DISTURBANCE ATTENUATION FOR A PERIODICALLY EXCITED PIECE-WISE LINEAR BEAM SYSTEM*

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Abstract

In this paper, we consider the problem of disturbance attenuation for a class of piece-wise linear systems. The proposed control design ensures that the closed-loop system is uniformly convergent. Uniform convergence guarantees the existence of a unique globally asymptotically stable steady-state solution for a given periodic disturbance. This property allows to uniquely assess the performance of the controller in terms of disturbance attenuation. Both state-feedback and output-feedback variants of the control design are presented. The effectiveness of the strategy is shown by application to a piece-wise linear beam system.

Key words

Convergence-based control, non-smooth dynamics

1 Introduction

The motivation for this work originates from the need to analyse and control the dynamics of complicated engineering constructions including structural elements with piece-wise linear (PWL) restoring characteristics, such as tower cranes, suspension bridges and solar panels on satellites [Heertjes, 1999]. More specifically, the disturbance attenuation problem is an important control problem to be solved to ensure the performance of these systems and to avoid damage to the structures. Since the dynamics of such systems are generally formulated as PWL systems, we will investigate the disturbance attenuation problem for PWL systems. PWL systems are currently receiving a great deal of attention.

In [Johansson and Rantzer, 1998], a new framework was developed, based on piece-wise quadratic Lyapunov functions, to analyse the stability of piece-wise affine (PWA) systems. In [Rantzer and Johansson, 2000] this framework was extended for performance analysis and optimal control. In [Hassibi and Boyd, 1998], a study related to stability analysis and controller design for PWL systems was presented. This study uses common and piece-wise quadratic Lyapunov functions for stability purposes. Here, in the case of a common quadratic Lyapunov function, both the stability analysis and the state-feedback synthesis can be expressed as a convex optimization problem based on constraints in linear matrix inequality (LMI) form. However, it has been pointed out that this is difficult in the case of a piece-wise quadratic Lyapunov function. A solution for this problem has been given in [Feng et al., 2002] and [Rodrigues et al., 2000]. [Feng et al., 2002] presents a H_∞ controller synthesis method based on a piecewise quadratic Lyapunov function that can be cast in the form of solving a set of LMIs using standard LMI solvers. [Rodrigues et al., 2000] shows a method used to design state- and output-feedback controllers with constraints on the smoothness and continuity of the piecewise quadratic Lyapunov function. However, the controller design of [Rodrigues et al., 2000] is restricted, as it is mentioned in [Rodrigues et al., 2000], by two fundamental assumptions: 1) there are no sliding modes at the hyperplane boundaries between regions with different affine dynamics, 2) the examined PWL system and the controller are always in the same region. [Rodrigues and How, 2001] examines the case where the assumptions in [Rodrigues et al., 2000] are violated and presents a general stability analysis of the closed-loop system for that case.

A common characteristic of the papers [Johansson and Rantzer, 1998], [Rodrigues et al., 2000], [Hassibi and Boyd, 1998] and [Rantzer and Johansson, 2000] is that they guarantee stability of a PWL system for zero inputs. In the papers [Hassibi and Boyd, 1998], [Feng et al., 2002], [Khalil, 2002] and [Rodrigues and How, 2001] it is assumed that given an initial condition for a PWL system, an input signal, and a disturbance, the systems has a unique solution for $t > 0$.

In [Demidovich, 1967] (see also [Pavlov et al., 2004]), the notion of convergence for nonlinear systems with inputs is introduced. A system with this property has

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a unique globally asymptotically steady-state solution which is determined only by the system input and does not depend on the initial conditions. In [Pavlov, 2004] and [Pavlov et al., 2005], the notion of convergent systems is extended to the notion of (uniformly, exponentially) convergent systems and input-to-state convergent systems (in section 2, further information about these notions is given). Based on the extensions made in [Pavlov, 2004], the design of a controller that renders a non-convergent system convergent, is pursued. Furthermore, in [Pavlov, 2004] the first result on convergence for PWA systems is published.

So far, results related to performance of PWL/PWA systems, in terms of disturbance attenuation, where given among others, in [Rantzer and Johansson, 2000], [Hassibi and Boyd, 1998] and [Feng et al., 2002]. The performance results of these papers, which are based on single or piecewise quadratic Lyapunov functions, provide an upper bound for the system output by bounding the L_2 gain from the system input to the system output. Nevertheless, these results are not very general, since they have been derived under the assumption of zero initial conditions.

In this paper we propose a controller design strategy for a class of bi-modal PWL systems, based on the extended notions of convergence, in order to study the performance of such systems for disturbance attenuation. The convergence property is beneficial in the scope of performance analysis of bi-modal PWL systems, because it ensures that these systems exhibit unique steady-state solutions. Due to the fact that convergence is based on a quadratic Lyapunov function, we can provide an upper bound for the system states in (steady-state) given a bounded input which is similar to the bounds presented in [Rantzer and Johansson, 2000], [Hassibi and Boyd, 1998] and [Feng et al., 2002], for any initial condition. In addition to that, the uniqueness of the system steady-state response allows for a more accurate evaluation of the performance based on computed responses. In this paper, we focus on a specific class of disturbances, namely harmonic disturbances. The motivation for this choice lies in the fact that in engineering practice many disturbances can be approximated by harmonic signals.

More specifically, this paper presents a controller design strategy for a class of bi-modal PWL systems and treats its application to a piece-wise linear model of an experimental beam system. This system consists of a flexible steel beam, which is clamped on two sides and is supported by a one-sided linear spring. Due to the one-sided spring the beam has two different dynamical regimes, which both can be well described as being linear. This system is excited by exogenous periodic disturbances.

The goal of the strategy is the performance of the closed-loop PWL beam system in terms of disturbance attenuation. In order to uniquely define the performance of the closed-loop system it should not have multiple steady-state solutions. This property can be attained by rendering the PWL beam system convergent by means of feedback.

The controller design strategy uses state- and output-feedback control laws in order to render the closed-loop system of the PWL beam convergent. The output-feedback controller is a combination of a model-based switching observer [Juloski et al., 2002] and a state-feedback controller.

The paper structure is as follows. The controller design strategy is introduced in section 2. In sections 3 and 4, state- and output-feedback controllers are designed for a bi-modal PWL system, respectively. A description of the PWL beam system is given in section 5. In section 6, simulation results related to the controller performance are presented. Conclusions and directions for future work are given in section 7.

2 Controller design strategy

We consider the following class of bi-modal time-continuous PWL systems:

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Bw(t) + B_1 u(t) \text{ for } H^T x(t) \le 0 \\ A_2 x(t) + Bw(t) + B_1 u(t) \text{ for } H^T x(t) > 0 \end{cases} \tag{1a}$$

$$y(t) = Cx(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^q$ and $w(t) \in \mathbb{R}^m$ are the state, the output, the control input and the exogenous input of the system, respectively, depending on time $t \in \mathbb{R}$. The input $w(t)$ acts as a disturbance on the system and it is considered to be periodic. The matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $B_1 \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$ and $H \in \mathbb{R}^n$. The hyperplane defined by $\ker H^T$ separates the state space $\mathbb{R}^n$ in two half-spaces. The considered class of bi-modal PWL systems has identical input matrices B, B_1 and an identical output matrix C for both modes.

The goal of the controller design strategy is the disturbance attenuation of such systems for a range of periodic excitations. Disturbance attenuation roughly measures to what extent the amplitude of a periodic disturbance $w(t) = Asin\omega t$ is amplified/suppressed in the output or in (each component of) the state $x(t)$. Obviously, such measure only makes sense if the steady-state response remains bounded and is unique under a periodic excitation.

Due to the fact that PWL systems are nonlinear, they often exhibit multiple steady-state solutions when excited by periodic disturbances. In order to uniquely define the performance of the closed-loop system it should not have multiple steady-state solutions. The present strategy focuses on attaining such property by making PWL systems globally, uniformly convergent. A detailed treatment of convergent systems was given in [Demidovich, 1967] (see also [Pavlov et al., 2004]).

Consider the system

$$\dot{z} = F(z, w(t)), \tag{2}$$

with state $z \in \mathbb{R}^d$ and input $w \in \mathbb{R}^m$, where $F(z, w)$ is locally Lipschitz in z and continuous in w. The input $w(t)$ is a piecewise continuous function of t defined for all $t \in \mathbb{R}$.

Definition 1. *System* (2) *with given input $w(t)$ is said to be (uniformly, exponentially) convergent if*

1. *all solutions $z(t)$ are well defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}$, $z(t_0) \in \mathbb{R}^m$;*
2. *there exists a unique solution $\bar{z}_w(t)$ defined and bounded for all $t \in (-\infty, +\infty)$;*
3. *the solution $\bar{z}_w(t)$ is globally (uniformly, exponentially) asymptotically stable.*

If system (2) is convergent for a class of inputs, then for every input from this class it has a unique bounded globally, asymptotically stable, steady-state solution $\bar{z}_w(t)$.

If the input of a convergent system is periodic with period T, then the corresponding $\bar{z}_w(t)$ is also periodic with the same period T, see [Pavlov et al., 2004].

In the present work, given the fact that the (convergent) closed-loop system exhibits periodic solutions with period T, we can define performance more specifically by saying that we want to minimize

$$\max_{s \in [t, t+T]} |x_i(s)|, \text{ for } i = 1, ...n, \tag{3}$$

over a specific (excitation) frequency range. Herein, $x_i(s)$ are the state components of system (1).

The problem at hand is to provide a suitable control input $u(t)$ to system (1) such that, for a given periodic, continuous and bounded input $w(t)$, 1) the closed-loop system exhibits a unique periodic steady-state solution and 2) the amplification of the bounded input amplitude in (each component of) the closed-loop system states is smaller than the amplification of the bounded input amplitude in (each component of) the open-loop system states. Note that we will ensure the first property of the closed-loop system by making it uniformly convergent by means of feedback.

In section 5, the output feedback control design will be based on the input-to-state0 convergence (ISC) property of the control system. Let us know introduce the ISC property.

Consider the system

$$\dot{z} = F(z, w, t), \tag{4}$$

$t \in \mathbb{R},\ z \in \mathbb{R}^d,\ w \in \mathbb{R}^m$, where $F(z, w, t)$ is piecewise continuous in t, continuous in w and locally Lipschitz in z. The input $w(t)$ is a piecewise continuous function of t.

Definition 2. *[Pavlov et al., 2004] System* (4) *is said to be input-to-state convergent (ISC) if it is globally uniformly convergent for a class of piece-wise continuous inputs, and for every input $w(t)$ taken from this class, the system is input-to-state stable [Khalil, 2002] with respect to the system's solution $\bar{z}_w(t)$, i.e. there exist a KL-function $\beta(r, s)$ [Khalil, 2002] and a class K_∞-function [Khalil, 2002] $\gamma(r)$ such that any solution of this system corresponding to some input $\tilde{w}(t) := w(t) + \Delta w(t)$ satisfies*

$$|z(t) - \bar{z}_w(t)| \leq \beta(|z(t_0) - \bar{z}_w(t_0)|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} |\Delta w(\tau)|). \tag{5}$$

3 State-feedback controller design

In the controller design strategy, a static state-feedback is chosen as the input for the system (1):

$$u(t) = -Kx(t), \tag{6}$$

where $u(t)$ is the control action and $K \in \mathbb{R}^{1\times n}$ is the controller gain. Consequently, the dynamics of the closed-loop system (1) and (6) can be written as:

$$\dot{x}(t) = \begin{cases} A_a x(t) + Bw(t) \text{ for } H^T x(t) \leq 0 \\ A_b x(t) + Bw(t) \text{ for } H^T x(t) > 0 \end{cases} \tag{7a}$$

$$y(t) = Cx(t), \tag{7b}$$

where $A_a = A_1 - B_1 K$ and $A_b = A_2 - B_1 K$. The closed-loop system described by (7) is also a bi-modal PWL system with an identical input matrix B and has an identical output matrix C for both modes. Furthermore, the hyperplane defined by $\ker H^T$ separates the state-space $\mathbb{R}^n$ of the closed-loop system in two half-spaces.

The controller design problem can now be formally stated as:

Problem: *Determine, if possible, the controller gain K in* (6) *such that 1) the closed-loop system* (7) *is globally, uniformly convergent for a class of piece-wise continuous inputs $w : \mathbb{R}^+ \longrightarrow \mathbb{R}^m$ and 2) for a given disturbance $w(t)$ the maximum absolute value of the state components of* (7)*, $max(|x_i|)$, $i = 1, ..., n$, is lower than the maximum absolute value of the uncontrolled state components $max(|x_i|)$, $i = 1, ..., n$.*

Note that here we consider a class of bounded periodic disturbances $w(t)$ and that the uncontrolled system derives from (1) when u=0.

The first part of this problem can be solved using a result in [Pavlov, 2004], which states conditions under which system (7) is globally uniformly convergent and ISC for all piece-wise continuous disturbances w:

Theorem 1. *Consider the state-space $\mathbb{R}^n$ which is divided into regions $\Lambda_i,\ i = 1, ..., l$, by hyperplanes given by equations of the form $H_j^T z + h_j = 0$, for some $H_j \in \mathbb{R}^n$ and $h_j \in \mathbb{R},\ j = 1, ...k$. Consider the piece-wise affine system*

$$\dot{z} = A_i z + b_i + Dw(t), \quad for \ z \in \Lambda_i,\ i = 1, ..., l. \tag{8}$$

Suppose that the right-hand side of (8) *is continuous and there exists a positive definite matrix* $Q = Q^T$ *such that*

$$QA_i + A_i^T Q < 0, \; i = 1, ..., l. \tag{9}$$

Then the system (8) *is globally exponentially convergent and ISC for piecewise continuous bounded inputs.*

This *Theorem 1* is based on a quadratic Lyapunov function. For the proof of *Theorem 1* the reader is referred to [Pavlov et al., 2004]. It should be noted that input-to-state convergence implies uniform convergence. Using *Theorem 1* for (7), the following LMI constraints are derived to guarantee global uniform convergence and input-to state convergence:

$$Q = Q^T > 0, \tag{10a}$$

$$A_a^T Q + QA_a < 0, \tag{10b}$$

$$A_b^T Q + QA_b < 0, \tag{10c}$$

with $A_a = A_1 - B_1 K$ and $A_b = A_2 - B_1 K$. The inequalities (10a)-(10c) are nonlinear matrix inequalities in $\{Q, K\}$ but are linear in $\{Q, K^T Q\}$, and thus can be efficiently solved using standard LMI solvers (such as the LMItool in Matlab). Note that these LMIs imply stability of (7) for zero input. In addition to that, based on the results on convergence [Pavlov et al., 2004] and [Pavlov et al., 2005], these LMIs also imply that the system (7) has a unique bounded globally asymptotically stable steady-state solution for every bounded input.

4 Output-feedback controller design

In general, the entire state of (1) will not be available for feedback. Therefore, the goal of this section is to construct an output-feedback controller that solves the problem stated in the previous section for the system (1).

This output-feedback controller consists of a state-feedback controller as in (6) and a switching model-based observer. This observer recovers the states of the system without any information on which linear dynamics of the system is currently active.

Now, we will propose such observer/controller combination such that the resulting closed-loop system, hereafter called the interconnected system, is globally, uniformly convergent. This will allow once more for a unique performance evaluation.

The choice of the observer/controller combination that renders the interconnected system globally, uniformly convergent is based on a property presented in [Pavlov, 2004]:

Property 1. *Consider the system*

$$\begin{cases} \dot{z} = F(z, y, w), \; z \in \mathbb{R}^d \\ \dot{y} = G(z, y, w), \; y \in \mathbb{R}^q. \end{cases} \tag{11}$$

Suppose that the z-subsystem is input-to state-convergent with respect to y and w. Assume that there exists a class KL function $\beta_y(r, s)$ such that for any piece-wise continuous input ($w(\cdot), \; z(\cdot)$), any solution of the y-subsystem satisfies

$$|y(t)| \leq \beta_y(|y(t_0)|, t - t_0). \tag{12}$$

Then the interconnected system (11) *is input-to-state convergent.*

In Figure 1 a schematic representation of the interconnected system (11) is depicted.

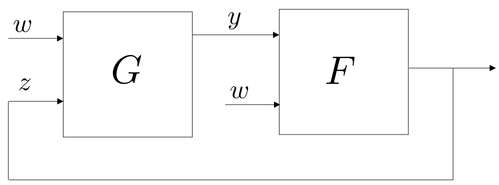

Figure 1. Schematic representation of the interconnected system (11).

In the following subsection, we will derive a set of LMIs that guarantee global, exponential stability of the observer error dynamics. Then, we will show that the closed-loop system consists of 1) the observer error dynamics (the y-subsystem) and 2) the PWL system in closed-loop with the controller (the z-subsystem) has the form of (11). Next, we will show that the LMIs (10) guarantee that the PWL system in closed-loop with the controller is ISC (Definition 2) with respect to both the exogenous input w and the observer error. Finally, we will combine the achieved results using Property 1 in order to prove that the interconnected system (11) is globally uniformly convergent.

4.1 Global exponential stability of the observer error

We consider a switching observer of the following structure

$$\dot{\hat{x}}(t) = \begin{cases} A_1 \hat{x}(t) + Bw(t) + B_1 u(t) + L_1 \Delta y(t), \; \text{if } H^T \hat{x} \leq 0 \\ A_2 \hat{x}(t) + Bw(t) + B_1 u(t) + L_2 \Delta y(t), \; \text{if } H^T \hat{x} > 0, \end{cases} \tag{13}$$

for the system (1), with $L_1, L_2 \in \mathbb{R}^{n \times p}$ and $\hat{x}(t) \in \mathbb{R}^n$. The observer output is $\hat{y}(t) = C \; \hat{x}(t)$ and $\Delta y(t) =$

$y(t) - \hat{y}(t)$. The model output y is used as observer output injection.

The dynamics of the observer error $\Delta x(t) = x(t) - \hat{x}(t)$ is described by

$$
\Delta\dot{x}(t) = \begin{cases} (A_1 - L_1C)\Delta x, & \text{if } H^T x \le 0 \ \wedge \ H^T\hat{x} \le 0 \\ (A_2 - L_2C)\Delta x + \Delta A x, & \text{if } H^T x \le 0 \ \wedge \ H^T\hat{x} > 0 \\ (A_1 - L_1C)\Delta x - \Delta A x, & \text{if } H^T x > 0 \ \wedge \ H^T\hat{x} \le 0 \\ (A_2 - L_2C)\Delta x, & \text{if } H^T x > 0 \ \wedge \ H^T\hat{x} > 0, \end{cases} \tag{14}
$$

where $\Delta A = A_1 - A_2$.

In [Juloski et al., 2002] a result is proposed that provides a set of LMI constraints that guarantees global asymptotic stability of the observer error dynamics described in (14). Unfortunately, these constraints are not sufficient in the present case. An extension of this theorem is given in order to provide a set of LMI constraints that guarantees global *exponential* stability of the observer error.

Theorem 2. *The observer error dynamics* (14) *is globally exponentially stable (GES) for all* $x : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ *(in the sense of Lyapunov), if there exist matrices* $P = P^T > 0$, L_1, L_2 *and constants* $\tau_1, \tau_2 \ge 0$, $\alpha > 0$ *such that the following set of matrix inequalities is satisfied:*

$$
\begin{bmatrix} (A_2 - L_2C)^T P + & P\Delta A + \\ +P(A_2 - L_2C) + \alpha P & +\frac{1}{2}\tau_1 HH^T \\ & \\ \Delta A^T P + & -\tau_1 HH^T \\ +\tau_1\frac{1}{2}HH^T & \end{bmatrix} \le 0 \tag{15a}
$$

$$
\begin{bmatrix} (A_1 - L_1C)^T P + & -P\Delta A + \\ +P(A_1 - L_1C) + \alpha P & +\frac{1}{2}\tau_2 HH^T \\ & \\ -\Delta A^T P + & -\tau_2 HH^T \\ +\tau_2\frac{1}{2}HH^T & \end{bmatrix} \le 0. \tag{15b}
$$

Hence, it can be very efficiently determined whether there exists a quadratic Lyapunov function that proves global exponential stability of the observer error. For the proof of *Theorem 2* the reader is referred to Appendix B. Note that L_1, L_2 are non unique. L_1, L_2 influence the rate of convergence of the observer error to zero. In case there is measurement noise in the observer output injection, the choice of L_1, L_2 should be a balance between convergence rate and noise amplification. The inequalities in (15) are nonlinear matrix inequalities in $\{P, L_1, L_2, \lambda_1, \lambda_2\}$, but are linear in $\{P, L_1^T P, L_2^T P, \tau_1, \tau_2\}$. Thus, they can be efficiently solved using linear matrix inequalities solvers (such as the software LMItool for Matlab).

4.2 Input-to-state convergence for the PWL system in closed-loop with the state-feedback controller

Using the control law

$$
u(t) = -K\hat{x}(t), \tag{16}
$$

in (1a) yields

$$
\dot{x}(t) = \begin{cases} A_a x(t) + Bw(t) - B_1 K\Delta x(t), & \text{if } H^T x \le 0 \\ A_b x(t) + Bw(t) - B_1 K\Delta x(t), & \text{if } H^T x > 0. \end{cases} \tag{17}
$$

Observing equations (14) and (17), it is straightforward that the corresponding systems constitute an interconnected system as in (11). Using *Theorem 1* for (17), we derive the inequalities (10a)-(10c). These inequalities guarantee that system (17) is input-to-state convergent with respect to $w(t)$ and $\Delta x(t)$.

4.3 Global uniform convergence of the interconnected system

By applying Property 1 to the interconnected system, we prove that the interconnected system is globally uniformly convergent. Hereto, we use that: 1) (17) is ISC with respect to $w(t)$ and $\Delta x(t)$ and 2) (14) is GES. This in fact means that the separation principle holds for the observer/controller combination. Due to the fact that 1) holds, the system state (17) always converges to a unique, bounded steady state solution for every finite initial condition and for bounded inputs $w(t)$ and $\Delta x(t)$. Therefore, the use of the observer (13), for system state reconstruction, has no influence to the stability of the interconnected system. Furthermore, due to the fact that 2) holds, $x_{w,\Delta x}$ will converge to the steady-state solution $x_{w,\Delta x=0}$ ($x_{w,\Delta x=0}$ is the steady-state solution of (17) for $\Delta x = 0$).

5 Application to a piece-wise linear beam system

In this section we introduce a PWL beam system depicted in Figure 2. The developed controller design strategy is applied to this system.

The PWL beam system consists of a steel beam supported at both ends by two leaf springs. The beam is excited by a force w generated by a rotating mass-unbalance, which is mounted at the middle of the beam, see Figure 3. A tacho-controlled motor, that enables a constant rotation speed, drives the mass-unbalance. An actuator applies a control force u to the beam. A second beam, that is clamped at both ends, is located parallel to the first one and represents a one-sided spring. This spring represents a non-smooth nonlinearity in the dynamics of the PWL beam system and as a result the beam system (beam and one-sided spring) has nonlinear and non-smooth dynamics. The restoring characteristic of the one-sided spring is assumed linear; consequently, the beam system can be described as a piece-wise linear system, as shown in the next section.

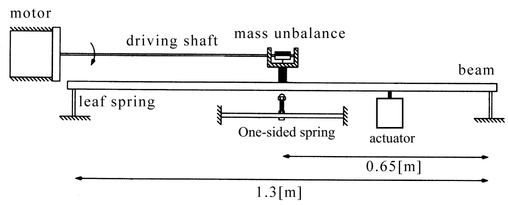

Figure 2. Schematic view of the PWL beam system.

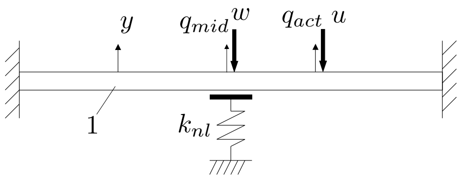

Figure 3. Elastic beam with one-sided support.

5.1 Dynamics of the PWL beam system

The dynamics of the PWL beam system can be described by a three-degree-of-freedom (3DOF) model [Doris et al., 2005] of the following form

$$M\ddot{q}+B_s\dot{q}+K_sq+f_{nl}(q) = h_1\ w(t)+h_2\ u(t), \quad (18)$$

where $h_1 = [1 \quad 0 \quad 0]^T$, $h_2 = [0 \quad 1 \quad 0]^T$ and $q = [q_{mid} \quad q_{act} \quad q_\xi]^T$. Herein, q_{mid} is the displacement of the middle of the beam and q_{act} is the displacement of the point of the beam at which the actuator is mounted, see in Figure 3. Moreover, q_ξ reflects the contribution of the first eigenmode of the beam and M, B and K_s are the mass, the damping and the stiffness matrices of the 3DOF model, respectively. We apply a periodic (harmonic) excitation force

$$w(t) = A\sin\omega t, \quad (19)$$

which is generated by the rotating mass-unbalance at the middle of the beam. Herein, ω is the excitation frequency and A the amplitude of the excitation force. Moreover, f_{nl} is the restoring force of the one-sided spring:

$$f_{nl}(q) = k_{nl}\ h_1\ min(0, h_1^Tq) = k_{nl}\ h_1\ min(0, q_{mid}), \quad (20)$$

where k_{nl} is the stiffness of the spring. The force f_{nl} acts when there is contact between the middle of the beam and the one-sided spring.

In a state-space formulation, the model takes the form of (1) and by using the observer-based state-feedback (16) it can be written in the form of (17), where $x = [q^T \quad \dot{q}^T]^T$ and $H = [h_1^T \quad 0^T]^T$. Furthermore,

$A_1 = \begin{bmatrix} 0 & I \\ -M^{-1}(K_s + k_{nl}\ h_1\ h_1^T) & -M^{-1}B_s \end{bmatrix}$,

$A_2 = \begin{bmatrix} 0 & I \\ -M^{-1}K_s & -M^{-1}B_s \end{bmatrix}$,

$B = \begin{bmatrix} 0 \\ M^{-1}h_1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ M^{-1}h_2 \end{bmatrix}$ and $0 = [0 \quad 0 \quad 0]^T$.

In the examined case, the output of (1), $y(t) = Cx(t)$, describes a transversal displacement of a point 1 on the beam, depicted in figure 3. The numerical values of M, B_s, K_s, k_{nl} and C are given in Appendix A.

6 Simulation of the PWL beam system

In order to illustrate the effectiveness of the control strategy proposed in sections 2, 3, 4 and 5, simulation results related to the PWL beam are presented.

In the first part of this section, it is shown that the observer error converges to zero exponentially and in the second part, it is shown that the interconnected system consisting of (14) and (17) is globally uniformly convergent. Note that in the examined case, the z- and y-subsystems of (11) are represented by (17) and (14), respectively.

6.1 Global exponential stability of the observer error

In order to design the observer (13) for the interconnected system, the transversal displacement of a properly chosen point on the beam is used as observer output injection. This displacement is the model output $y(t) = Cx(t)$, see figure 3. The position of this point should be chosen such that the LMIs (15) are feasible. By solving these LMIs the gains L_1, L_2, that guarantee global asymptotic stability of the observer error, are calculated. The numerical values of these gains are given in the Appendix A.

In figure 4, the observer error states $\Delta x_4(t) = \dot{q}_{mid}(t) - \hat{\dot{q}}_{mid}(t)$, $\Delta x_5(t) = \dot{q}_{act}(t) - \hat{\dot{q}}_{act}(t)$ and $\Delta x_6(t) = \dot{q}_\xi(t) - \hat{\dot{q}}_\xi(t)$ and an exponential boundary of the observer error are depicted. This boundary (dashed line) has the form $1/\sqrt{\lambda_{min}(P)}\ |\Delta x(t_0)|_P e^{-\frac{\alpha t}{2}}$ and it is derived from (31). The values for P, α and $\Delta x(t_0)$ are given in Appendix A. Based on this figure, the observer error converges to zero exponentially. Therefore, the Property 1 can be applied to the interconnected system.

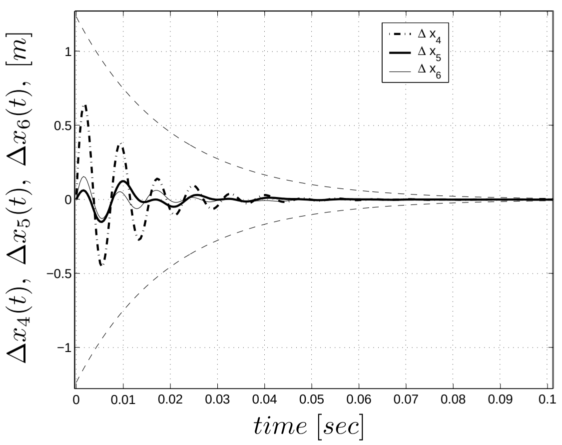

Figure 4. $\Delta x_4(t)$ (dashed-dotted line), $\Delta x_5(t)$ (thick solid line) and $\Delta x_6(t_0)$ (solid line) for an excitation frequency $\omega = 2\pi 55\ rad/s$ and an excitation amplitude $A = 121\ N$.

6.2 Global uniform convergence of the interconnected system and attained disturbance attenuation

In this subsection, we show that 1) the PWL beam system in closed-loop with the observer-based controller exhibits a unique asymptotically stable steady-state solution and 2) the effect of the excitation force w on the systems response is significantly smaller in the closed-loop system than in the open-loop system. More specifically, we show that the maximum value of the transversal displacement of the points on the beam are significantly smaller when a control force u is acting on the beam than in the open-loop case.

Numerical computation of the periodic solutions of the open-loop PWL beam system ((1) with $u = 0$) for harmonic disturbances, as in (19), shows that this system is not globally uniformly convergent. Hereto, the collocation method [Doedel et al., 1998] and the path-following procedure [Ascher et al., 1995] are used.

More specifically, in figures 5, 7, and 8, the plots of $max(|q_{mid}|)$, $max(|q_{act}|)$ and $max(|q_\xi|)$ for such periodic solutions are depicted for an excitation frequency range of $10 - 60$ $[Hz]$. q_{mid}, q_{act} and q_ξ are derived from the open-loop system and they are divided by the the input amplitude A in order to take a normalized form. Based on these figures, q_{mid}, q_{act} and q_ξ exhibit two steady-state solutions for excitation frequencies within the frequency range of $39 - 56$ $[Hz]$. In this frequency range, the dashed line is an unstable harmonic solution and the solid line is a stable $\frac{1}{2}$ subharmonic solution. Due to the fact that the open-loop system exhibits two steady-state solutions, it is not convergent.

By using numerical analysis for the PWL beam closed-loop system (interconnected system (14) and (17)) for such periodic disturbances, we show that this system is globally uniformly convergent, as guaranteed by the theory. In figures 5, 7, and 8, the plots of $max(|q_{mid}|)$, $max(|q_{act}|)$ and $max(|q_\xi|)$ of the closed-loop system are depicted (dash-dotted lines). Based on these figures, q_{mid}, q_{act} and q_ξ exhibit a unique steady-state solution in the frequency range of $10 - 60$ $[Hz]$. This fact indicates that the controlled system is convergent and indeed a unique performance assessment in terms of disturbance attenuation can now be performed. For a better understanding of these results also a time response of q_{mid} is shown in figure 6. In this figure the time response of q_{mid} is depicted for three different initial conditions $x_{0i},\ i = 1, 2, 3$ (for the numerical values of x_{0i} see Appendix A). The excitation frequency and the force amplitude for the examined case are $f = 45$ Hz and $A = 81$ N, respectively. Figure 6 shows that the time response of q_{mid} converges to a unique steady-state solution for different initial conditions.

The comparison of the plots of $max(|q_{mid}|)$, $max(|q_{act}|)$ and $max(|q_\xi|)$ calculated for the open- and closed-loop systems shows that the closed-loop system responses are significantly smaller than those of the open-loop system. Based on this comparison, it is concluded that the effect of the disturbances w to the PWL beam is attenuated due to the control force u. Note that especially the nonlinear resonances are suppressed. This can also be noticed in figure 9, where the time response of q_{mid} in steady-state is shown. In this figure the dashed line is the open-loop solution of q_{mid}, while the solid line is the closed-loop solution. The excitation frequency for this case is 22 Hz and the force amplitude is $A = 18$ N (see also the vertical dashed line in Figure 5).

Remark: The control gain K is calculated initially by solving LMI (10) using the toolbox LMItool of Matlab. The elements of K derived in this way are in the order of 10^9. Applying a high gain control in an experimental system may firstly, lead to noise amplification, which is undesirable for the system performance, and secondly, lead to actuator saturation. In addition to that, high control gain implies big control effort for the suppression of the system resonance peaks. Therefore, a more sophisticated way to overcome such high gain controller design is followed. Due to the fact that LMI (10) provides sufficient conditions for convergence, K is not unique. Based on engineering insight, we choose a control gain that adds damping to the nonlinear resonances of the system. In this way, the system resonance peaks are suppressed. By using LMI constraints (10) we check whether the system remains convergent. Based on trial and error technique, we notice that by adding damping in q_{mid}), we render the system convergent and reduce the resonance peaks in all system states (see figures 5, 7, and 8). Based on this approach, we achieve small control gain values with respect to the initial ones. These values are in the order of 10^2. A more constructive way to choose a control gain K is by using an LMI condition that ensures bounds on the control action. The development of such LMI is subject of future work.

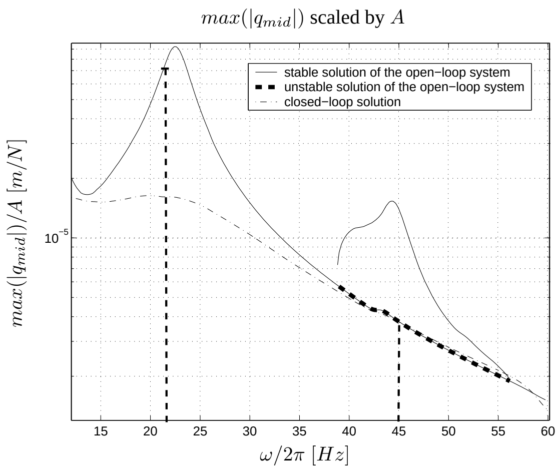

Figure 5. Scaled maximum absolute values of the transversal displacement of the middle of the beam, based on the open-loop system (solid line, dashed line) and the interconnected system (14) and (17) (dashed-dotted line).

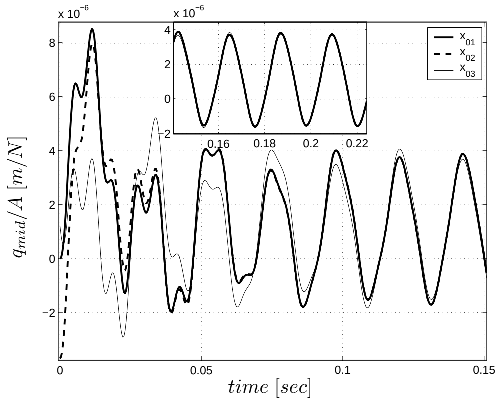

Figure 6. The transversal displacement of the middle of the beam, for the interconnected system (14) and (17) and for different initial conditions x_{0i} ($\omega = 2\pi 45\ rad/s$ and $A = 81\ N$).

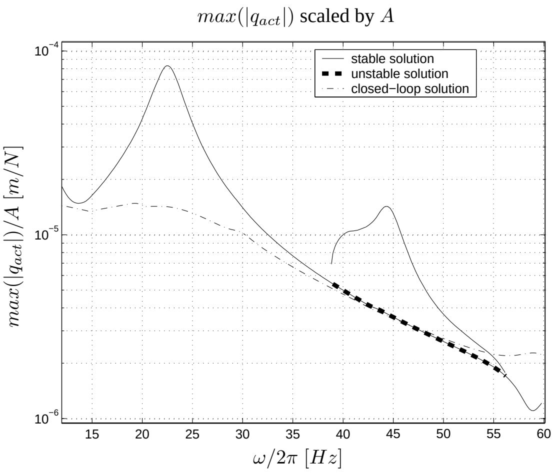

Figure 7. Scaled maximum absolute values of the transversal displacement of q_{act}, based on the open-loop system (solid line, dashed line) and the interconnected system (14) and (17) (dashed-dotted line).

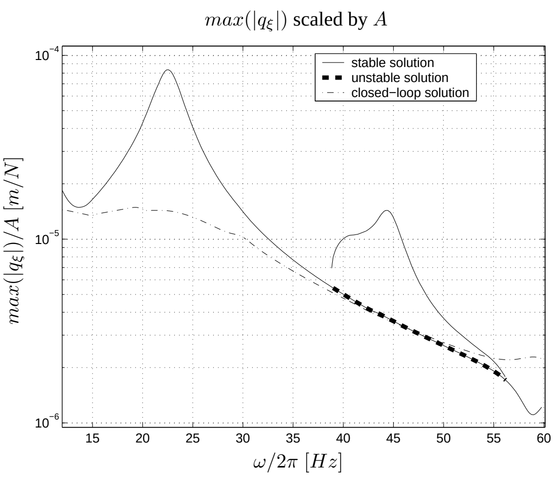

Figure 8. Scaled maximum absolute values of the transversal displacement of q_{ξ}, based on the open-loop system (solid line, dashed line) and the interconnected system (14) and (17) (dashed-dotted line).

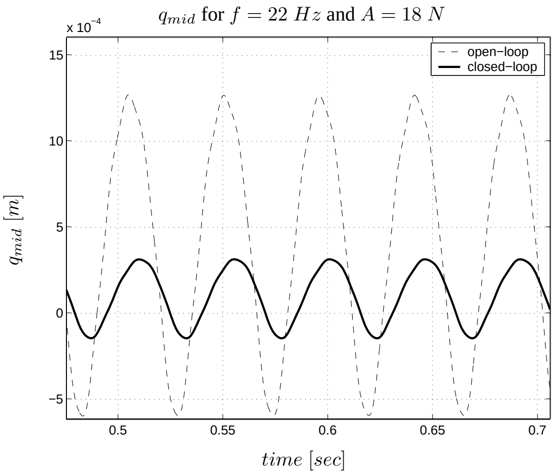

Figure 9. The steady-state solution of the transversal displacement of the middle of the beam, based on the closed-loop (solid line) and the open-loop system (dashed line) for $\omega = 2\pi 22\ rad/s$ and $A = 18\ N$.

7 Conclusions and Future work

The controller design strategy developed in the present work has proven to be suitable for disturbance attenuation of bi-modal piece-wise linear (PWL) systems excited by periodic disturbances.

We propose a convergence-based controller design for disturbance attenuation. More specifically, we use the fact that a nonlinear system has a unique globally asymptotically stable solution when it is uniformly convergent. Convergence has been used in this paper in order to uniquely define the performance of the closed-loop system.

In the present paper, we define disturbance attenuation as the suppression of the vibrations of a PWL system, caused by exogenous periodic disturbances, over a specific frequency range. By performance we indicate the ability of the controller to achieve such disturbance attenuation.

The strategy is applied to a bi-modal PWL beam system. The control laws proposed to render the closed-loop system of the PWL beam convergent and to attain disturbance attenuation are 1) a static state-feedback controller and 2) an output-feedback controller. For the output-feedback controller, a model-based switching observer is used.

The simulation results show that the interconnected system, consisting of the PWL beam in closed-loop with the observer-based controller and the observer, is globally uniformly convergent. In addition, the designed controller has been shown to perform well, since it suppresses all the (nonlinear) resonance peaks of the beam's transversal vibrations considerably in the presence of periodic disturbances.

Interesting extensions of the present work may include the experimental implementation of the proposed control strategy for PWL systems; especially on the PWL beam system.

References

Ascher U.M., Mattheij R.M.M., Russel R.D. (1995) *Numerical Solution of Boundary Value Problem for Ordinary Differential Equations*. Society for Industrial and Applied Mathematics, Philadelphia.

Demidovich B.Pb., (1967) *Lectures on stability theory (in Russian)*. Nauka, Moscow, 1967.

Doedel E.J., Champneys A.R., Fairgrieve T.F., Kuznetsov Y.A., Sandstede B., Wang X. AUTO97: Continuation and bifurcation software for ordinary differential equations, User Manual, ftp://ftp.cs.concordia.ca/pub/doedel/auto, 1998,13.

Doris A., Juloski A.Lj., Heemels W.P.M.H., Wouw N. van de, Nijmeijer H. (2005). *Switching observer design for an experimental piece-wise linear beam system*. IFAC 2005. Prague.

Feng G., Lu G. P. and S. S. Zhou (2002), *An approach to H_∞ controller synthesis of piece-wise linear systems*, communications in information and systems, vol 2, no. 3, pp. 245-254, International Press 002.

Hassibi A., Boyd S., (1998) *Quadratic Stabilization and Control of Piece-wise-Linear Systems*, Proc. American Control Conf., pages 3659-64, Philadelphia, PA.

Heertjes M.F., (1999) *Controlled Stabilization of Long-Term Solutions in a Piece-wise Linear Beam System*, PhD thesis, Eindhoven University of Technology.

Johansson M., Rantzer A., (1998), *Computation of piece-wise quadratic Lyapunov functions for hybrid systems*, IEEE transactions on automatic control, vol 43, pp. 555-559.

Juloski A.Lj., Heemels W.P.M.H., Weiland S., (2002). *Observer Design for a Class of Piece-wise Affine Systems*, Proc. of the 41st IEEE Conference: Decision and Control, Las Vegas, Nevada, USA.

Khalil H.K. (2002). *Nonlinear systems*, 3rd ed. Prentice Hall, Upper Saddle River, 2002.

Pavlov A., Progromsky A., Wouw N. van de, Nijmeijer H., (2005) *On convergence properties of piecewise affine systems* IFAC conference (2005), Accepted.

Pavlov A., Progromsky A., Wouw N. van de, Nijmeijer H., (2004). *Convergent dynamics, a tribute to Boris Pavlovich Demidovich* Systems & Control Letters 52 (2004) 257-261.

Pavlov A., (2004). *The output regulation problem: a convergent dynamics approach*, PhD thesis, Eindhoven University of Technology.

Rantzer A., Johansson M., (2000), *Piece-wise Linear Quadratic Optimal Control*, IEEE transactions on automatic control, vol 45, no. 4.

Rodrigues L., Hassibi A., How Jonathan P., *Output Feedback Controller Synthesis for Piecewise-Affine Systems with Multiple Equilibria*, Proceedings of the American Control Conference 2000, pp.1784-1789, Chicago, Illinois, USA, June 28-30 2000.

Rodrigues L., How Jonathan P., (2001) *Observer-Based Control of Piecewise-Affine Systems*, Proceedings of the 40th IEEE Conference of Decision and Control, Orlando, Florida USA, December 2001.

Appendix A

The matrices M, K_s, B_s, K, C and the values of k_{nl} and $x_{01},\ x_{02},\ x_{03}$ are

$$M = \begin{bmatrix} 4.494 & -2.326 & 0.871 \\ -2.326 & 7.618 & 2.229 \\ 0.871 & 2.229 & 2.374 \end{bmatrix},$$

$$K_s = 10^6 \begin{bmatrix} 2.528 & \text{-}0.345 & 1.026 \\ -0.345 & 1.082 & 0.296 \\ 1.026 & 0.296 & 0.613 \end{bmatrix},$$

$$B_s = 10^2 \begin{bmatrix} 1.173 & \text{-}0.298 & 0.416 \\ -0.298 & 2.012 & 0.314 \\ 0.416 & 0.314 & 0.365 \end{bmatrix},$$

$K = [0 \quad 0 \quad 0 \quad 535 \quad 0 \quad 0]$,
$C = [-0.9579 \quad 1.2165 \quad -0.2642 \quad 0 \quad 0 \quad 0]$,
$k_{nl} = 198000\ N/m$,
$x_{01} = [10^{-3} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$,
$x_{02} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$ and
$x_{03} = 10^{-3}[-0.3 \quad -0.3 \quad 0.7 \quad 2.1 \quad 3.7 \quad -4.5]$.

The values of $\Delta x(t_0)$, L_1 and L_2 are
$\Delta x(t_0) = [0.001 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$,
$L_1 = 10^4[0.0322 \quad 0.0468 \quad -0.1110 \quad -8.9161 \quad 3.3834 \quad -5.7828]$,
$L_2 = [0.0329 \quad 0.0472 \quad -0.1121 \quad -8.7315 \quad 3.5947 \quad -6.1488]$.

The values of P, α and $|\Delta x(t_0)|_P$ are:

$$P = 10^{-7} \begin{bmatrix} 2333 & -2074 & 2.19.98 & -0.31 & -1.08 & 0.59 \\ -2074 & 6531 & 14.85.85 & 2.18 & -1.76 & -1.95 \\ 220 & 1486 & 8.64.59 & -0.33 & 1.47 & -0.37 \\ -0.31 & 0.02 & -0.00.33 & 0.01 & -0.02 & 0.00 \\ -1.09 & -1.76 & 0.01.48 & -0.02 & 0.05 & 0.01 \\ 59 & -1.95 & -0.00.37 & 0.00 & 0.01 & 0.01 \end{bmatrix},$$

$\alpha = 100$ and $|\Delta x(t_0)|_P = 1.53\ 10^{-5}$.

Appendix B

Proof of theorem 2

We propose a Lyapunov candidate function V of the following form:

$$V(\Delta x) = \Delta x^T P \Delta x, \tag{21}$$

with $P = P^T > 0$.

Based on [Juloski et al., 2002], we can show that if $H^T x \leq 0$ and $H^T(x - \Delta x) \leq 0$ then

$$\dot{V}(\Delta x) = \Delta x^T((A_1 - L_1C)^T P + P(A_1 - L_1C))\Delta x, \tag{22a}$$

if $H^T x \leq 0$ and $H^T(x - \Delta x) > 0$ then

$$\dot{V}(\Delta x) = \begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2C)^T P + \ +P(A_2 - L_2C) & P\Delta A \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix}, \tag{22b}$$

if $H^T x > 0$ and $H^T(x - \Delta x) \leq 0$ then

$$\dot{V}(\Delta x) = \begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} (A_1 - L_1 C)^T P + & -P\Delta A \\ +P(A_1 - L_1 C) & \\ -\Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix}, \tag{22c}$$

and if $H^T x > 0$ and $H^T(x - \Delta x) > 0$ then

$$\dot{V}(\Delta x) = \Delta x^T((A_2 - L_2 C)^T P + P(A_2 - L_2 C))\Delta x. \tag{22d}$$

Multiplication of $H^T x \leq 0$ and $H^T(x - \Delta x) > 0$ or $H^T x > 0$ and $H^T(x - \Delta x) \leq 0$ leads to:

$$\begin{aligned} &H^T x \leq 0 \text{ and } H^T(x - \Delta x) > 0 \Rightarrow \\ &H^T x H^T(x - \Delta x) \leq 0 \end{aligned} \tag{23}$$

and

$$\begin{aligned} &H^T x > 0 \text{ and } H^T(x - \Delta x) \leq 0 \Rightarrow \\ &H^T x H^T(x - \Delta x) \leq 0. \end{aligned} \tag{24}$$

We can rewrite the inequality in (23) and (24) as follows:

$$\begin{aligned} &H^T x H^T(x - \Delta x) \leq 0 \Rightarrow \\ &\begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}HH^T \\ -\frac{1}{2}HH^T & HH^T \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix} \leq 0 \end{aligned} \tag{25}$$

Moreover $V(\Delta x)$, given by (21) can be written as:

$$V(\Delta x) = \begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix}. \tag{26}$$

It is known that the inequality

$$\dot{V}(\Delta x) \leq -\alpha V(\Delta x) \tag{27}$$

implies global exponential stability of $V(\Delta x)$. Therefore, there exists a $U(t) = U(t_0)e^{-\alpha t}$, with $U(t_0) = \Delta x(t_0)^T P \Delta x(t_0)$ such that:

$$\begin{aligned} &V(\Delta x(t)) \leq U(t) \Rightarrow \\ &\Delta x(t)^T P \Delta x(t) \leq U(t_0)e^{-\alpha t} \Rightarrow \\ &\Delta x(t)^T P \Delta x(t) \leq \Delta x(t_0)^T P \Delta x(t_0) e^{-\alpha t} \Rightarrow \\ &|\Delta x(t)|_P^2 \leq |\Delta x(t_0)|_P^2 e^{-\alpha t} \Rightarrow \\ &|\Delta x(t)|_P \leq |\Delta x(t_0)|_P e^{-\frac{\alpha t}{2}}, \end{aligned} \tag{28}$$

where $|\Delta x(t)|_P$ is a norm of $\Delta x(t)$ with the form

$$|\Delta x|_P = \sqrt{\Delta x^T P \Delta x}, \text{ for } \Delta x \in \mathbb{R}^n \text{ and } P = P^T > 0. \tag{29}$$

This norm is called the *P-norm* of Δx.
It is also known that,

$$\lambda_{min}(P)|\Delta x(t)|^2 \leq |\Delta x(t)|^T P |\Delta x(t)|, \tag{30}$$

where $\lambda_{min}(P)$ is the minimum eigenvalue of P and $|\Delta x(t)|$ is the *Euclidean norm* of $\Delta x(t)$.
The combination of (29) and (30) yields

$$\begin{aligned} &\lambda_{min}(P)|\Delta x(t)|^2 \leq |\Delta x(t)|^T P |\Delta x(t)| \Rightarrow \\ &\sqrt{\lambda_{min}(P)|\Delta x(t)|^2} \leq \sqrt{|\Delta x(t)|^T P |\Delta x(t)|} \\ &\qquad \leq |\Delta x(t_0)|_P e^{-\frac{\alpha t}{2}} \Rightarrow \\ &\sqrt{\lambda_{min}(P)}\ |\Delta x(t)| \leq |\Delta x(t_0)|_P e^{-\frac{\alpha t}{2}} \Rightarrow \\ &|\Delta x(t)| \leq 1/\sqrt{\lambda_{min}(P)}\ |\Delta x(t_0)|_P e^{-\frac{\alpha t}{2}}. \end{aligned} \tag{31}$$

Substituting (22) and (26) into (27) yields

$$\Delta x^T((A_1 - L_1 C)^T P + P(A_1 - L_1 C) + \alpha P)\Delta x \leq 0, \tag{32a}$$

if $H^T x \leq 0$ and $H^T(x - \Delta x) \leq 0$,

$$\begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} (A_2 - L_2 C)^T P + & P\Delta A \\ +P(A_2 - L_2 C) + \alpha P & \\ \Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix} \leq 0 \tag{32b}$$

if $H^T x \leq 0$ and $H^T(x - \Delta x) > 0$,

$$\begin{bmatrix} \Delta x \\ x \end{bmatrix}^T \begin{bmatrix} (A_1 - L_1 C)^T P + & -P\Delta A \\ +P(A_1 - L_1 C) + \alpha P & \\ -\Delta A^T P & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ x \end{bmatrix} \leq 0 \tag{32c}$$

if $H^T x > 0$ and $H^T(x - \Delta x) \leq 0$ and

$$\Delta x^T((A_2 - L_2 C)^T P + P(A_2 - L_2 C) + \alpha P)\Delta x \leq 0 \tag{32d}$$

if $H^T x > 0$ and $H^T(x - \Delta x) > 0$.
Applying the S-procedure to the sets of inequalities {(32b), (25)} and {(32c), (25)} the LMI constraints (15a) and (15b) are derived, respectively.