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Convergent systems and the output regulation problem

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Abstract

The notion of convergent systems is revisited from a control perspective. Sufficient conditions for a system to be convergent in a set are given. It is shown that a nonlinear system with an asymptotically stable linearization at the origin is convergent in some neighborhood of the origin. These results are applied to the output regulation problem. Based on the convergence property, a procedure for estimating the set of admissible initial conditions for a solution to the local output regulation problem is presented. An application of the procedure is illustrated by an example.

1 Introduction

In this paper, we consider the problem of asymptotic regulation of the output of a dynamical system, which is subject to disturbances generated by an external system. This problem is known as the output regulation problem. Many problems in control theory can be considered as particular cases of this problem: tracking of a class of reference signals, rejecting a class of disturbances, stabilization, partial stabilization or controlled synchronization. For nonlinear systems, a complete solution to the *local* output regulation problem was given in [1]. In that work, necessary and sufficient conditions for the solvability of the problem in some neighborhood of the origin were obtained and a procedure for designing a controller, which solves the problem, was presented. The paper was followed by a number of improvements concerning different aspects of the output regulation problem for nonlinear systems: regulation in presence of uncertainties, approximate, semiglobal output regulation (see [2], [3] and the references therein). At the same time, one problem regarding the *local* output regulation problem remained open: given a controller solving the problem in *some* neighborhood of the origin, how to determine (or estimate) this neighborhood of admissible initial conditions? Without answering this question, the solution to the local output regulation problem may not be satisfactory from an engineering point of view.

An answer to the above question can be found using the so-called convergence property of dynamical systems. Roughly, a convergent system is a system, which, being excited by a bounded signal, has a unique asymptotically stable bounded response. The concept of convergent systems was introduced by V.A. Pliss and then generalized by B.P. Demidovich [4]. In [4], a simple sufficient condition for convergence of a general nonlinear system was presented. For systems with nonlinearities subject to a sector-bounded growth condition, sufficient conditions for convergence, based on absolute stability theory, were obtained in [5]. Convergence proved to be useful for the problem of controlled synchronization of oscillatory systems [6] and in the analysis of cooperative oscillatory behavior of mutually coupled dynamical systems [7]. Since controlled synchronization can be considered as a particular case of the output regulation problem, the last one may also gain from using the convergence property.

The paper is organized as follows. In Section 2, the definition of convergent systems and sufficient conditions for convergence are given. The output regulation problem is formulated in Section 3. In Section 4, we demonstrate an application of the results on convergence to the output regulation problem and give a procedure for estimating the set of admissible initial conditions for a solution to the local output regulation problem. This procedure is illustrated by an example in Section 5. Conclusions are contained in Section 6.
The notations used in the paper are the following. $\mathcal{A}^T$ is the transposed matrix $\mathcal{A}$. The norm of a vector is denoted $|x| = (x^T x)^{1/2}$. For a positive definite matrix $P = P^T > 0$ the ellipsoid $E_P(\mathcal{R})$ is defined by $E_P(\mathcal{R}) = \{x \in \mathbb{R}^n : x^T P x < \mathcal{R}^2\}$. For $\mathcal{R} = \infty$ we define $E_P(\infty) = \mathbb{R}^n$. An open ball is denoted $B_w(r) = \{w : |w| < r\}$. $\|P\|$ denotes the operator norm of the matrix P induced by the vector norm. By I we denote the identity matrix. The largest eigenvalue of a symmetric matrix $J = J^T$ is denoted as $\Lambda(J)$. $C^{1,0}_{x,t}$ is the class of functions $f(x,t)$, which are continuously differentiable with respect to x and continuous with respect to t. The Jacobian matrix of $f(x,t)$ with respect to x is denoted $\mathcal{D}f_x(x,t)$.

2 Convergent systems

Following Demidovich [4], we give the following definition of convergent systems (slightly more general than in [4]):

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Definition 1 *The system*

$$\dot{x} = f(x,t), \quad f \in C^{1,0}_{x,t}, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \qquad (1)$$

has the convergence property in a set $\mathcal{E}$ if
1. All solutions $x(t,x_0,t_0)$ starting in $(x_0,t_0) \in \mathcal{E} \times \mathbb{R}$ are defined for all $t_0 \le t < \infty$ and do not leave $\mathcal{E}$.
2. In the set $\mathcal{E}$ there exists a unique bounded solution $\bar{x}(t)$, defined for $t \in \mathbb{R}$, i.e. $\bar{x}(t) \in \mathcal{E}$, $t \in \mathbb{R}$, $\sup_{t\in\mathbb{R}} |\bar{x}(t)| < \infty$.
3. The solution $\bar{x}(t)$ is asymptotically stable in $\mathcal{E}$ for $t \to +\infty$, i.e. $\bar{x}(t)$ is stable and for any solution $x(t,x_0,t_0)$ starting in $(x_0,t_0) \in \mathcal{E} \times \mathbb{R}$ the following relation holds:

$$\lim_{t\to+\infty} |x(t,t_0,x_0) - \bar{x}(t)| = 0.$$

If, additionally, it holds that for any $(x_0,t_0) \in \mathcal{E} \times \mathbb{R}$

$$|x(t,x_0,t_0) - \bar{x}(t)| \le C|x_0 - \bar{x}(t_0)|e^{-\alpha(t-t_0)}$$

for some $\alpha > 0$, $C > 0$, then we say that system (1) has the exponential convergence property in the set $\mathcal{E}$. If C and α do not depend on (x_0,t_0) then we call it uniform exponential convergence.

Remark 1. It can be easily checked that convergence in $\mathcal{E}$ is preserved under a smooth coordinate transformation $y = \psi(x)$, such that ψ is Lipschitz in $\mathcal{E}$. That means that if system (1) is convergent (exponentially or uniformly exponentially convergent) in the set $\mathcal{E}$ then the transformed system is convergent (exponentially or uniformly exponentially convergent) in $\psi(\mathcal{E})$.

The following theorem states sufficient conditions for a nonlinear system to be uniformly exponentially convergent in an ellipsoid (or in $\mathcal{E} = \mathbb{R}^n$).

Theorem 1 *Consider the system (1). Let*

$$J(x,t) = \frac{1}{2}\left(P\mathcal{D}f_x(x,t) + \mathcal{D}f_x{}^T(x,t)P\right),$$

where $P = P^T > 0$ is some positive definite matrix. Suppose, for some $\mathcal{R} > 0$, $\mathcal{R} \le \infty$ and $\alpha > 0$, the following conditions are satisfied

$$\sup_{t\in\mathbb{R},\ x\in E_P(\mathcal{R})} \Lambda(J(x,t)) \le -\alpha < 0, \qquad (2)$$

$$k := \sup_{t\in\mathbb{R}} |f(0,t)| < \alpha\mathcal{R}\|P\|^{-3/2}. \qquad (3)$$

Then, system (1) is uniformly exponentially convergent in $E_P(\mathcal{R})$.

The proof of this theorem is given in the Appendix. For $\mathcal{R} = \infty$ and $P = I$ the statement of the theorem was proved in [4]. For a linear system with inputs $\dot{x} = Ax + Bw(t)$ with Hurwitz A and bounded $w(t)$ the conditions of the theorem are satisfied for $\mathcal{R} = \infty$ and any $P > 0$ such that $PA + A^TP < 0$. Thus, for a given $w(t)$ such system is uniformly exponentially convergent in $\mathbb{R}^n$ and the unique limit solution $\bar{x}_w(t)$ is determined by the input $w(t)$. It is natural to expect that a nonlinear system $\dot{x} = f(x,w(t))$ with asymptotically stable linearization at $(x,w) = (0,0)$ and a small input $w(t)$ is locally (in some neighborhood of the origin) exponentially convergent and its limit solution $\bar{x}_w(t)$ is determined by the input $w(t)$. This is stated in the next assertion.

Corollary 1 *Consider the system with inputs*

$$\dot{x} = f(x,w), \quad x \in \mathbb{R}^n, \ w \in \mathbb{R}^m, \ f \in C^{1,0}_{x,w} \qquad (4)$$

such that $f(0,0) = 0$ and $\mathcal{D}f_x(0,0)$ is Hurwitz. Then, for any continuous input $w(t)$ defined for all $t \in \mathbb{R}$ and such that $\sup_{t\in\mathbb{R}} |w(t)|$ is small enough, system (4) is uniformly exponentially convergent in some neighborhood of the origin.

Proof of Corollary 1: Since $\mathcal{D}f_x(0,0)$ is Hurwitz, then there exists a positive definite matrix $P = P^T > 0$ such that

$$\frac{1}{2}(P\mathcal{D}f_x(0,0) + \mathcal{D}f_x{}^T(0,0)P) =: -Q < 0. \qquad (5)$$

Denote $J(x,w) = 1/2(P\mathcal{D}f_x(x,w) + \mathcal{D}f_x{}^T(x,w)P)$. Since $\mathcal{D}f_x(x,w)$ is continuous, then $J(x,w)$ and $\Lambda(J(x,w))$ are also continuous. Since $\Lambda(J(0,0)) = \Lambda(-Q) < 0$ and $f(0,0) = 0$, then, due to continuity of $\Lambda(J(x,w))$ and $f(x,w)$, we can find such $\mathcal{R} > 0$, $\delta > 0$ that the following inequalities hold:

$$\sup_{|w|<\delta,\ x\in E_P(\mathcal{R})} \Lambda(J(x,w)) = -\alpha < 0, \qquad (6)$$

$$\sup_{|w|<\delta} |f(0,w)| < \alpha\mathcal{R}\|P\|^{-3/2}. \qquad (7)$$

Thus, for system (4) with $w(t)$ satisfying $\sup_{t\in\mathbb{R}} |w(t)| < \delta$ both conditions of Theorem 1 are satisfied for $P > 0$ and $\mathcal{R} > 0$ found above. By Theorem 1, system (4) with $w(t)$ satisfying $\sup_{t\in\mathbb{R}} |w(t)| < \delta$ is uniformly exponentially convergent in $E_P(\mathcal{R})$.□

3 The output regulation problem

Following [1], we consider systems modelled by equations of the form

$$\dot{x} = f(x,u,w), \qquad (8)$$
$$e = h(x,w), \qquad (9)$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, regulated output $e \in \mathbb{R}^l$ and exogenous input $w \in R^p$ generated by the exosystem

$$\dot{w} = s(w). \qquad (10)$$

The exogenous signal $w(t)$ can be viewed as a disturbance in equation (8) or as a reference signal in (9). It is assumed that $f(0,0,0) = 0$, $h(0,0) = 0$, $s(0) = 0$; functions f, h, s are C^k functions for some large k. Denote $A = \mathcal{D}f_x(0,0,0)$, $B = \mathcal{D}f_u(0,0,0)$. We assume that exosystem (10) is *neutrally stable*. Neutral stability means that the equilibrium $w = 0$ is Lyapunov stable in forward and backward time [3]. An important representative of neutrally stable exosystems is a linear harmonic oscillator.

The state-feedback local output regulation problem is formulated in the following way. Given a nonlinear system of the form (8), (9) and a neutrally stable exosystem (10), find, if possible, a mapping $\beta(x,w)$, $\beta(0,0) = 0$, such that

A) The system

$$\dot{x} = f(x, \beta(x,0), 0) \tag{11}$$

has an asymptotically stable linearization at $x = 0$,

B) There exists a neighborhood $X \times W$ of $(0,0)$ such that for each initial condition $(x(0), w(0)) \in X \times W$ the solution of

$$\begin{aligned} \dot{x} &= f(x, \beta(x,w), w), & (12)\\ \dot{w} &= s(w) & (13) \end{aligned}$$

satisfies $e(t) = h(x(t), w(t)) \to 0$ as $t \to \infty$.

Conditions for the solvability of this problem are given by the following theorem.

Theorem 2 [1] *The state-feedback local output regulation problem is solvable if and only if the pair (A,B) is stabilizable and there exist mappings $x = \pi(w)$ and $u = c(w)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood W_0 of the origin, satisfying the conditions*

$$\begin{aligned} \frac{\partial \pi}{\partial w}(w) s(w) &= f(\pi(w), c(w), w),\\ 0 &= h(\pi(w), w) \end{aligned} \tag{14}$$

for all $w \in W_0$. A controller solving the problem is given by

$$u = \beta(x,w) = c(w) + K(x - \pi(w)), \tag{15}$$

where K is such that $A + BK$ is Hurwitz.

Remark 2. Under conditions of Theorem 2, a controller $u = \beta(x,w)$ solves the local output regulation problem if and only if

$$\beta(\pi(w), w) = c(w) \tag{16}$$

and $A + B\mathcal{D}\beta_x(0,0)$ is a Hurwitz matrix.

A controller resulting from Theorem 2 solves the output regulation problem for initial conditions in some neighborhood $X \times W$ of the origin. From an engineering point of view, such solution may not be satisfactory, since this region of admissible initial conditions $X \times W$ is not specified. Thus, once a controller solving the local output regulation problem is found, there is a need to estimate this region.

4 Estimates of $X \times W$

In this section, we give a procedure for estimating the set $X \times W$ for system (8) in closed-loop with a controller $u = \beta(x,w)$ solving the local output regulation problem. For convenience, the right-hand side of the closed-loop system is denoted $F(x,w)$:

$$\dot{x} = f(x, \beta(x,w), w) =: F(x,w). \tag{17}$$

Note, that due to condition **A)** the Jacobian matrix $\mathcal{D}F_x(0,0)$ is Hurwitz.

Procedure 1 *(Estimation of $X \times W$)*

1) *Find a positive definite matrix P such that*

$$P\mathcal{D}F_x(0,0) + \mathcal{D}F_x^T(0,0)P < 0.$$

Such P exists, because $\mathcal{D}F_x(0,0)$ is Hurwitz.

2) *Find $\delta > 0$, $\mathcal{R} > 0$ such that the following inequalities are satisfied for some $\alpha > 0$:*

$$\sup_{|w|<\delta,\ x \in E_P(\mathcal{R})} \Lambda(J(x,w)) \le -\alpha < 0 \tag{18}$$

$$\sup_{|w|<\delta} |F(0,w)| < \alpha \mathcal{R} \|P\|^{-3/2}, \tag{19}$$

where $J(x,w) = 1/2(P\mathcal{D}F_x(x,w) + \mathcal{D}{F_x}^T(x,w)P)$. Such δ and $\mathcal{R}$ exist by Corollary 1.

3) *Find $r > 0$ such that if $w_0 \in B_w(r)$ then the solution of the exosystem (10) with $w(0) = w_0$ satisfies $w(t) \in \mathcal{W}_0$, $|w(t)| < \delta$ and $\pi(w(t)) \in E_P(\mathcal{R})$ for all $t \in \mathbb{R}$. Such r exists due to neutral stability of the exosystem and continuity of $\pi(w)$.*

Then, $E_P(\mathcal{R}) \times B_w(r)$ is an estimate of the set $X \times W$. Moreover, for any solution of (17), (10) starting in $(x_0, w_0) \in E_P(\mathcal{R}) \times B_w(r)$ the regulated output $e(t) = h(x(t), w(t))$ exponentially converges to zero.

Proof of the procedure: Let $w(t)$ be a solution of (10) such that $w(0) \in B_w(r)$. It follows from (14) and (16) that $\bar{x}_w(t) = \pi(w(t))$ is a solution of (17). Due to the choice of r in step **3)**, the solution $\bar{x}_w(t)$ is bounded and lies in $E_P(\mathcal{R})$ for all $t \in \mathbb{R}$. By Theorem 1, conditions (18) and (19) guarantee, that system (17) is exponentially convergent in $E_P(\mathcal{R})$. Due to convergence, any solution of (17) starting in $E_P(\mathcal{R})$ exponentially tends to $\bar{x}_w(t) = \pi(w(t))$. Thus, $e(t) = h(x(t), w(t)) \to h(\pi(w(t)), w(t)) = 0$ as $t \to \infty$ and the convergence is exponential. Hence, $E_P(\mathcal{R}) \times B_w(r)$ is an estimate of the set $X \times W$.□

The matrix inequality in step **1)** admits multiple positive definite solutions P. At the moment it is an open question how to choose P in order to obtain the best (in some sense) estimate of $X \times W$.

The estimates resulting from the procedure are conservative, since they contain only the initial conditions for which the regulated output $e(t)$ tends to zero exponentially. In the formulation of the output regulation problem the rate of convergence is not specified. Thus, the set $X \times W$ may also contain initial states for which the convergence of $e(t)$ to zero is not exponential.

The proposed procedure can be applied to system (8) in closed loop with any controller solving the local output regulation problem. In particular, it can be applied to system (8) in closed loop with the controller (15). Actually, for the case of controller (15), this forms an alternative proof of the "if" part of Theorem 2. The original proof of Theorem 2 ([1]) is based on center manifold theory. The proof presented above is based on the convergence property. This new approach allows to find estimates of the region of admissible initial conditions $X \times W$.

The procedure and its proof also show how to apply the results on convergence to the output regulation problem: first, ensure that for every solution of the exosystem $w(t)$, the closed-loop system (17) is convergent in some set $\mathcal{E}$ and then show that in the set $\mathcal{E}$ there exists a bounded trajectory $\bar{x}_w(t)$ defined for all $t \in \mathbb{R}$, on which the regulated output is zero: $h(\bar{x}_w(t), w(t)) \equiv 0$. Then, due to convergence, any other solution of (17) starting in $\mathcal{E}$ will tend to $\bar{x}_w(t)$ and, hence, $e(t) = h(x(t), w(t)) \to h(\bar{x}_w(t), w(t)) = 0$ as $t \to \infty$.

Procedure 1 provides estimates of $X \times W$ for system (8) with a static controller solving the *state-feedback* local output regulation problem. It can be easily updated for the *error-feedback* case, in which a controller, solving the problem, incorporates a dynamic feedback.

5 Example

Consider the controlled Van der Pol oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1-x_1^2)x_2 - x_1 + u \end{aligned} \qquad (20)$$

and the exosystem

$$\begin{aligned} \dot{w}_1 &= \Omega w_2 \\ \dot{w}_2 &= -\Omega w_1. \end{aligned} \qquad (21)$$

It can be easily checked that by Theorem 2 the feedforward controller $u = (1-\Omega^2)w_1 + \Omega w_2(1-w_1^2)$, solves the local output regulation problem for the regulated output $e = x_1 - w_1$. The corresponding solutions to the regulator equations are given by $\pi_1(w) = w_1$, $\pi_2(w) = \Omega w_2$ and $c(w) = (1-\Omega^2)w_1 + \Omega w_2(1-w_1^2)$. The linearized system (20) is asymptotically stable and thus a stabilizing feedback term is not required. Let us estimate the set $X \times W$ for the closed-loop system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1-x_1^2)x_2 - x_1 + c(w) \end{aligned}\right. =: F(x,w). \qquad (22)$$

Following Procedure 1, we first pick a positive definite solution to the matrix inequality $PA + A^TP < 0$: $P = (4, 1;\ 1, 3)$. Second, we find the maximal $\mathcal{R}$ and r such that

a) $\sup_{x \in E_P(\mathcal{R})\ |w|<r} \Lambda(J(x,w)) =: -\alpha(\mathcal{R}) < 0$, where $J(x,w) = 1/2(P\mathcal{D}F_x(x,w) + \mathcal{D}F_x^T(x,w)P)$,

b) $\sup_{|w|<r} |c(w)| < \alpha(\mathcal{R})\mathcal{R}\|P\|^{-3/2}$,

c) $(w_1, \Omega w_2)P(w_1, \Omega w_2)^T < \mathcal{R}^2$ for $|w| < r$.

In condition **a)** the supremum $-\alpha(\mathcal{R})$ does not depend on r, because in our case $J(x,w)$ depends only on x. Condition **c)** is the only condition we need to check in step **3)** of Procedure 1. This is due to the fact that the solutions to the regulator equations are globally defined ($\mathcal{W}_0 = \mathbb{R}^2$) and that $|w(t)|$ remains constant along solutions of the exosystem.

Inequalities **a)**, **b)** and **c)** are solved semi-analytically (for $\Omega = 1$) resulting in the family of estimates $E_P(\mathcal{R}(r)) \times B_w(r)$ with $\mathcal{R}(r)$ shown in Fig. 1. According to simulations of systems (22) and (21), the

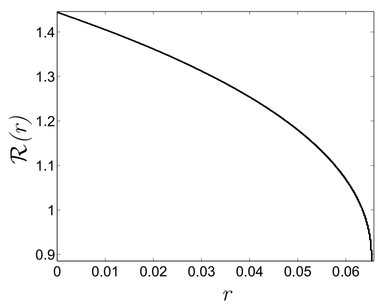

Fig.1 $\mathcal{R}(r)$ for the estimates $E_P(\mathcal{R}(r)) \times B_w(r)$.

obtained estimates are rather conservative. It appears that output regulation still occurs if, for given values of $\mathcal{R}$, the magnitude r of the reference signal is approximately 10 times larger than shown in Fig. 1. There may be several reasons for such conservativeness. The first one is (possibly) the bad choice of the matrix P. Different P's may result in a better estimate. Another reason is that the procedure gives only an estimate of initial conditions, for which the tracking error $e(t)$ tends to zero exponentially, while not necessarily exponential tracking can occur for a much larger set of initial conditions. These facts indicate that further improvements of the procedure are still possible.

6 Conclusions

In this paper, we have revisited the notion of convergent systems from the perspective of its possible application to control problems. It has been shown that the results on the convergence property obtained by Demidovich and extended in this paper can be applied to the output regulation problem. The approach to the output regulation problem based on the convergence property is an alternative to the original approach from [1] based on center manifold theory and it may have some advantages. In particular, within this new approach, a procedure for estimating the set of admissible initial conditions for a solution to the local output regulation problem has been proposed. Without such estimates the solution to the local output regulation problem may not be satisfactory from an engineering point of view. An application of the algorithm has been demonstrated by an example. Further investigation is needed to improve the procedure in order to make the estimates less conservative. Since a lot of control problems can be viewed as variants of the output regulation problem, the notion of convergent systems has great potential in application to control.

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Appendix: Proof of Theorem 1

We first prove the theorem for $P = I$ and then extend the result to the case of general $P > 0$. Actually, it is possible to skip the first step and prove the statement right away for general P. But in that case we would have to reformulate and prove the two following technical lemmas, which for $P = I$ were proved in [4]:

Lemma 1 *Consider an $n-$dimensional vector-function $f(x) \in C^1$. Let $\Lambda(J(x))$ and $\lambda(J(x))$ be the highest and the lowest eigenvalues of*

$$J(x) = \frac{1}{2}\left(\mathcal{D}f_x(x) + \mathcal{D}{f_x}^T(x)\right),$$

respectively. Then, for any $h \in \mathbb{R}^n$

$$\lambda_m |h|^2 \le h^T(f(x+h) - f(x)) \le \Lambda_m |h|^2,$$

where

$$\lambda_m = \inf_{\xi\in[0,1]} \lambda(J(x+\xi h)), \quad \Lambda_m = \sup_{\xi\in[0,1]} \Lambda(J(x+\xi h)).$$

Lemma 2 *Consider the system (1). Suppose*

$$\frac{d}{dt}|x(t)|^2 = 2x^T f(x,t) < 0, \quad \forall t \in \mathbb{R}, \quad |x| = \mathcal{R}.$$

Then there exists at least one solution $\bar{x}(t)$ defined for all $t \in \mathbb{R}$ and such that $|\bar{x}(t)| < \mathcal{R}$, $\forall t \in \mathbb{R}$.

The next lemma proves the assertion of Theorem 1 for the case $P = I$. In the proof we essentially use the ideas of B.P. Demidovich [4].

Lemma 3 *Consider system (1). Let*

$$\tilde{J}(x,t) = \frac{1}{2}\left(\mathcal{D}f_x(x,t) + \mathcal{D}{f_x}^T(x,t)\right).$$

Suppose, for some $\mathcal{R} > 0$ and $\mathcal{R} \le \infty$, the maximal eigenvalue of $\tilde{J}(x,t)$ satisfies

$$\sup_{t\in\mathbb{R},\ |x|<\mathcal{R}} \Lambda(\tilde{J}(x,t)) \le -\tilde{\alpha} < 0 \tag{23}$$

and

$$\tilde{k} := \sup_{t\in\mathbb{R}} |f(0,t)| < \tilde{\alpha}\mathcal{R}. \tag{24}$$

Then, system (1) is uniformly exponentially convergent in $\mathcal{E} = \{x : |x| < \mathcal{R}\}$.

Proof of Lemma 3: For $\mathcal{R} = \infty$ the lemma was proved in [4]. Let us prove it for $\mathcal{R} < \infty$. Denote $V(x) = 1/2|x|^2$. Consider $dV/dt = x^T f(x,t)$ for $|x| = \mathcal{R}$:

$$\begin{aligned}
\frac{dV}{dt}(x,t) &= x^T(f(x,t) - f(0,t)) + x^T f(0,t) \\
&\le -\tilde{\alpha}|x|^2 + |x^T f(0,t)| \\
&\le -\tilde{\alpha}|x|^2 + \tilde{k}|x| = |x|(-\tilde{\alpha}|x| + \tilde{k})\big|_{|x|=\mathcal{R}} \\
&= \mathcal{R}(-\tilde{\alpha}\mathcal{R} + \tilde{k}) < 0.
\end{aligned}$$

In the first inequality we use Lemma 1 and condition (23), in the second – the Cauchy inequality and the definition of $\tilde{k}$ (see (24)) and in the last one – condition (24). Thus, the condition of Lemma 2 is satisfied and by this lemma there exists a solution $\bar{x}(t)$ defined for all $t \in \mathbb{R}$ and such that $|\bar{x}(t)| < \mathcal{R}$, $\forall t \in \mathbb{R}$. Moreover, the set $|x| < \mathcal{R}$ is invariant because $d/dt|x(t)|^2 < 0$ for $|x| = \mathcal{R}$. Let us show asymptotic stability of $\bar{x}(t)$. Let $x(t)$ be another solution of (1) starting in (x_0, t_0) such that $|x_0| < \mathcal{R}$. Consider the difference $\epsilon(t) = x(t) - \bar{x}(t)$. Let $\mathcal{V}(\epsilon) = 1/2|\epsilon|^2$. Since $d\epsilon/dt = f(x,t) - f(\bar{x},t)$, then

$$\begin{aligned} \frac{d\mathcal{V}}{dt} &= \epsilon^T(f(x,t) - f(\bar{x},t))\big|_{\text{Lemma 1}} \\ &\leq \sup_{\xi\in[0,1],\ t\in\mathbb{R}} \Lambda(\tilde{J}(\bar{x} + \xi(x - \bar{x}), t))|\epsilon|^2. \end{aligned}$$

The solutions $x(t)$ and $\bar{x}(t)$ belong to the convex set $|x| < \mathcal{R}$. Hence, $\bar{x}(t) + \xi(x(t) - \bar{x}(t))$ belongs to this set for any $\xi \in [0,1]$. Thus, it follows from condition (23) that

$$\begin{aligned} &\sup_{\xi\in[0,1], t\in\mathbb{R}} \Lambda(\tilde{J}(\bar{x}(t) + \xi(x(t) - \bar{x}(t)), t)) \\ &\leq \sup_{|x|<\mathcal{R}, t\in\mathbb{R}} \Lambda(\tilde{J}(x,t)) \leq -\tilde{\alpha} < 0. \end{aligned}$$

This implies $d\mathcal{V}/dt \leq -2\tilde{\alpha}\mathcal{V}$ and finally

$$|x(t) - \bar{x}(t)| \leq |x(t_0) - \bar{x}(t_0)|e^{-\tilde{\alpha}(t-t_0)}. \qquad (25)$$

It follows from (25) that any other solution $\tilde{x}(t)$ lying in the ball $|x| < \mathcal{R}$ for all $t \in \mathbb{R}$ satisfies

$$|\tilde{x}(t) - \bar{x}(t)| \leq |\tilde{x}(t_0) - \bar{x}(t_0)|e^{-\tilde{\alpha}(t-t_0)} \leq 2\mathcal{R}e^{-\tilde{\alpha}(t-t_0)}.$$

In the limit for $t_0 \to -\infty$ we obtain $|\tilde{x}(t) - \bar{x}(t)| \leq 0$. Hence, $\bar{x}(t) \equiv \tilde{x}(t)$ and the solution $\bar{x}(t)$ lying for all $t \in \mathbb{R}$ in the ball $|x| < \mathcal{R}$ is unique.□

Prior to proving the case of general $P > 0$, we introduce some notations and formulate one more technical lemma. The numbers $\overline{\sigma}(A)$ and $\underline{\sigma}(A)$ denote the largest and the lowest singular values of A, respectively. If S is an invertible matrix then S^{-T} denotes $(S^{-1})^T$.

Lemma 4 *Consider a matrix $A = A^T$ such that its maximal eigenvalue satisfies $\Lambda(A) \leq -\alpha < 0$. Let $S = P^{1/2}$, where P is a positive definite matrix. Then, the maximal eigenvalue of $B = S^{-T}AS^{-1}$ satisfies*

$$\Lambda(B) \leq -\frac{\alpha}{\|P\|}. \qquad (26)$$

Proof of Lemma 4: It is known from linear algebra that for any invertible matrix S the number of positive, negative and zero eigenvalues of the symmetric matrices A and $S^{-T}AS^{-1}$ are the same [8]. Since all the eigenvalues of A are negative, then all the eigenvalues of B are also negative. Notice that the eigenvalues of B coincide with the eigenvalues of AP^{-1}. This fact follows from the following manipulations:

$$\begin{aligned} &\det(S^{-T}AS^{-1} - \lambda I) = 0 \Leftrightarrow \det(AS^{-1} - \lambda S^T) = 0 \\ &\Leftrightarrow \det(AS^{-1}S^{-T} - \lambda I) = \det(AP^{-1} - \lambda I) = 0. \end{aligned}$$

Furthermore, we use the following properties of eigenvalues and singular values, which can be found or easily derived from the found ones in [9]:

1) $\underline{\sigma}(\mathcal{A}) \leq |\lambda(\mathcal{A})| \leq \overline{\sigma}(\mathcal{A})$, where $\lambda(\mathcal{A})$ is any eigenvalue of $\mathcal{A}$,
2) if $\mathcal{A}^{-1}$ exists then $\underline{\sigma}(\mathcal{A}) = 1/\overline{\sigma}(\mathcal{A}^{-1})$,
3) $\overline{\sigma}(\mathcal{AB}) \leq \overline{\sigma}(\mathcal{A})\overline{\sigma}(\mathcal{B})$,
4) if $P = P^T > 0$ then $\overline{\sigma}(P) = \|P\|$,
5) if $A = A^T < 0$ then $\underline{\sigma}(A) = |\Lambda(A)|$.

Thus, $|\Lambda(B)| = |\Lambda(AP^{-1})| \geq \underline{\sigma}(AP^{-1}) =$

$$= \frac{1}{\overline{\sigma}(PA^{-1})} \geq \frac{1}{\overline{\sigma}(A^{-1})\overline{\sigma}(P)} = \frac{\underline{\sigma}(A)}{\overline{\sigma}(P)} = \frac{|\Lambda(A)|}{\|P\|}.$$

Finally,

$$\Lambda(B) = -|\Lambda(B)| \leq -\frac{|\Lambda(A)|}{\|P\|} \leq -\frac{\alpha}{\|P\|}.\square$$

To prove the case of general $P > 0$, we perform the coordinate transformation $y = Sx$, where $S = P^{1/2}$, and then apply Lemma 3. This trick was proposed in [7]. Let us check the conditions of the lemma. After the change of coordinates system (1) takes the form

$$\dot{y} = Sf(S^{-1}y, t) = \tilde{f}(y,t). \qquad (27)$$

The symmetrized Jacobian of $\tilde{f}(y,t)$, $\tilde{J}(y,t) = 1/2\left(\mathcal{D}\tilde{f}_y(y,t) + (\mathcal{D}\tilde{f}_y(y,t))^T\right)$, equals to $\tilde{J}(y,t) =$

$$\begin{aligned} &= S^{-T}\tfrac{1}{2}\left(P\mathcal{D}f_x(S^{-1}y,t) + \mathcal{D}f_x{}^T(S^{-1}y,t)P\right)S^{-1} \\ &= S^{-T}J(S^{-1}y,t)S^{-1}. \end{aligned}$$

Due to condition (2), the largest eigenvalue of $J(S^{-1}y,t)$ satisfies

$$\begin{aligned} \sup_{t\in\mathbb{R},\ |y|<\mathcal{R}} \Lambda(J(S^{-1}y,t)) &= \sup_{t\in\mathbb{R},\ |Sx|<\mathcal{R}} \Lambda(J(x,t)) \\ &= \sup_{t\in\mathbb{R},\ x\in E_P(\mathcal{R})} \Lambda(J(x,t)) \leq -\alpha < 0. \end{aligned}$$

Hence, by Lemma 4 the largest eigenvalue of $\tilde{J}(y,t)$ satisfies

$$\sup_{t\in\mathbb{R},\ |y|<\mathcal{R}} \Lambda(\tilde{J}(y,t)) \leq -\frac{\alpha}{\|P\|} =: -\tilde{\alpha} < 0.$$

Thus, condition (23) is satisfied. Condition (24) is also satisfied, because $\tilde{k} = \sup_{t\in\mathbb{R}}|\tilde{f}(0,t)|$

$$\begin{aligned} &= \sup_{t\in\mathbb{R}}|Sf(0,t)| \leq \|S\|\sup_{t\in\mathbb{R}}|f(0,t)| \\ &= \|P\|^{1/2}k \leq [\text{condition (3)}] \leq \alpha\mathcal{R}/\|P\| = \tilde{\alpha}\mathcal{R}. \end{aligned}$$

Hence, both conditions of Lemma 3 are satisfied. By this lemma, system (27) is uniformly exponentially convergent in $\tilde{\mathcal{E}} = \{y : |y| < \mathcal{R}\}$. Since the coordinate transformation $x = S^{-1}y$ is Lipschitz, then by Remark 1 the initial system (1) is also uniformly exponentially convergent in the set $\mathcal{E} = \{x : |Sx| < \mathcal{R}\} = E_P(\mathcal{R})$. This completes the proof of Theorem 1.□