin{aligned}
\widehat{\upsilon} &= k(\eta_1, \zeta_1, \bar{\upsilon}) = -\lambda(\zeta_1 - \psi(\eta_1)) + \frac{\partial \psi}{\partial y_1}\dot{\eta}_1 + \bar{\upsilon} \\
&= -\lambda\left(\zeta_1 + \eta_1\right) - \left(\mathsf{sat}(\eta_1) + \eta_1 + 5\zeta_1\right) + \bar{\upsilon},
\end{aligned}$$

makes the control system $\widehat{\Sigma}$ $\delta_\exists$-ISS with respect to input $\bar{\upsilon}$, for any $\lambda \in \mathbb{R}^+$. Therefore, the state feedback control law

$$\upsilon = \widehat{k}(\eta_1, \zeta_1, \bar{\upsilon}) = k(\eta_1, \zeta_1, \bar{\upsilon}) - \eta_1^2 - \zeta_1^2, \tag{IV.2}$$

makes the control system Σ $\delta_\exists$-ISS with respect to input $\bar{\upsilon}$.

Using Theorem 3.5, we conclude that the function $V : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_0^+$, defined by:

$$\begin{aligned}
V(x, x') &= V_1(y_1, y_1') + \left| (z_1 - \psi(y_1)) - (z_1' - \psi(y_1')) \right|^2 \\
&= (x - x')^T P (x - x') \\
&= (x - x')^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} (x - x'),
\end{aligned}$$

where $x = [y_1, z_1]^T$ is the state of Σ, is a $\delta_\exists$-ISS Lyapunov function for the control system Σ equipped with the state feedback control law $\widehat{k}$ in (IV.2) with $\lambda > \frac{25+5+1}{2}$. Here, we choose $\lambda = 16$.

It can be readily verified that the function $\widehat{V}(x, x') = \sqrt{V(x, x')}$ is also a $\delta_\exists$-ISS Lyapunov function for the control system Σ equipped with the state feedback control law $\widehat{k}$ in (IV.2) with $\lambda > \frac{25+5+1}{2}$, satisfying:

(i) for any $x, x' \in \mathbb{R}^2$,
$\sqrt{\lambda_{\min}(P)}\|x - x'\| \leq \widehat{V}(x, x') \leq \sqrt{\lambda_{\max}(P)}\|x - x'\|$;

(ii) for any $x, x' \in \mathbb{R}^2$, such that $x \neq x'$, and for any $\bar{u}, \bar{u}' \in \mathsf{U} \subseteq \mathbb{R}$,
$\frac{\partial \widehat{V}}{\partial x} f\left(x, \widehat{k}(x, \bar{u})\right) + \frac{\partial \widehat{V}}{\partial x'} f\left(x', \widehat{k}(x', \bar{u}')\right) \leq -2.5\widehat{V}(x, x') + \frac{|\bar{u} - \bar{u}'|}{\lambda_{\min}(P)}$;

(iii) for any $x, y, z \in \mathbb{R}^2$
$\left|\widehat{V}(x, y) - \widehat{V}(x, z)\right| \leq \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}}\|y - z\|$,

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ stand for minimum and maximum eigenvalues of P. Note that the property (iii) is a consequence of Proposition 10.5 in [22].

For constructing a bisimilar finite abstraction, using the results in [4, Theorem 4.1] which does not impose any restriction on the sampling time, the control system is required to be incrementally stable and to exhibit an incremental Lyapunov function.

Now, we construct a finite abstraction $S(\Sigma)$ for the control system Σ, equipped with the control input υ in (IV.2), using the results in [4, Theorem 4.1]. We assume that $\bar{\upsilon}(t) \in \mathsf{U} = [-10,\ 10]$, for any $t \in \mathbb{R}_0^+$, and $\bar{\upsilon}$ belongs to set $\mathcal{U}$ that contains piecewise constant curves of duration $\tau = 0.1$ second (τ is the sampling time) taking values in $[\mathsf{U}]_{0.5} = \{\bar{u} \in \mathsf{U} \mid \bar{u} = 0.5k,\ k \in \mathbb{Z}\}$. We work on the subset $\mathsf{D} = [-1,\ 1] \times [-1,\ 1]$ of the state space Σ. For a given precision[8] $\varepsilon = 0.1$ and using properties (i), (ii), and (iii) of $\widehat{V}$, we conclude that D should be quantized with resolution of $\eta = 0.009$, using the results in Theorem 4.1 in [4]. The state set of $S(\Sigma)$ is $[\mathsf{D}]_\eta = \{x \in \mathsf{D} \mid x_i = k_i\eta,\ k_i \in \mathbb{Z},\ i = 1, 2\}$. It can be readily seen that the set $[\mathsf{D}]_\eta$ is finite. The computation of the finite abstraction $S(\Sigma)$ was performed using the tool Pessoa [13]. Using the computed finite abstraction, we can synthesize controllers, providing $\bar{\upsilon}$ in (IV.2), satisfying specifications difficult to enforce with conventional controller design methods. Here, our objective is to design a controller navigating the trajectories of Σ, equipped with the control input υ in (IV.2), to reach the target set $W = [-0.05,\ 0.05] \times [-0.05,\ 0.05]$, indicated with a dark gray box in Figure 2, while avoiding the obstacles, indicated as light gray boxes in Figure 2, and remain indefinitely inside W. If we denote by ϕ and ψ the predicates representing the target and obstacles, respectively, this specification can also be expressed by the LTL formula $\Diamond\Box\phi \wedge \Box\neg\psi$ [22]. Furthermore, to add a discrete component to the problem, we assume that the controller is implemented on a microprocessor, which is executing other tasks in addition to the control task. We consider a schedule with epochs of three time slots in which the first slot is allocated to the control task and the other two to other tasks. A time slot refers to a time interval of the form $[k\tau, (k+1)\tau[$ with $k \in \mathbb{N} \cup \{0\}$ and where τ is

[8] The parameter ε is the maximum error between a trajectory of the control system and its corresponding trajectory from the finite abstraction at times $k\tau$, $k \in \mathbb{N}_0$, with respect to the Euclidean metric.

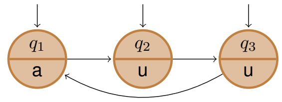

Fig. 1. Finite system describing the schedulability constraint. The lower part of the states are labeled with the outputs a and u denoting availability and unavailability of the microprocessor, respectively.

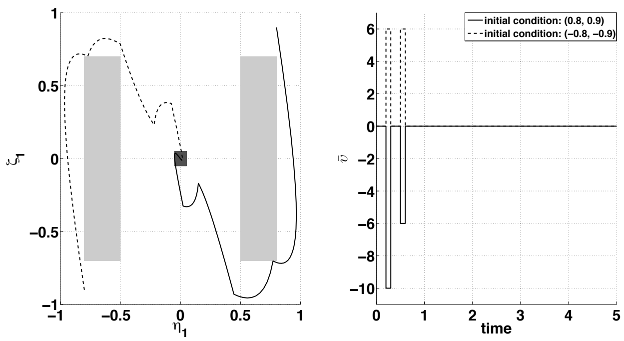

Fig. 2. Evolutions of the closed-loop system with initial conditions $(0.8,\ 0.9)$, and $(-0.8,\ -0.9)$ (left panel) and evolutions of the corresponding input signals.

the sampling time. Therefore, the microprocessor schedules is given by (depending on the initial slot):

$$|\mathsf{auu}|\mathsf{auu}|\mathsf{auu}|\cdots,\quad |\mathsf{uua}|\mathsf{uua}|\mathsf{uua}|\cdots,\quad |\mathsf{uau}|\mathsf{uau}|\mathsf{uau}|\cdots,$$

where a denotes a slot available for the control task and u denotes a slot allotted to other tasks. We assume that in unallocated time slots, the input $\bar{\upsilon}$ is identically zero. The schedulability constraint on the microprocessor can be represented by the finite system in Figure 1.

A controller, providing $\bar{\upsilon}$ in (IV.2) and enforcing the specification has been designed by using standard algorithms from game theory, implemented in Pessoa, where the finite system is initialized from state q_2, see second sequence above. In Figure 2, we show the closed-loop trajectories of Σ, equipped with the control input υ in (IV.2) (including the additional controller for $\bar{\upsilon}$) and stemming from the initial conditions $[0.8, 0.9]$ and $[-0.8, -0.9]$ as well as the evolution of the corresponding input signals $\bar{\upsilon}$. It is readily seen that the specifications are satisfied. It can be easily seen that the schedulability constraint is also satisfied, implying that the control input $\bar{\upsilon}$ is identically zero at unallocated time slots.

V. Discussion

In this paper, we developed a synthesis approach for controllers enforcing incremental input-to-state stability. The proposed approach in this paper generalizes the work in [6], [20], [19], [25], [24] by being applicable to larger classes of control systems and the work in [14] by enforcing incremental input-to-state stability rather than input-to-state convergence. Moreover, in contrast to the proposed design approach in [14], here we provided a way of constructing incremental Lyapunov functions which are known to be a key tool in the analysis provided in [4], [8]. As we showed in the example, the explicit existence of an incremental Lyapunov function helps us to use the results in [4, Theorem 4.1] to construct a finite bisimilar abstraction for a resulting incrementally stable closed-loop (non-smooth) control system. Using the constructed finite abstraction, we have synthesized another controller for the incrementally stable closed-loop system enforcing the satisfaction of logic specifications, which is difficult (or even impossible) to enforce using conventional approaches.

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