ANALYSIS OF FRICTION COMPENSATION EFFECTS IN CONTROLLED MECHANICAL SYSTEMS

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Abstract

In this paper, the effects of friction compensation, including exact compensation, undercompensation and overcompensation, in PD controlled 1DOF mechanical systems are investigated. The friction force that is acting on the mechanical system and the friction compensation term in the feedback loop are described by a class of discontinuous friction models consisting of static, Coulomb and viscous friction, and including the Stribeck effect. In order to capture the sticking phenomenon at zero velocity, the friction model is expressed as a multi-valued map. Lyapunov's stability theorem, LaSalle's invariance principle and the properties of the ω-limit set of trajectories of a 2-dimensional differential inclusion are employed to show that firstly, exact compensation leads to global exponential stability of the setpoint, secondly, undercompensation leads to steady state errors, and thirdly, overcompensation conditionally induces limit cycling.

1 Introduction

Friction occurs in many controlled positioning systems and it can deteriorate performance of those controlled systems in terms of large steady-state errors and limit cycling, see for example [4]. Friction compensation is, therefore, needed in order to improve the system performance. Satisfactory friction compensation can be obtained if a good friction model is available. However, friction is a highly nonlinear phenomenon, which is difficult to be completely described by a simple model [4, 12]. Because of such modeling errors and parameter estimation errors, inexact friction compensation is inevitable.

The limit cycling effect that is induced by the overcompensation of friction in PD and PID controlled 1-degree-of-freedom (1DOF) systems has been analyzed in [6] by means of the describing function method. Papadopoulos and Chasparis [13] validate the predicted limit cycle on an experimental setup but, at the same time, they also show that the prediction of the describing function is not always accurate. The numerical and the experimental results in [11] also indicate that overcompensation of friction induces limit cycling and undercompensation of friction leads to steady-state errors. This manuscript is intended to provide a rigorous mathematical analysis of those observations for a class of discontinuous friction models and for more general cases of undercompensation and overcompensation of friction. Our analysis is based on Lyapunov's stability theorem [15], LaSalle's invariance principle [1, 2, 16] and the properties of the ω-limit set of trajectories of a 2-dimensional differential inclusion [8]. In this study, we consider friction compensation in PD controlled 1DOF systems.

This paper is organized as follows. Section 2 explains the model of the controlled system with friction compensation. The effects of exact compensation, undercompensation and overcompensation of friction to the performance of the controlled system are studied in sections 3, 4 and 5, respectively. Section 6 provides numerical demonstrations of the analytical results. Finally, conclusions are drawn in section 7.

2 Controlled 1DOF frictional systems

We consider 1DOF frictional mechanical systems that can be described by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{F_v}{J}y - \frac{1}{J}F(y) + \frac{1}{J}u, \end{aligned} \tag{2.1}$$

where x, y and J are the position, the velocity and the inertia of the mechanical system, respectively, $F_v > 0$ is the linear viscous friction damping, u is the input force and $F(y)$ is the nonlinear friction force given by

$$F(y) \in \begin{cases} g(y)\mathrm{sign}(y) & \text{if} \quad y \neq 0 \\ [-F_s, F_s] & \text{if} \quad y = 0, \end{cases} \tag{2.2}$$

with $F_s > 0$ the static friction level and $g(y)$ a Stribeck function, which represents the continuous decay[1] of the friction curve from F_s to a Coulomb friction level $F_c > 0$. Notice that the set-valued nature of (2.2) at $y = 0$ allows to model the sticking phenomenon. The friction model (2.2) is reduced to a Coulomb friction model if $g(y) = F_s = F_c$. Other commonly used Stribeck functions in the control literature [4, 12] are of the forms

$$g(y) = F_c + (F_s - F_c)e^{-(|y|/v_s)^\delta} \tag{2.3}$$

and

$$g(y) = F_c + (F_s - F_c)\frac{1}{1 + (|y|/v_s)^\delta}, \tag{2.4}$$

[1]The continuous decay implies $g(0) = F_s$, $\lim_{y\downarrow 0} F(y) = F_s$ and $\lim_{y\uparrow 0} F(y) = -F_s$.

where $v_s > 0$ is called the Stribeck velocity and $\delta > 0$ is the shaping parameter of the Stribeck curve. The combination of a linear viscous friction and the nonlinear friction F as considered in (2.1) is able to represent a rather general class of static friction models [4, 12]. However, the friction model (2.2) excludes the friction model with a discontinuous drop of the friction curve from F_s to F_c, which is shown in [3] to be inadequate for describing the possible disappearance of the friction-induced stick-slip phenomenon. Here we opted for a static friction model since we focus on the effect of friction on the global dynamics; however, when the behaviour for very small velocities is particularly important one could opt for a dynamic friction model, see e.g. [7].

In order to regulate the frictional mechanical system (2.1) towards a setpoint x_s, we consider a PD controller with friction compensation of the form

$$u = K_p(x_s - x) + K_d(0 - y) + \widetilde{F}(y, \bar{u}), \tag{2.5}$$

where $K_p > 0$ is the proportional gain, $K_d > 0$ is the derivative gain, and $\widetilde{F}(y, \bar{u})$ is a friction compensation term given by

$$\widetilde{F}(y, \bar{u}) \in \begin{cases} \widetilde{g}(y)\mathrm{sign}(y) & \text{if } y \neq 0 \\ \widetilde{F}_s\mathrm{Sign}(\bar{u}) & \text{if } y = 0, \end{cases} \tag{2.6}$$

where $\bar{u} = K_p(x_s - x)$. The presence of the set-valued Sign function, where $\mathrm{Sign}(0) \in [-1, 1]$, allows the friction compensation term $\widetilde{F}$ to take any value between $[-\widetilde{F}_s, \widetilde{F}_s]$ if both $y = 0$ and $\bar{u} = 0$, and provides alternatives in the implementation.

Without loss of generality, the setpoint is assumed to be the origin, i.e. $x_s = 0$, such that the control input u becomes

$$u = -K_p x - K_d y + \widetilde{F}(y, -K_p x). \tag{2.7}$$

Substitution of the feedback (2.7) into the system (2.1) results in the closed-loop system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{K_p}{J}x - \frac{(K_d + F_v)}{J}y + \frac{1}{J}\Delta F(y, x), \end{aligned} \tag{2.8}$$

where $\Delta F(y, x) = \widetilde{F} - F$ is the friction compensation error given by

$$\Delta F(y, x) \in \begin{cases} (\widetilde{g}(y) - g(y))\,\mathrm{sign}(y) & \text{if } y \neq 0 \\ [-F_s, F_s] - \widetilde{F}_s\mathrm{Sign}(x) & \text{if } y = 0. \end{cases} \tag{2.9}$$

To study the effects of friction compensation on the dynamics of the closed-loop system (2.8), the following definition is adopted. The friction force F is said to be exactly compensated if

$$\widetilde{F}_s = F_s \;\text{ and }\; \widetilde{g}(y) = g(y), \forall\, y \neq 0, \tag{2.10}$$

undercompensated if

$$\widetilde{F}_s < F_s \;\text{ and }\; \widetilde{g}(y) < g(y), \forall\, y \neq 0, \tag{2.11}$$

and overcompensated if

$$\widetilde{F}_s > F_s \;\text{ and }\; \widetilde{g}(y) > g(y), \forall\, y \neq 0. \tag{2.12}$$

Notice that in the exact compensation and the undercompensation cases the friction compensation error ΔF given by (2.9) is an upper semi-continuous, convex and bounded multi-valued map. In these cases, the closed-loop system (2.8) is, likewise the open-loop system (2.1), a system of differential inclusion of Filippov-type [8], for which existence of solutions is guaranteed [5, Theorem 3, p.98]. However, for the overcompensation case ΔF is not convex. As an illustration for $x > 0$ following (2.9) $\Delta F(0, x > 0)$ belongs to the interval $[-(F_s + \widetilde{F}_s), F_s - \widetilde{F}_s]$ but this interval does not include $\lim_{y \downarrow 0} \Delta F(y, x > 0) = \widetilde{F}_s - F_s$ if $\widetilde{F}_s > F_s$. In this case, the closed-loop system (2.8) is a discontinuous system, which does not belong to Filippov-type differential inclusions. For consistency of solutions definition, we can apply Filippov's convex method to render the discontinuous system to be a Filippov-type differential inclusion. The non-convex map (2.9) is, then, extended by including the closed convex hull of all the limits of ΔF, which results in

$$\overline{\Delta F}(y, x) \in \begin{cases} (\widetilde{g}(y) - g(y))\,\mathrm{sign}(y) & \text{if } y \neq 0 \\ \left[F_s - \widetilde{F}_s, F_s + \widetilde{F}_s\right] & \text{if } y = 0 \text{ and } x < 0 \\ \left[-(F_s + \widetilde{F}_s), F_s + \widetilde{F}_s\right] & \text{if } y = 0 \text{ and } x = 0 \\ \left[-(F_s + \widetilde{F}_s), \widetilde{F}_s - F_s\right] & \text{if } y = 0 \text{ and } x > 0. \end{cases} \tag{2.13}$$

3 Exact compensation case

In this section, it will be proven that in the case of exact friction compensation the closed-loop system (2.8) has a unique equilibrium point at the origin, which is globally exponentially stable.

Equilibria of the closed-loop system (2.8) satisfy

$$y = 0 \text{ and } K_p x \in [-F_s, F_s] - \widetilde{F}_s\mathrm{sign}(x). \tag{3.1}$$

Since in the exact compensation case $\widetilde{F}_s = F_s$, (3.1) yields a unique equilibrium point at the origin. In order to prove that the origin is a globally exponentially stable equilibrium of (2.8), consider the Lyapunov function candidate

$$V(x, y) = \frac{1}{2}(F_v x + Jy)^2 + \frac{1}{2}(K_p J + K_d F_v)x^2 \tag{3.2}$$

that is radially unbounded. Its time-derivative along trajectories of (2.8) is given by

$$\dot{V}(x, y) = -K_p F_v x^2 + F_v \Delta F x - K_d J y^2 + J \Delta F y. \tag{3.3}$$

From (2.9) and (2.10), we have $\Delta F x \leq 0$ and $\Delta F y = 0$ such that

$$\dot{V}(x, y) \leq -K_p F_v x^2 - K_d J y^2. \tag{3.4}$$

The existence of the quadratic Lyapunov function (3.2) with its time-derivative satisfying (3.4) guarantees that the origin is a globally exponentially stable equilibrium point of (2.8) [15].

By using a similar method, the result on exact friction compensation can be extended to multi-degree-of-freedom (MDOF) systems with multiple friction forces. The key feature of this method is that at zero velocity the friction compensation term $\widetilde{F}$ depends on the controlled position error, see (2.6).

4 Undercompensation case

The objective of this section is to show that in the undercompensation case the closed-loop system (2.8) has a globally attractive equilibrium set containing the origin. The method to achieve this objective is based on Lyapunov's stability theorem [15] and LaSalle's invariance principle [1, 2, 16].

As mentioned in the previous section, equilibria of the closed-loop system (2.8) satisfy (3.1). From the undercompensation definition (2.11), (3.1) results in the equilibrium set

$$S_E = \left\{ (x,y) \in \mathbb{R}^2 : |x| \leq \frac{F_s - \widetilde{F}_s}{K_p}, y = 0 \right\}. \tag{4.1}$$

Obviously, the equilibrium set S_E contains the origin, which is the setpoint of the controlled system (2.8).

Since the invariance principle requires uniqueness of solutions in forward time, firstly we need to verify whether the closed-loop system (2.8) satisfies this condition. A sufficient condition for uniqueness of solutions of Filippov-type differential inclusions is the absence of repulsive sliding modes [9]. Sliding mode behaviors can be investigated using the projections of vector fields onto the normal vector to a switching surface. The switching surface of (2.8) is $S = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ and $n = [\ 0 \quad 1\ ]^T$ is the normal vector to S. Notice that $S_E \subset S$. The switching surface S partitions the state space into $G^- = \{(x,y) \in \mathbb{R}^2 : y < 0\}$ and $G^+ = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. The projections of the vector field in G^+ and G^- on the normal vector n at the switching surface S are given by

$$n^T f^+(x,y) = -\frac{K_p}{J}x + \frac{\widetilde{F}_s - F_s}{J}, \forall (x,y) \in S \tag{4.2}$$

and

$$n^T f^-(x,y) = -\frac{K_p}{J}x - \frac{\widetilde{F}_s - F_s}{J}, \forall (x,y) \in S, \tag{4.3}$$

respectively. Repulsive sliding modes occur at the switching surface S if $n^T f^+(x,y) > 0$ and $n^T f^-(x,y) < 0$, which results in

$$K_p|x| < (\widetilde{F}_s - F_s). \tag{4.4}$$

Since $K_p > 0$ and in the undercompensation case $\widetilde{F}_s - F_s < 0$, the inequality (4.4) never holds such that repulsive sliding modes never occur. Therefore, uniqueness of solutions in forward time of the closed-loop system (2.8) is guaranteed in the undercompensation case.

Theorem 1 *The origin of the closed-loop system (2.8) is globally stable and the equilibrium set S_E given by (4.1) is globally attractive if the friction force is undercompensated, i.e. if condition (2.11) holds.*

Proof Consider the Lyapunov function candidate

$$V(x,y) = \frac{K_p}{2}x^2 + \frac{J}{2}y^2. \tag{4.5}$$

The time-derivative of $V(x,y)$ along trajectories of (2.8) is given by

$$\dot{V}(x,y) = -(K_d + F_v)y^2 + \Delta F(y)y. \tag{4.6}$$

From the condition (2.11), it can be shown that

$$\Delta F(y)y < 0, \forall\, y \neq 0 \text{ and } \Delta F(y)y = 0 \text{ iff } y = 0. \tag{4.7}$$

Substitution of (4.7) into (4.6), yields

$$\dot{V}(x,y) \leq -(K_d + F_v)y^2. \tag{4.8}$$

The existence of the Lyapunov function (4.5) with its time derivative satisfying (4.8) proves that the origin is globally stable [15]. Furthermore, $\dot{V}(x,y) = 0$ only in the set S and the equilibrium set S_E is the largest invariant set of (2.8) contained in the set S. Because the closed-loop system (2.8) has unique solutions in forward time, LaSalle's invariance principle [1, 2, 16] can be applied to conclude that all trajectories of (2.8) converge to the equilibrium set S_E. Hence, the equilibrium set S_E is globally attractive.□

Theorem 1 indicates that undercompensation of friction leads to steady-state errors, which are bounded by $(F_s - \widetilde{F}_s)/K_p$ due to the size of the equilibrium set S_E. Limit cycling, however, never occurs. This result on the undercompensation case can be extended to MDOF systems with multiple friction forces because in this case the equilibrium set is due to the remaining friction forces. By choosing an appropriate Lyapunov function, for example as proposed in [16], and applying LaSalle's invariant principle a similar result can be obtained.

5 Overcompensation case

This section provides a rigorous analysis showing that overcompensation of friction in the closed-loop system (2.8) may provoke limit cycling around the setpoint. As discussed at the end of Section 2, in the overcompensation case the term ΔF in (2.8) is replaced by $\overline{\Delta F}$ given by (2.13) to render (2.8) a Filippov-type differential inclusion.

The analysis is based on the properties of the ω-limit set of trajectories of a 2-dimensional differential inclusion. Here, we adopt the definition of ω-limit sets given in [8, p.129]. In order to prove that the system (2.8) exhibits limit cycling, it is sufficient to show that the ω-limit set of its trajectories is a closed orbit. The following theorem, which is proven in [8, Theorem 3, p.137], is useful for achieving this goal.

Theorem 2 *Consider a 2-dimensional autonomous differential inclusion*

$$\dot{z} \in F(z) \tag{5.1}$$

with $F(z)$ a set-valued function that is closed, convex and bounded for all $z \in \mathbb{R}^2$ and the function F is upper semi-continuous. Suppose that uniqueness of solutions in forward time holds at any point on a trajectory $\Gamma = \{z \in \mathbb{R}^2 : z = \varphi(t), t \in [0,\infty)\}$ of (5.1). If the ω-limit set of Γ is bounded and contains no equilibrium points then it consists of one closed orbit.

Since the closed-loop system (2.8) with ΔF replaced by $\overline{\Delta F}$ is a Filippov-type differential inclusion, it satisfies the conditions of Theorem 2. Next, we state a result on boundedness of trajectories of the closed-loop system (2.8).

Proposition 3 *The ω-limit set of all trajectories of the closed-loop system (2.8) - with ΔF replaced by $\overline{\Delta F}$ as in (2.13) - is bounded if the friction force is overcompensated, i.e. condition (2.12) holds.*

Proof Consider the positive definite function

$$V(x,y) = \frac{1}{2}(F_v x + Jy)^2 + \frac{1}{2}(K_p J + K_d F_v)x^2 \tag{5.2}$$

that is radially unbounded. Its time-derivative along trajectories of (2.8) is given by

$$\begin{aligned}\dot{V}(x,y) = \quad & -K_p F_v x^2 + F_v \overline{\Delta F}(y,x)x - K_d J y^2 + \\ & J\overline{\Delta F}(y,x)y,\end{aligned} \tag{5.3}$$

with $\overline{\Delta F}$ as in (2.13). From the property of the Stribeck curve, $F_c \le g(y) \le F_s$, the condition (2.12) and the friction compensation error (2.13), it can be shown that

$$\overline{\Delta F}(y,x)x \le (\widetilde{F}_s - F_c)|x| \text{ and } \overline{\Delta F}(y,x)y \le (\widetilde{F}_s - F_c)|y|. \tag{5.4}$$

Substitution of (5.4) into (5.3) yields

$$\begin{aligned}\dot{V}(x,y) \le \quad & -K_p F_v x^2 + F_v(\widetilde{F}_s - F_c)|x| - \\ & K_d J y^2 + J(\widetilde{F}_s - F_c)|y|.\end{aligned} \tag{5.5}$$

Following (5.5), $\dot{V}(x,y) < 0$ if

$$K_p F_v x^2 + K_d J y^2 > (\widetilde{F}_s - F_c)\left(F_v|x| + J|y|\right). \tag{5.6}$$

Since $\widetilde{F}_s - F_c > 0$, inequality (5.6) holds for all pairs (x,y) that are sufficiently separated from the origin because the left-hand side of the inequality is a quadratic function of x and y while the right-hand side is a linear function of the absolute values of x and y. Therefore, trajectories of the closed-loop system (2.8) cannot grow unbounded in forward time.□

In the following, we find the conditions on the closed-loop system (2.8) such that the ω-limit set of its trajectories does not contain any equilibrium points. Equilibria of the closed-loop system (2.8) in the overcompensation case satisfy $y = 0$ and $K_p x = \overline{\Delta F}(y,x)$, which results in

$$S_\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : |x| \le \frac{\widetilde{F}_s - F_s}{K_p}, y = 0\right\}. \tag{5.7}$$

It has been shown in the previous section that repulsive sliding modes occur if the inequality (4.4) holds. Since in the overcompensation case $\widetilde{F}_s - F_s > 0$, following (4.4) repulsive sliding modes occur at the segment

$$\Psi = \left\{(x,y) \in \mathbb{R}^2 : |x| < \frac{\widetilde{F}_s - F_s}{K_p}, y = 0\right\} \tag{5.8}$$

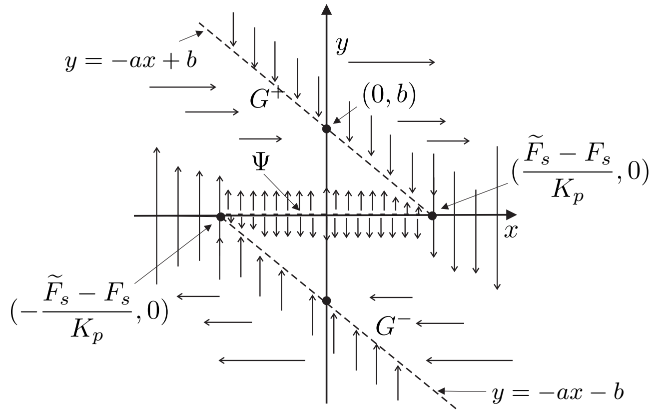

Figure 1: The vector field of (2.8) in the case of overcompensation of friction, where $a = \dfrac{K_p}{K_d + F_v}$ and $b = \dfrac{\widetilde{F}_s - F_s}{K_d + F_v}$.

of the switching surface S. From (5.7) and (5.8), the equilibrium set S_ε can be rewritten as

$$S_\varepsilon = (-\frac{\widetilde{F}_s - F_s}{K_p}, 0) \cup \Psi \cup (\frac{\widetilde{F}_s - F_s}{K_p}, 0). \tag{5.9}$$

Hence, the set Ψ is an unstable equilibrium set of (2.8) and it can be concluded that the ω-limit set of all trajectories of (2.8) starting at $(x_0, y_0) \in \mathbb{R}^2 \setminus S_E$ does not contain the set Ψ. However, the ω-limit set may contain one or both of the extremal equilibrium points $(-\dfrac{\widetilde{F}_s - F_s}{K_p}, 0)$ and $(\dfrac{\widetilde{F}_s - F_s}{K_p}, 0)$.

Next, the possible convergence of trajectories to the two extremal equilibrium points is investigated through a phase-plane analysis as depicted in Figure 1. The projection of the vector field of (2.8) on the normal vector $m = [\,1 \quad 0\,]^T$ to the y-axis is $m^T f(x,y) = y$ such that $m^T f(x,y) < 0$ for all $(x,y) \in G^-$ and $m^T f(x,y) > 0$ for all $(x,y) \in G^+$. The projection of the vector field on the normal vector $n = [\,0 \quad 1\,]^T$ to the x-axis is given by

$$n^T f(x,y) = -\frac{K_p}{J}x - \frac{(K_d + F_v)}{J}y + \frac{1}{J}\overline{\Delta F}(y,x). \tag{5.10}$$

Since in the overcompensation case $\overline{\Delta F} < (\widetilde{F}_s - F_c)$, $\forall\, y > 0$ and $\overline{\Delta F} > -(\widetilde{F}_s - F_c)$, $\forall\, y < 0$, (5.10) yields

$$n^T f(x,y) < 0 \text{ if } y > \frac{-K_p}{K_d + F_v}x + \frac{\widetilde{F}_s - F_c}{K_d + F_v} \text{ and } y > 0,$$

$$n^T f(x,y) > 0 \text{ if } y < \frac{-K_p}{K_d + F_v}x - \frac{\widetilde{F}_s - F_c}{K_d + F_v} \text{ and } y < 0.$$

Trajectories of the closed-loop system (2.8) cross the switching surface S, which is the x-axis, transversally if and only if the inequality

$$n^T f^+(x,y) \cdot n^T f^-(x,y) > 0, \tag{5.11}$$

holds, where $n^T f^+(x,y)$ and $n^T f^-(x,y)$ are given (4.2) and (4.3), respectively. From the condition (5.11), (4.2) and (4.3), the transversal intersections occur at the segment $S_T = \left\{(x,y) \in \mathbb{R}^2 : |x| > (\widetilde{F}_s - F_s)/K_p, y = 0\right\}$ of the switching surface S.

The phase-plane analysis shows that the extremal equilibrium points $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ and $(-\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ can only be reached from G^+ and G^-, respectively. The dynamics of (2.8) in G^+ reduces to

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{K_p}{J}x - \frac{K_d + F_v}{J}y + \frac{\Delta g(y)}{J}, \end{aligned} \tag{5.12}$$

with $\Delta g(y) = \widetilde{g}(y) - g(y)$. Let approximate the Stribeck functions $\widetilde{g}(y)$ and $g(y)$ by a Taylor expansion such that $\Delta g(y)$ can be approximated by

$$\Delta g(y) = \widetilde{g}(0) - g(0) + (\widetilde{g}'(0) - g'(0))y + \text{h.o.t.}, \tag{5.13}$$

where $\widetilde{g}(0) = \widetilde{F}_s$, $g(0) = F_s$, $\widetilde{g}'(y) = \frac{\partial \widetilde{g}(y)}{\partial y}$ and $g'(y) = \frac{\partial g(y)}{\partial y}$. Note that for the Stribeck functions (2.3) and (2.4) this approximation is possible only for $\delta \geq 1$ because $g'(0) = 0$ if $\delta > 1$, $g'(0) = -\frac{F_s - F_c}{v_s}$ if $\delta = 1$ and $g'(0)$ is not defined if $\delta < 1$. Hence, for the case where $\widetilde{g}'(0)$ and $g'(0)$ are well-defined, the system (5.12) around $y \gtrapprox 0$ can be approximated by the linear system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{K_p}{J}x - \frac{K_d + F_v - \widetilde{g}'(0) + g'(0)}{J}y + \frac{\widetilde{F}_s - F_s}{J}. \end{aligned} \tag{5.14}$$

Notice that the extremal equilibrium point $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ coincides with the equilibrium point of the linear system (5.14). Because of the symmetry of the vector field about the y-axis a similar linear approximation also holds for the extremal equilibrium point $(-\frac{\widetilde{F}_s - F_s}{K_p}, 0)$. The linear approximation allows to investigate the possible convergence of trajectories of the closed-loop system (2.8) to the extremal equilibrium points such that the following result can be concluded.

Theorem 4 *Consider the closed-loop system (2.8) - with ΔF replaced by $\overline{\Delta F}$ as in (2.13) - in the case where the friction force is overcompensated, i.e. condition (2.12) holds, and assume that $\widetilde{g}'(0) = \frac{\partial \widetilde{g}(y)}{\partial y}|_{y=0}$ and $g'(0) = \frac{\partial g(y)}{\partial y}|_{y=0}$ are well-defined. The ω-limit set of any trajectory of the closed-loop system (2.8), starting away from the equilibrium set S_ε given by (5.7), consists of one closed orbit that encircles the equilibrium set S_ε if the inequality*

$$K_d + F_v - \widetilde{g}'(0) + g'(0) > 0 \tag{5.15}$$

is violated or if both (5.15) and

$$(K_d + F_v - \widetilde{g}'(0) + g'(0))^2 < 4K_pJ \tag{5.16}$$

hold. But if only (5.16) is violated, the ω-limit set does not contain such a closed orbit but it may consist of one of the extremal equilibrium points $(-\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ or $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$.

Proof Since $J, K_p > 0$, applying the Hurwitz condition, the linear approximation (5.14) is stable if and only if the inequality (5.15) holds. Thus, if (5.15) is violated the linear system (5.14) becomes unstable and, following the phase plane analysis, trajectories of the closed-loop system (2.8) will not converge to the extremal equilibrium points. Consequently the ω-limit set of any trajectory of (2.8), starting outside the equilibrium set S_ε, does not contain any equilibrium points. Applying Theorem 2 and Proposition 3, the ω-limit set consists of one closed orbit if uniqueness of solutions holds at any point along those trajectories. It has been shown that transversal intersections occur at the segment S_T of the switching surface S and uniqueness of solutions in forward time holds at any point on S_T. Since $S = S_\varepsilon \cup S_T$ and the trajectories do not contain any point in S_ε, we can conclude that uniqueness of solutions in forward time holds at any point along those trajectories and the first part of the theorem is proven.

If the Hurwitz condition (5.15) holds, trajectories in G^+ may eventually converge to the extremal equilibrium point $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$. However, if the inequality (5.16) holds the dynamics of the linear system (5.14) are undercritically damped, i.e. it has a pair of complex eigenvalues, such that those trajectories will oscillate before converging to the equilibrium point. Note that the dynamics (5.14) hold only in G^+ and once a trajectory crosses the x-axis it will move away from the x-axis towards the region G^- as depicted in Figure 1. Because the vector field in G^- and in G^+ are symmetric the same scenario takes place and the cycle repeats such that the two extremal equilibrium points cannot be reached neither in finite time nor in infinite time. Hence, the ω-limit set of those trajectories does not contain any equilibrium points. By applying the same reasoning as in the first part, we can conclude the second part of the theorem.

If only the inequality (5.16) is violated, the dynamics of the linear system (5.14) become supercritically damped, i.e. it has two real eigenvalues, such that trajectories in G^+ converge exponentially to the extremal equilibrium point $(\frac{\widetilde{F}_s - F_s}{K_p}, 0)$ without oscillation. Therefore, the extremal equilibrium point can be reached in infinite time. This result also holds for the other extremal equilibrium point due to the symmetry of the vector field. Hence, the ω-limit set of trajectories of the closed-loop system (2.8) may consist of one of the two extremal equilibrium points.$\square$

Theorem 4 indicates that overcompensation of friction may provoke limit cycling and that the limit cycling effect can be eliminated by tuning the gains of the PD controller, i.e. choose

K_p and K_d satisfying the Hurwitz condition (5.15) and violating the inequality (5.16). This limit cycling result cannot be extended to a multi-degree of freedom system because it is based on Theorem 5.1, which is valid only for 2-dimensional systems. However, the result on boundedness of the ω-limit set, Proposition 3, can be extended to MDOF systems by using a similar approach. The sliding-mode analysis on the switching surface, see for example [9], and the local stability analysis of the extremal equilibrium points are also applicable to investigate possible convergence of trajectories of a MDOF frictional system to an equilibrium point. Those extended analysis could predict whether trajectories of a controlled system converge to an attractor - not necessarily a closed orbit - or to an equilibrium point as a result of overcompensation of friction.

6 A numerical example

This section provides numerical illustrations of the theoretical results obtained in the previous three sections. For this purpose, we consider the 1DOF mechanical system studied in [14]. The dynamics of the system can be described by (2.1), (2.2) with $g(y) = F_c + (F_s - F_c)e^{-(y/v_s)^2}$ and the parameter values: $J = 0.0260$ kgm^2, $F_v = 0.0710$ Nms/rad, $F_c = 0.4195$ Nm, $F_s = 0.5005$ Nm and $v_s = 0.15$ rad/s. The friction compensation is given by (2.6) with $\widetilde{F}_s = \alpha F_s$ and $\widetilde{g}(y) = \alpha g(y)$, where $\alpha > 0$ is a scaling factor. Following the definitions in Section 2, we have exact compensation case if $\alpha = 1$, undercompensation case if $\alpha < 1$ and overcompensation case if $\alpha > 1$.

Solutions of the closed-loop system are obtained numerically using the so-called switch-model approximation for the dynamics around the switching surface S, see e.g. [10, 14]. Figure 2 shows that trajectories of the closed-loop system (2.8) with $K_p = 0.1$ and $K_d = 0.1$ converge exponentially to the origin in the exact compensation case as predicted in Section 3. A phase portrait showing an attracting equilibrium set of the controlled system (2.8), with $\alpha = 0.8$ (20% undercompensation) and the PD controller gains set as in the case of exact compensation, is depicted in Figure 3. This simulation result agrees with Theorem 1. Figure 4(a) depicts a phase portrait showing an asymptotically stable closed orbit of the closed-loop system (2.8) with $\alpha = 1.2$ (20% overcompensation) in the undercritically damped case, with $K_p = 1$ and $K_d = 0.2$. The closed orbit comes closer to the extremal points of the equilibrium set S_ε, when it crosses the y-axis but does not hit these points such that the closed orbit encircles the equilibrium set S_ε. On the other hand, Figure 4(b) depicts a phase portrait of (2.8) with the same value of α in the supercritically damped case, with $K_p = 1$ and $K_d = 0.8$. The phase portrait shows two at-

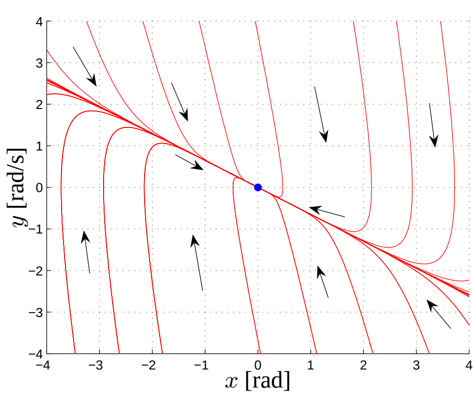

Figure 2: Phase portrait of the controlled system (2.8) with $\alpha = 1$ (exact compensation), $K_p = 0.1$ and $K_d = 0.1$.

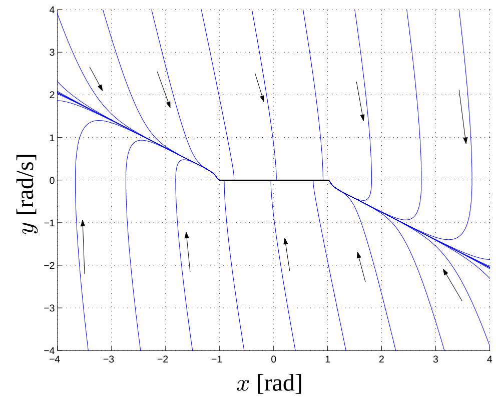

Figure 3: Phase portrait of the controlled system (2.8) with $\alpha = 0.8$ (undercompensation), $K_p = 0.1$ and $K_d = 0.1$.

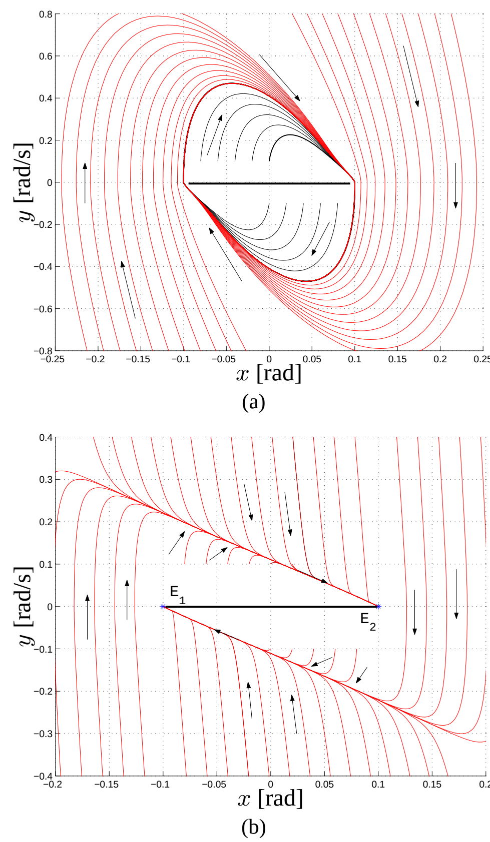

Figure 4: Phase portrait of the system (2.8) with $\alpha = 1.2$ (overcompensation): (a) the undercritically damped case with $K_p = 1$ and $K_d = 0.2$, and (b) the supercritically damped case with $K_p = 1$ and $K_d = 0.8$, E_1 and E_2 are the extremal equilibrium points.

tracting extremal equilibrium points of the equilibrium set S_ε. These simulation results confirm the prediction of Theorem 4.

7 Conclusions

We have investigated the positive effect of exact friction compensation and the negative effects of undercompensation and overcompensation of friction in PD controlled 1DOF mechanical systems for a class of discontinuous friction models consisting of static, Coulomb and viscous friction, and including the Stribeck effect. It is proven that exact friction compensation in the 1DOF mechanical systems makes the closed-loop system behaves as the linear system without friction even though the friction value at zero velocity is not explicitly known. It has been shown that undercompensation of friction in the 1DOF controlled mechanical systems results in a globally attractive equilibrium set containing the setpoint, which is globally stable. This result indicates that the controlled systems may exhibit steady-state errors and that limit cycling never occurs. The steady-state error is bounded by the size of the equilibrium set, which can be influenced by tuning the proportional gain of the PD controller.

It also has been proven that overcompensation of friction in the same controlled mechanical systems provokes limit cycling in case the linearized dynamics of the controlled systems around the extremal equilibrium points are undercritically damped. However, such a limit cycling effect disappears if the PD controller gains are tuned such that the linearized dynamics become supercritically damped. Since the analysis involves the linearized dynamics around the extremal equilibrium points, this result is valid only for discontinuous friction models whose the first partial derivative of the Stribeck function is well-defined locally at zero velocity. The predictions of the theoretical results have been demonstrated by a numerical example. Furthermore, possible extensions of the results to MDOF frictional systems are also indicated.

References

[1] S. Adly and D. Goeleven. A stability theory for second-order nonsmooth dynamical systems with application to friction problems. *Journal de Mathematiques Pures et Appliquees*, 83:17–51, 2004.

[2] J. Alvarez, I. Orlov, and L. Acho. An invariant principle for discontinuous dynamic systems with application to a coulomb friction oscillator. *ASME Journal of Dynamic Systems, Measurement, and Control*, 122:687–690, 2000.

[3] B. Armstrong-Hélouvry and B. Amin. PID control in the presence of static friction: A comparison of algebraic and describing function analysis. *Automatica*, 32(5):679–692, 1996.

[4] B. Armstrong-Hélouvry, P. Dupont, and C. Canudas de Wit. A survey of models, analysis tools, and compensation methods for the control of machines with friction. *Automatica*, 30(7):1083–1138, 1994.

[5] J. P Aubin and A. Cellina. *Differential Inclusions*. Springer-Verlag, 1984.

[6] C. Canudas de Wit. Robust control for servo-mechanisms under inexact friction compensation. *Automatica*, 29(3):757–761, 1993.

[7] C. Canudas de Wit, H. Olsson, K. J. Åström, and P. Lischinsky. A new model for control of systems with friction. *IEEE Transaction on Automatic Control*, 40(3):419–425, 1995.

[8] A. F. Filippov. *Differential equations with discontinuous right-hand sides*. Kluwer Academic Publishers, 1988.

[9] R. I. Leine and H. Nijmeijer. *Dynamics and Bifurcations of Non-smooth Mechanical Systems*. Springer, 2004.

[10] R. I. Leine, D. H. van Campen, A. de Kraker, and L. van den Steen. Stick-slip vibrations induced by alternate friction models. *Nonlinear Dynamics*, 16:41–54, 1998.

[11] N. Mallon, N. van de Wouw, D. Putra, and H. Nij-meijer. Friction compensation in a controlled one-link robot using a reduced-order observer. *IEEE Transaction on Control Systems Technology*, 14(2):374–383, 2006.

[12] H. Olsson, K. J. Åström, C. Canudas de Wit, M. Gäfvert, and P. Lischinsky. Friction models and friction compensations. *European Journal of Control*, 4(3):176–195, 1998.

[13] E. G. Papadopoulos and G. C. Chasparis. Analysis and model-based control of servomechanism with friction. In *Proceedings of the 2002 IEEE/RSJ Int. Conf. on Intelligent Robots and Systems*, pages 2109–2114, Lausanne, Switzerland, 2002.

[14] D. Putra and H. Nijmeijer. Limit cycling in an observer-based controlled system with friction: Numerical analysis and experimental validation. *International Journal of Bifurcation and Chaos*, 14(9):3083–3093, 2004.

[15] D. Shevitz and B. Paden. Lyapunov stability theory of nonsmooth systems. *IEEE Transactions on Automatic Control*, 39(9):1910–1914, 1994.

[16] N. Van de Wouw and R. I. Leine. Attractivity of equilibrium sets of systems with dry friction. *Nonlinear Dynamics*, 35:19–39, 2004.