Hybrid Systems With State-Triggered Jumps: Sensitivity-Based Stability Analysis With Application to Trajectory Tracking

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***Abstract*—The definition of asymptotic stability for a trajectory of a hybrid system with state-triggered jumps is not straightforward. Nearby solutions jump at close but noncoincident times, making the standard notion of closeness, based on vector difference, unsuitable to compare trajectories point-wise in time. With tracking control as ultimate goal, we propose a notion of stability and a constructive stability proof based on sensitivity analysis applicable to single-jump-flow trajectories. A key role in the analysis is played by a time-triggered linear system, associated with the discontinuous trajectory of interest, whose uniform asymptotic stability suffices to guarantee the asymptotic stability of the original discontinuous trajectory. As an illustrative example, the stability analysis is applied to guarantee closed-loop stable tracking for a trajectory with velocity jumps of a 2 DoF mechanical system with unilateral constraint.**

***Index Terms*—Hybrid systems, stability, sensitivity analysis, trajectory tracking, trajectories with jumps.**

I. Introduction

THIS article studies the problem of defining and assessing local asymptotic stability of a trajectory of a hybrid dynamical system. These systems show both continuous (flow) and discrete (jump) dynamics [1]. Our analysis concerns, in particular, hybrid systems with (time- and) state-triggered jumps, where the state trajectory becomes discontinuous under the effect of the discrete dynamics. We will refer to this class of systems as hybrid systems with state-triggered jumps.

Mechanical systems performing motions with hard impacts are necessarily modeled using the framework of nonsmooth mechanics [2], [3] in which complementarity conditions restrict the configuration space and at the same time, enforce feasibility of contact forces. In many cases of practical interest, e.g., for juggling and walking robots [4]–[6], the nonsmooth dynamics can locally be fitted in the hybrid system formalism [1], an approach that can also often be taken in trajectory tracking of mechanical systems with state-triggered jumps [7], [8]. Whereas the stabilization of jumping trajectories of a mechanical system with unilateral constraints is the main motivation for our investigation, the approach and obtained results are applicable to a larger class of hybrid systems and are, therefore, presented as such. More specifically, the proposed stability notion and sensitivity-based stability analysis concerns the specific type of trajectories termed *single-jump-flow* trajectories, characterized by continuous flow phases followed by single discrete jumps.

In earlier investigations [7], [9], [10], tracking problems for hybrid systems have been solved under the assumption that the jump times of the system and reference trajectory coincide. In that case, standard Lyapunov methods can be employed in terms of the classical Euclidean tracking error to perform stability analysis. The requirement that the jump times of the trajectories coincide with those of a reference trajectory is, however, stringent and this coincidence can generally not be assumed: this is not the case, in general, for hybrid systems with state-triggered jumps and, in particular, for hybrid systems that represent mechanical systems with unilateral constraints.

When reference and closed-loop jump times do not coincide, the Euclidean error between two trajectories shows a big increase whenever the system trajectory jumps and the reference trajectory does not or vice versa. This phenomenon is usually referred to as "peaking" [8], [11], [12]. A few approaches have been recently proposed in the literature to deal with the mismatch in the jump times by defining stability on the basis of a different notion of error/distance between single-jump-flow trajectories. In [13], the times belonging to the infinitesimal intervals around the jump times are neglected in defining the tracking problem. In [14], the reference trajectory together with a *mirrored* version of it has been used to construct a new error notion for the tracking problem of a ball in a polyhedral billiard. In [1, Sec. 5.3], the concept of graphical closeness of solutions is considered. In [8], [15], and [16], Biemond *et al.* proposed to simplify the stability analysis and tracking control design by suggesting to use a distance function between two trajectories that is invariant with respect to the discrete jump dynamics. In [17], Kim *et al.* used gluing functions to perform a state-transformation turning the hybrid dynamics into (piecewise) continuous dynamics, removing the state jumps. Morarescu and Brogliato [18]

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tackled the trajectory tracking problem for mechanical systems with frictionless unilateral constraints by adapting the reference motion online to enforce smooth transitions between free and constrained motion.

In this article, we employ the notion of error introduced in [19] to define and analyze asymptotic stability of discontinuous trajectories and to propose a possible solution to the problem of tracking a reference trajectory for hybrid systems with state-triggered jumps. The employed error notion is based on extending the (reference) trajectory about the jump times and considering the distance between the state of a trajectory and that at the particular segment of the extended (reference) trajectory that has encountered the same number of jumps. This error will not, in particular, show any peaking and presents the basis for an effective trajectory tracking control approach named *reference spreading control* [19]–[21].

For smooth nonlinear control systems, the open- and closed-loop local stability of a continuous reference trajectory can be assessed via its associated time-varying linearization. The key contribution of this article is to show that, for hybrid systems with *state-triggered* jumps, the local stability of a *discontinuous* single-jump-flow trajectory can be assessed via the study of its *time-triggered* linearization, a linear time-triggered hybrid system (LTTHS) that emerges from the sensitivity analysis originally developed in [19] and that is independent of the reference extensions. The reference trajectory is assumed to satisfy a set of assumptions (in particular, transversality and absence of Zeno behavior) ensuring continuous dependence of impact times (and the solution away from impact times) with respect to variations of initial conditions and control inputs.

This article is organized as follows. In Section II, hybrid systems with state-triggered jumps are reviewed and the problem definition is precisely stated together with key regularity assumptions. Section III reviews the concept of extended reference trajectory, that leads to the error notion used to define stability of discontinuous trajectories. This section also presents the main result of this article: the ability to infer stability of a discontinuous trajectory of a hybrid system with state-triggered jumps by analysis of the stability of an associated time-triggered linear system (the hybrid linearization). Section IV applies the obtained results to a mechanical system with a unilateral constraint. Conclusion is presented in Section V.

II. Preliminaries and Problem Statement

A. Hybrid Systems

A hybrid dynamical system can be represented schematically as in Fig. 1(a). The system has a state $x \in \mathbb{R}^n$ that continuously evolves according to a control vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, parameterized by the external input $u \in \mathbb{R}^m$. Continuous evolution is only possible when the state x at a given time is in a closed set $C \subseteq \mathbb{R}^n$ called the *flow set*. Explicitly, the state evolution satisfies the differential equation

$$\dot{x} = f(x, u), \quad x \in C. \tag{1a}$$

A *jump* in the state can occur whenever the state reaches a set $D \subseteq \mathbb{R}^n$, called the *jump set*. A jump implies an instantaneous state change according to the jump map $g : \mathbb{R}^n \to \mathbb{R}^n$.

In this article, we consider unique solutions of hybrid systems. Aside from some basic regularity assumptions on the flow map f, uniqueness requires that whenever a jump occurs ($x \in D$)

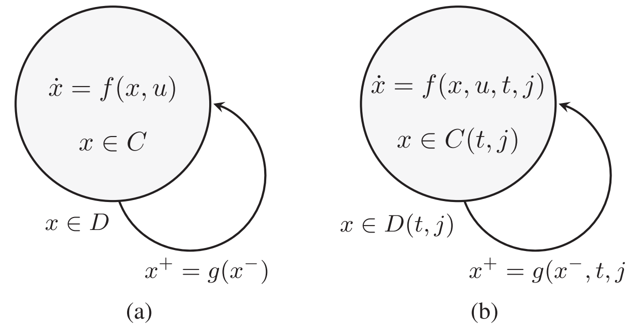

Fig. 1. Hybrid system with one mode of execution without (a) and with (b) explicit dependency on hybrid time (t, j).

evolving in C is also no longer possible [1, Prop. 2.11]. A way to enforce this is to assume that $D \subseteq \partial C$, with ∂C denoting the boundary of C, together with some transversality assumptions on the continuous flow to avoid that the flow is tangent to D when it reaches it (grazing). Transversality (cf. [22]) plays an important role in this article and will be discussed in more detail in the problem formulation in Section II-B.

We largely adopt the hybrid system notation from [1]. In particular, we employ the notion of *hybrid time*, which merges regular time $t \in \mathbb{R}$ with discrete time $j \in \mathbb{N}$. Discrete time should be thought of as a jump counter, indicating how many times the state has jumped, so that the state jumps satisfy

$$x(t, j+1) = g(x(t, j)), \qquad x(t, j) \in D. \tag{1b}$$

All these notions are standard and we refer to [1, Def. 2.6] for the definition of a solution to (1a), (1b).

To allow for time-varying vector fields, jump maps, and time-varying flow and jump sets, the hybrid dynamical systems that we consider in this article are (with slight abuse of notation) written as follows:

$$\dot{x} = f(x, u, t, j), \qquad x \in C(t, j) \tag{2a}$$

$$x^+ = g(x^-, t, j), \qquad x^- \in D(t, j) \tag{2b}$$

where $x^+ := x(t, j+1)$ and $x^- := x(t, j)$. We refer to (2) as a *nonlinear state-triggered hybrid system (NSTHS)* and represent it as shown in Fig. 1(b).

Remark 1: In (2), one could define a new state (x, t, j) showing that (2) is just a special case of (1). However, we found that keeping the hybrid time (t, j) explicit leads to a more intuitive understanding of the stability analysis and reference spreading control. In Section III, we show that the control law is of the form $u = u(x, t, j)$, depending, therefore, explicitly on the continuous flow state x and hybrid time (t, j).

Another reason to keep (t, j) explicitly in (2) is that otherwise the proposed definition of discontinuous trajectory stability would require to treat (t, j) differently than the other part of the state (otherwise, when time would be included in the state, a state perturbation would also perturb time). △

In Section II-B and Appendix B, we will use the sets

$$C_j := \bigcup_{t \in \mathbb{R}} C(t, j) \times \{t\} \tag{3}$$

$$D_j := \bigcup_{t \in \mathbb{R}} D(t, j) \times \{t\} \tag{4}$$

to ease the derivations of the results. Note that C_j and $D_j \subseteq \mathbb{R}^{n+1}$. For a given $j \in \mathbb{N}$, $C(t,j)$ and $D(t,j)$ can be interpreted as the "slices" of C_j and D_j at time t.

We adopt basic regularity assumptions for (2). To be precise, the flow map $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}^n$ is assumed to be locally Lipschitz with respect to the state x on the set $\mathbb{R}^n$ and input u in $\mathbb{R}^m$, continuous and bounded in t for each x and u and every fixed j. The jump map $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}^n$ is assumed to be continuous with respect to x and t.

For the NSTHS (2), one may aim at designing a control law to achieve tracking of a given reference trajectory. A generic time-varying state feedback to achieve this goal is

$$u = \kappa(x, t, j). \tag{5}$$

The *closed-loop* NSTHS (*cl*-NSTHS) resulting from substituting (5) into (2) is given by

$$\dot{x} = f_{\mathrm{cl}}(x, t, j), \qquad x \in C(t, j) \tag{6a}$$

$$x^+ = g(x^-, t, j), \qquad x^- \in D(t, j) \tag{6b}$$

where $f_{\mathrm{cl}}(x, t, j) := f(x, \kappa(x, t, j), t, j)$.

Suppose $x(t, j)$ is the solution to (6) for a given initial condition $x(t_0, 0) = x_0 \in \mathrm{int}\, C(t_0, 0)$ (int means interior). The hybrid domain of $x(t, j)$ is written as (cf. [1, Def. 2.3])

$$\mathrm{dom}\, x = \bigcup_{j=0}^{J_x-1} I_x^j \times \{j\}, \qquad J_x \in \mathbb{N} \cup \{\infty\} \tag{7}$$

with I_x^j the closed (continuous-time) interval between the jth and $(j+1)$th jump events and J_x the number of time intervals ($J_x = \infty$, when an infinite number of jumps occur). The first interval I_x^0 starts at t_0. If $J_x < \infty$, the last interval $I_x^{J_x-1}$ ends at $t_f \in \mathbb{R} \cup \{\infty\}$. The jth jump time is denoted t_j so that

$$I_x^j = [t_j, t_{j+1}] \tag{8}$$

and also

$$t_j = \min\ I_x^j. \tag{9}$$

The set of jump times associated with $x(t, j)$ is denoted

$$E_x := \bigcup_{j=1}^{J_x-1} \{t_j\} \times \{j-1\}. \tag{10}$$

We conclude this section by introducing several notational conventions used to indicate the flow map. We will use $\mathcal{H}$ to indicate the closed-loop hybrid system defined by the quadruple $(f_{\mathrm{cl}}, g, C, D)$, representing (6). Sometimes, we regard the solution to $\mathcal{H}$ starting from x_0 at hybrid time $(t_0, 0)$, just as a function of continuous time t instead of hybrid time $(t, j) \geq (t_0, 0)$. To this end, with a slight abuse of notation, we will write $x(t)$ or, more explicitly, $\phi_{\mathcal{H}}(t, t_0, x_0)$ to indicate

$$x(t) = \phi_{\mathcal{H}}(t, t_0, x_0) := x(t, j_{\mathcal{H}}(t, t_0, x_0)) \tag{11}$$

with $x(t, j)$ as in (6) for $x(t_0, 0) = x_0$ and where $j_{\mathcal{H}}(t, t_0, x_0) := \max\{j \mid (t, j) \in \mathrm{dom}\ x(\cdot, \cdot)\}$ indicates the discrete time corresponding to assuming that, at time t, all discrete-time transitions have already occurred. We will also employ

$$\varphi_j(t, s, x) \tag{12}$$

to denote the flow (with no jumps) of the time-varying vector field $f_{\mathrm{cl}}(\cdot, \cdot, j)$ in (6a) in the time interval $[s, t]$ with initial condition $x(s) = x$. Employing (11) and (12), and assuming to have just one jump at a time (this will be stated more formally in Assumption 1), one can write with no ambiguity $x(t, j) = \varphi_j(t, t_j, x(t_j, j)) = \varphi_j(t, t_j, \phi_{\mathcal{H}}(t_j, t_0, x_0))$, as $j_{\mathcal{H}}(t_j, t_0, x_0) = j$ by definition of t_j. Furthermore, it holds that $x(t_j, j) = \phi_{\mathcal{H}}(t_j, t_0, x_0) = g(x(t_j, j-1), t_j, j-1)$.

B. Problem Formulation

Consider a reference trajectory with jumps and denote it $\alpha(t, j)$. Assume that α is both t-complete ($\sup_t \mathrm{dom}\ \alpha = \infty$) and the unique solution to (2) with $\alpha(t_0, 0) = \alpha_0 \in \mathrm{int} C(t_0, 0)$ and $u = \mu(t, j)$, continuous in t for each j. The jth event time of α is denoted τ_j and $I_\alpha^j = [\tau_j, \tau_{j+1}]$ is the jth time interval between two consecutive events ($\tau_0 = t_0$). Denote J_α the number of intervals and E_α the set of event times. We consider the problem of assessing the stability of $\alpha(t, j)$, both in open and closed loop. Our stability analysis applies, in particular, to a single-jump-flow reference trajectory that is t-complete, non-Zeno, has a bounded interjump time, and intersects the jump set transversally. Furthermore, some minimal and easily encountered regularity conditions are also required, leading in total to six assumptions, detailed below. These assumptions are, for example, already satisfied for the simulation examples in [8], [11], [13]–[15], [17], [20], [23]–[25]. Mechanical systems with smooth unilateral constraints in a neighborhood of trajectories with nonaccumulating (partially) elastic impacts fit the considered system class as well. We suggest the reader to skip the definition of the assumptions and explanatory remarks at first read, returning to them when necessary (in particular, when willing to understand the details of the proof of our main result in Section III-D).

Assumption 1 (t-complete, Non-Zeno, Bounded Interjump Time): The reference α is defined $\forall t > t_0$ (t-complete) and $\exists \underline{\delta}_t > 0$ such that $\tau_{j+1} - \tau_j \geq \underline{\delta}_t$, $\forall j \in \{0, 1, \ldots, J_\alpha - 1\}$ (non-Zeno). If $J_\alpha = \infty$, then moreover $\exists \overline{\delta}_t > 0$ such that $\tau_{j+1} - \tau_j \leq \overline{\delta}_t$, $\forall j \in \{0, 1, \ldots, J_\alpha - 1\}$ (bounded interjump time). ▲

As α is non-Zeno, J_α can become infinite only for $t \to \infty$. Completeness implies that $\alpha \in C(t, j_{\mathcal{H}}(t, t_0, \alpha_0)), \forall t \geq t_0$. We require the jumps to be *transversal* to the boundary of C. In (nonsmooth) mechanics, for example, a jump is transversal when the impact between two convex bodies occurs with nonzero relative normal velocity (otherwise, a grazing impact occurs). To this end, we make use of *guard functions* γ_α that need to be defined only in a ball about each reference event.

Assumption 2 (Existence of a Guard Function): Given Assumption 1, $\exists \varepsilon_\gamma > 0$ and $c_1 > 0$, and a real-valued guard function $\gamma_\alpha(x, t, j)$, C^1 with respect to both x and t, $\forall j \in \{0, 1, \ldots, J_\alpha - 1\}$, such that

$$\begin{aligned} &\gamma_\alpha(x, t, j) > 0, && (x, t) \in B_j \cap \mathrm{int}(C_j) \\ &\gamma_\alpha(x, t, j) = 0, && (x, t) \in B_j \cap \partial C_j =: Z_j \\ &\gamma_\alpha(x, t, j) < 0, && (x, t) \in B_j \cap ((\mathbb{R}^n \times \mathbb{R}) \backslash C_j) \end{aligned} \tag{13}$$

where $B_j := B_{\varepsilon_\gamma}(\alpha(\tau_{j+1}, j), \tau_{j+1})$. In case J_α is finite, we pose $B_{J_\alpha - 1} = \emptyset$. In (13), int and ∂ denote, respectively, the set's interior and boundary. We require $\gamma_\alpha(\cdot, \cdot, j) = 0$ to define a codimension 1 manifold (γ_α is the first coordinate of a C^1-diffeomorphism between B_{ε_γ} and an open neighborhood of the origin on $\mathbb{R}^{n+1}$). We assume that

$$Z_j \subset D_j \tag{14}$$

and also

$$\|\mathbf{D}_1\gamma_\alpha(\alpha(t,j),t,j)\| \le c_1 \tag{15}$$

uniformly $\forall(t,j) = (\tau_{j+1}, j) \in E_\alpha$. In (15), $\mathbf{D}_1$ denotes partial differentiation with respect to the first argument. ▲

In the assumption above, $B_{\varepsilon_\gamma}(x,t)$ denotes an open ball of radius ε_γ about (x,t), that is, $B_{\varepsilon_\gamma}(x,t) := \{(y,s) \in \mathbb{R}^n \times \mathbb{R} \mid \|(y,s)-(x,t)\| < \varepsilon_\gamma\}$.

Let us now formalize the assumption on the transversality property of the jumps of α.

Assumption 3 (Transversality): Let Assumption 2 hold, implying the existence of γ_α. There exist $c_2 > 0$ such that

$$\begin{aligned}\mathbf{D}_1\gamma_\alpha(\alpha(t,j),t,j)\cdot f(\alpha(t,j),\mu(t,j),t,j)\\ +\mathbf{D}_2\gamma_\alpha(\alpha(t,j),t,j)\cdot 1 \le -c_2\end{aligned} \tag{16}$$

for every event time $(t,j) = (\tau_{j+1}, j) \in E_\alpha$. ▲

Aside from the assumptions on α, we also pose continuity conditions on the jump map g and vector field f.

Assumption 4 (Locally Differentiable Jump Map): Given Assumptions 1 and 2, we require $g(\cdot,\cdot,j)$ to be C^1 in the open ball $B_{\varepsilon_\gamma}(\alpha(\tau_{j+1},j),\tau_{j+1})$, $\forall j \in \{0,1,\ldots,J_\alpha-1\}$. ▲

Assumption 5 (Uniform Lipschitz Condition on f): In a neighborhood of the reference state-input trajectory (α,μ), $f(x,u,t,j)$ is Lipschitz with respect to x and u, uniformly in t and j. Namely, we assume that $\exists\varepsilon_L > 0$ independent of (t,j) and $\exists L$ for which, $\forall j \in \{0,1,\ldots,J_\alpha-1\}$, $\|f(x,u,t,j) - f(y,v,t,j)\| < L(\|x-y\| + \|u-v\|)$, $\forall t \in (\tau_j - \varepsilon_L, \tau_{j+1}+\varepsilon_L)$, $x,\ y \in B_{\varepsilon_L}(\overline{\alpha}(t,j))$, and $u,\ v \in B_{\varepsilon_L}(\overline{\mu}(t,j))$. ▲

Remark 2: The Lipschitz constant L is defined for time intervals $t \in (\tau_j - \varepsilon_L, \tau_{j+1}+\varepsilon_L)$ that are not contained in dom α. Therefore, an *extended* reference state-input trajectory $(\overline{\alpha},\overline{\mu})$, for which dom $\alpha \subset$ dom $\overline{\alpha}$, is used in its definition. See Section III-A for further details. △

The following assumption, imposing natural conditions on α and the state-triggered hybrid system, ensures that trajectories sufficiently close to $\alpha(t,j)$ are also t-complete. In the assumption, $\mathcal{T}_p S$ denotes the tangent cone to S at p as defined, e.g., in [1, Def. 5.12].

Assumption 6 (Local Existence of t-complete Solutions): Let Assumptions 1 to 3 hold. Every state-time pair $(x,t) \in Z_j$ of the reference event $(\alpha(\tau_{j+1},j),\tau_{j+1})$, with Z_j defined as in (13) and $j \in \{0,1,\ldots,J_\alpha-1\}$, is mapped by the jump map g to the subsequent flow set, while avoiding the jump set, i.e.,

$$(x,t) \in Z_j \Rightarrow (g(x,t,j),t) \in C_{j+1} \setminus D_{j+1}.$$

Furthermore, we assume that $f(x,u,t,j)$ satisfies

$$(f(x,u,t,j),1) \in \mathcal{T}_{(x,t)}C_j \tag{17}$$

$\forall j \in \{0,1,\ldots,J_\alpha-1\}$, $\forall u$ in a uniform neighborhood of $\overline{\mu}(t,j)$, and for $(x,t) \in \partial C_j \cap (U_j \setminus B_j)$, with U_j defined below and B_j as in (13). In (17)

$$U_j := \bigcup_{t\in[\tau_j-\varepsilon_C,\tau_{j+1}]} (B_{\varepsilon_C}(\overline{\alpha}(t,j)) \times \{t\}) \tag{18}$$

that is, U_j denotes the set of all state-time pairs (x,t) contained in the tube of size $\varepsilon_C > 0$ about the extended reference trajectory $\overline{\alpha}$. ▲

Remark 3: The assumption allows to handle solutions close to the reference trajectory that are only defined on a finite time domain, without resorting to advanced concepts such as preasymptotic stability [1, Ch. 7]. The required properties guarantee that, away from the jump event times (τ_{j+1},j) (hence, the asymmetry in the definition of U_j), trajectories in a neighborhood of $\alpha(t,j)$ remain in the flow set and that, after each jump, flowing is always possible (for the points that are the image through the impact map of the prejump states, the vector field is directed inward the flow set). To account for any differences in jump time of the nearby solutions, the given properties are required to hold on a larger time domain than dom α, requiring (similarly to what is discussed for the previous assumption) the availability of an extended reference trajectory $\overline{\alpha}$ (see Section III-A). △

In the next section, we introduce the concept of *reference spreading error* between two trajectories and we proceed with the definition of time-triggered linearization of the NSTHS (2) about a reference trajectory α. Furthermore, we show that the stability of this time-triggered linear hybrid system (about the origin) implies local stability of the NSTHS about α.

III. Sensitivity-Based Stability Analysis of Jumping Trajectories

In this section, the stability properties of the reference trajectory $\alpha(t,j)$ of the *cl*-NSTHS in (6) are analyzed. First, the notion of extended (reference) trajectory will be introduced to define a useful error measure on the basis of which a definition of stability of the jumping reference trajectory will be given. Secondly, a time-triggered linear hybrid system will be defined, the trajectories of which can be used to approximate those of the NSTHS starting near the reference trajectory. This section is concluded by showing that a switching controller can be designed using this linear hybrid system, as asymptotic stability of this system in a closed loop implies asymptotic stability of the reference trajectory of the *cl*-NSTHS in (6).

A. Error Definition

Any difference between the initial conditions x_0 and α_0 will most likely result in differences between the jump times of the reference and those of the closed-loop system as illustrated in Fig. 2. Therefore, as mentioned in the introduction, designing a tracking controller based on the Euclidean error defined as the difference between the current state x and α (for the same t only) may easily result in poor tracking performance (see [14] and [20] for simulation examples supporting this statement). In order to suitably compare both trajectories at each point in time, as suggested in [19], each segment of the reference trajectory α discriminated by the counter j (that is, corresponding to the interval $I_\alpha^j \times \{j\}$) is extended offline both forward and backward using the vector field (2a) (see the dashed lines in Fig. 2). In this, the input $u = \overline{\mu}(t,j)$ that is the continuously extended version (by design) of $\mu(t,j)$ is used such that each reference segment is defined for all $t \in [t_0,t_f]$. This *extended reference trajectory* is denoted $\overline{\alpha}$ and is, thus, defined for all $(t,j) \in [t_0,t_f] \times \{0,1,\ldots,J_\alpha-1\} =: \overline{I}_\alpha$. Note that $(\overline{\alpha},\overline{\mu})$ coincides with (α,μ) for $(t,j) \in$ dom α. Formally,

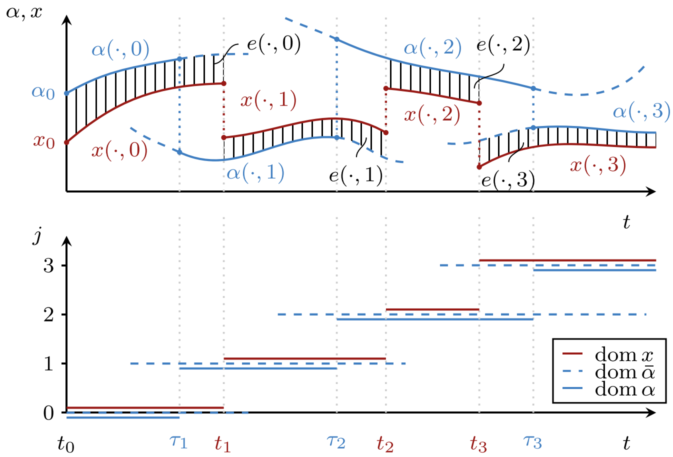

Fig. 2. Extended reference $\overline{\alpha}$, the tracking error e for a trajectory x, and their corresponding time domains.

$\forall j \in \{0, 1, \ldots, J_\alpha - 1\}$, we define $t \mapsto \overline{\alpha}(t, j)$, $t \in [t_0, t_f]$, as the solution to

$$\dot{\overline{\alpha}} = f(\overline{\alpha}, \overline{\mu}(t, j), t, j), \qquad (t, j) \in \overline{I}_\alpha \tag{19}$$

for $\overline{\alpha}(\tau_j, j) = \alpha(\tau_j, j)$, where τ_j denotes the jth jump time ($\tau_0 = t_0$). The construction of the extended trajectory requires both forward and backward integration of the vector field to extend the jth reference segment outside the original interval $[\tau_j, \tau_{j+1}]$. The forward extensions are needed when $t_j > \tau_j$ and the backward extensions when $t_j < \tau_j$ for the jth jump. The forward and backward calculations cannot, in most cases, be computed analytically but can quite easily be constructed using numerical integration.

Remark 4: Tracking the reference trajectory α using the notion of extended trajectories as proposed in [19] requires more than just the state of the reference at the current time, it requires knowledge of the future reference. The trajectory α needs to be known beforehand or, at least, the segment up to the next jump of α. If the closed-loop system encounters a jump prior to the reference trajectory, for example, it already needs to know what the desired motion is after the jump event. This knowledge comes from the backwards integration of the vector field from the time where the jump is expected to occur (for incremented jump counter), i.e., the reference jump time.

Philosophically, we also think that in proximity of an expected impact, it is indeed very natural to imagine that there are two references to be tracked and that the system should switch from one to the other only when the actual impact has occurred: if a jump has not occurred as expected, what should the system do in order to try to make it happen (ante-impact forward extension)? If a jump occurred earlier than expected, what should the system do to get where it is intended to be (postimpact backward extension)? △

Note that when a hybrid trajectory that is a solution to (2) is projected onto the continuous time domain, it is single valued for all t in its domain except for the jump times. This is not the case for $\overline{\alpha}$, as now for each time t, there are J_α extended "reference" trajectories discriminated by the counter j. This counter will be used to compare the state x at any given time to the relevant branch of the extended reference $\overline{\alpha}$. Note furthermore that the domain $\overline{I}_\alpha$ is *not* a hybrid time domain as in [1], because the natural ordering of its points is lost, i.e., the boundaries of the time intervals do not form an increasing sequence. The construction of the extended trajectory $\overline{\alpha}(t, j)$ requires the vector field f in (2a), with $u = \overline{\mu}(t, j)$, to be defined (by design) outside the domain $C(t, j)$, ignoring the presence of the jump set and allowing the time integration to continue beyond it (see [20] and [21] for examples). The extended reference trajectory $\overline{\alpha}$ allows to define the tracking error as

$$e(t, j) := x(t, j) - \overline{\alpha}(t, j), \qquad (t, j) \in \operatorname{dom} x. \tag{20}$$

A graphical representation of $\overline{\alpha}$, its "extended" hybrid time domain $\overline{I}_\alpha$, and the tracking error $e(t, j)$ is given in Fig. 2.

B. Stability Definition

We provide here the definition of stability and asymptotic stability for a single-jump-flow trajectory α of the NSTHS (6), using the notion of error introduced in the previous section.

Definition 1 (Stability): Given t_0, a trajectory α of (6) that is t-complete is said to be **stable** if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for every trajectory x of (6) satisfying $\|x(t_0, 0) - \alpha(t_0, 0)\| < \delta$, it holds that 1) $J_x = J_\alpha$, 2) for all $(t, j) \in \operatorname{dom} x$, $\|x(t, j) - \overline{\alpha}(t, j)\| < \varepsilon$, and 3) for all $j \in \{1, \ldots, J_x - 1\}$, $|t_j - \tau_j| < \varepsilon$, where $\overline{\alpha}$ is the extension of α defined in (19) and J_x and J_α are, respectively, the number of time intervals I_x^j and I_α^j defined as in (8) for the trajectories $x(t, j)$ and $\alpha(t, j)$, possibly infinite. ■

Remark 5: This definition of stability makes use of the error definition in (20), which is the Euclidean distance between the state x and the extended reference trajectory $\overline{\alpha}$ for each $(t, j) \in \operatorname{dom} x$. Note that $e(t, j)$ has the same time domain as $x(t, j)$ and requires $\operatorname{dom} x \subset \overline{I}_\alpha$ for it to be defined. The latter is guaranteed when $J_x \leq J_\alpha$ ($J_x = J_\alpha$ in the case α is stable, for sufficiently small δ, as in Definition 1). △

Definition 2 (Attractivity): Given a trajectory α of (6) that is t-complete, we say that α is **attractive** if there exists a $\delta > 0$ such that $\|x(t_0, 0) - \alpha(t_0, 0)\| < \delta$ implies, first, that $J_x = J_\alpha$ and, second, that $\|x(t, j) - \overline{\alpha}(t, j)\| \to 0$ for $t \to \infty$ with $(t, j) \in \operatorname{dom} x$, where $\overline{\alpha}$ is the extension of α defined in (19) and J_x and J_α are, respectively, the number of time intervals I_x^j and I_α^j defined as in (8) for the trajectories $x(t, j)$ and $\alpha(t, j)$. If $J_\alpha = \infty$, we require that furthermore $|t_j - \tau_j| \to 0$ for $j \to \infty$. ■

Definition 3 (Asymptotic stability): A trajectory α of the *cl*-NSTHS (6) is **asymptotically stable** if it is stable and attractive, respectively in the sense of Definitions 1 and 2. ■

C. Linear Time-Triggered Hybrid System

As mentioned in the previous section, a nonzero tracking error will likely result in a mismatch between the closed-loop jump times t_j and the reference jump times τ_j. The times t_j, with $j \in \{1, 2, \ldots, J_x - 1\}$ are not known in advance. Instead, as the reference α is assumed to be known, the event times τ_j are known. Next, we construct a linear time-triggered system, that jumps at the reference jump times τ_j, and solutions of which can be used to approximate the error in the state evolution for (6) in a neighborhood of α. We can use this time-triggered linear system to design a stabilizing feedback of the form (5) and we show that asymptotic stability of this closed-loop linear hybrid system *implies* (local) asymptotic stability of the reference trajectory α for the original *cl*-NSTHS (6). In [20] and [26], the time-triggered linear system is used in a linear quadratic regulator setting for mechanical systems with elastic impacts, respectively, for systems with multiple modes with different state dimension.

We refer here to this linear system with jumps at the times τ_j as the LTTHS associated with the reference trajectory α. The key feature of this LTTHS is that it converts the state-triggered behavior of (2) to a time-triggered one (cf., [19]) and incorporates a first-order approximation of the state-jumps (originally at slightly different times) in the definition of the jump map, as explained in detail in [19], [27], [28], and [29, Sec. 5.2]. We will show that asymptotic stability of α in the sense of Definition 3 can be assessed by studying the LTTHS corresponding to (2) and the state-input trajectory $(\alpha(t,j), \mu(t,j))$, with $(t,j) \in \text{dom }\alpha$. Hence, the stability analysis is significantly simplified as $\alpha(t,j)$ and the LTTHS jump at the same time. Let us now formally define the LTTHS (whose full derivation is given in Appendix D).

Definition 4 (LTTHS): The LTTHS associated with trajectory α and NSTHS (2) is given by

$$\begin{aligned} \dot{z} &= A(t,j)z + B(t,j)v, && (t,j) \in \text{dom }\alpha \\ z^+ &= G(j)z^-, && (t,j) \in E_\alpha \end{aligned} \tag{21}$$

with initial condition $z(t_0, 0) = z_0$ and where $z^+ := z(t, j+1)$, $z^- := z(t,j)$

$$A(t,j) := \mathbf{D}_1 f(\alpha(t,j), \mu(t,j), t, j) \tag{22}$$

$$B(t,j) := \mathbf{D}_2 f(\alpha(t,j), \mu(t,j), t, j) \tag{23}$$

and

$$G(j) := \frac{f^+ - \dot{g}^-}{\dot{\gamma}_\alpha^-}\mathbf{D}_1\gamma_\alpha^- + \mathbf{D}_1 g^- \tag{24}$$

with

$$f^+ = f(\alpha^+, \mu^+, \tau, j+1) \tag{25}$$

$$f^- = f(\alpha^-, \mu^-, \tau, j) \tag{26}$$

$$g^- = g(\alpha^-, \tau, j) \tag{27}$$

$$\dot{g}^- = (\mathbf{D}_1 g^-)f^- + \mathbf{D}_2 g^- \tag{28}$$

$$\gamma_\alpha^- = \gamma_\alpha(\alpha^-, \tau, j) \tag{29}$$

$$\dot{\gamma}_\alpha^- = (\mathbf{D}_1\gamma_\alpha^-)f^- + \mathbf{D}_2\gamma_\alpha^- \tag{30}$$

where $\tau = \tau_{j+1}$, $\alpha^+ = \alpha(\tau, j+1)$, $\alpha^- = \alpha(\tau, j)$, $\mu^+ = \mu(\tau, j+1)$, $\mu^- = \mu(\tau, j)$, and $\gamma_\alpha(\cdot,\cdot,j)$ denotes the guard functions (see Section II-B). ■

As will be clarified later on, the linear hybrid system (21)–(30) provides an approximation of the NSTHS in the sense that a trajectory of the NSTHS starting at a perturbed initial condition $x_0 = \alpha_0 + z_0$ with perturbed input $\mu(t,j) + v(t,j)$ can be approximated as $x(t,j) = \overline{\alpha}(t,j) + \overline{z}(t,j) + o(\|z_0\|)$ for $(t,j) \in \text{dom }x$. In this, $\overline{z}$ is the extended trajectory of z obtained in the same way as $\overline{\alpha}$, that is, for each j, it follows from integrating the vector field $\dot{z} = A(t)z + B(t)v(t,j)$ with $t \in [t_0, t_f]$ forward and backward in time from initial condition $\overline{z}(\tau_j, j) = z(\tau_j, j)$. The term $o(\|z_0\|)$ denotes a perturbation that is of order higher than one. The state $\overline{z}$ is, thus, a first-order approximation of the error (20) if $J_x \leq J_\alpha$.

The continuous dynamics in (21) and, in particular, the construction of the time-varying A and B matrices are well known. The precise form of the discrete dynamics in (21), i.e., the reset map G in (24), is less well familiar (the reader is referred to [19], [27], [28], [29, Sec. 5.2], and Appendix D for details). Here, instead, we present an intuitive description of the components of G.

The part $\mathbf{D}_1 g^-$ is the traditional sensitivity of the jump map g to a perturbation in the state. As jumps in x and z take place at different time instances, the effect of time mismatch needs to be included in the state jump as well. The term $-(1/\dot{\gamma}_\alpha^-)\mathbf{D}_1\gamma_\alpha^-$ when multiplied with z^- gives an approximation of the event time mismatch $t_j - \tau_j$. The term $\dot{g}^-$ appears in the jump map G to capture the change in the jump during this time mismatch. The term f^+ is there to incorporate the change in the state due to flow in the mismatch interval. The latter is required to make sure the approximation is still correct after the mismatch interval.

Given the time-triggered "linearization" of the NSTHS about a trajectory α, one can attempt to design a control law to make the origin of the LTTHS uniformly asymptotically stable. The uniformity property will be required for showing that α is an asymptotically stable trajectory of the NSTHS if the origin of the linearization indeed satisfies the posed stability properties (this will be discussed in Section III-D). Considering the fact that the LTTHS is a "linearization" and since this article deals with a local stability property, we restrict attention to feedbacks of the form

$$v(t,j) = -K(t,j)z(t,j). \tag{31}$$

By addition of higher-order terms in the control law, we expect that the basin of attraction of a stable solution could be enlarged, but such study is out of scope of the present article. In many cases, by suitably designing the time-varying feedback gain K, uniform asymptotic stability of the origin of the closed-loop LTTHS (*cl*-LTTHS) can be achieved, by which we mean the following (see [30, Def. 4.4]).

Definition 5 (LTTHS: Uniform asymptotic stability): The origin of the *cl*-LTTHS (21)–(31) is **uniformly asymptotically stable** if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, for every $T_0 \geq t_0$, $|z(T_0, j_{T_0})| \leq \delta$ implies $|z(t,j)| \leq \varepsilon$ for all $(t,j) \in \text{dom }\alpha$ with $t \geq T_0$ and that $\lim_{t\to\infty}|z(t,j)| = 0$. In this, j_{T_0} is the counter j corresponding to the time T_0, that is, the largest j such that $T_0 \geq \tau_j$. ■

The stability assessment of (21) in closed loop [i.e., with feedback (31)] is well established in literature (see, e.g., [31, Secs. 3.2 and 6.4] and [32]).

Using (31) and the fact that $\overline{z}(t,j)$ is a local approximation of the error (20), we obtain a *cl*-NSTHS with input

$$u = \kappa(x,t,j) = \overline{\mu}(t,j) - \overline{K}(t,j)(x(t,j) - \overline{\alpha}(t,j)). \tag{32}$$

Note that where z is defined for all $(t,j) \in \text{dom }\alpha$, the state x and error e have a different hybrid time domain that is not known in advance. For the feedback law to be well defined, we, thus, require the time-varying feedback gain to be defined for a larger time domain than dom α, i.e., for all $(t,j) \in \overline{I}_\alpha = [t_0, t_f] \times \{0, 1, \ldots, J_\alpha - 1\}$. Therefore, in (32), we introduced $\overline{K}(t,j)$ representing the feedback gain $K(t,j)$, but extended such that it is defined for all $(t,j) \in \overline{I}_\alpha$. Due to this extension, the feedback control is defined for all $(t,j) \in \text{dom }x$ (as long as $J_x \leq J_\alpha$). Several approaches are possible in constructing these extensions. However, the extension of the feedback gains $\overline{K}(t,j)$ does not influence the LTTHS since it only depends on the perturbation input $v(t,j)$ for $(t,j) \in \text{dom }\alpha$ (see [20]). This property is explained further in Appendix D.

D. Main Stability Result

The problem considered in this article is that of assessing the stability properties of a jumping reference trajectory $\alpha(t,j)$ of the *cl*-NSTHS (6) under Assumptions 1 to 6. As a stepping stone, a key fact that we will exploit is that, for any finite time $T > t_0$, the jump times of the *cl*-NSTHS (6) in dom $x \cap [t_0, T] \times \mathbb{N}$ depend in a continuously differentiable fashion on the initial condition x_0 as long as x_0 is chosen in a sufficiently small neighborhood of α_0 dependent on T. This result follows from Assumptions 1 to 4 and Assumption 6 as well as some minimal regularity assumptions on the vector field f_{cl} and jump map g, and it is proved in Lemma 1. Note that the size of this neighborhood might vanish for $T \to \infty$. In order to conclude stability, this dependency of jump times on initial condition, Assumption 5, and stability of the linear error dynamics are key. Let us first consider the former property for which we define a jump counter function for the reference trajectory α.

Definition 6 (Jump counter function): For a given $T > t_0$ and hybrid trajectory α, denote with $j_\alpha(T)$ the number of encountered jumps of the reference trajectory for $t \le T$. ■

The jump counter function $j_\alpha : (\tau_1, \infty) \to \mathbb{N}$ is right continuous and satisfies the inequality $\tau_{j_\alpha(T)} \le T$. It is equal to the jump counter $j_{\mathcal{H}}$, introduced in Section II, when evaluated along the specific trajectory α, i.e., $j_\alpha(T) = j_{\mathcal{H}}(T, t_0, \alpha_0)$. The following lemma now holds.

Lemma 1: For the hybrid system (6), assume that the vector field f_{cl} is locally Lipschitz with respect to x and continuous and bounded in t. Let α denote a reference trajectory of the NSTHS (6) satisfying Assumptions 1, 2, 3, 4, and 6, with initial condition $\alpha(t_0, 0) = \alpha_0 \in \text{int}(C(t_0, 0))$. As before, τ_j, $j \in \{0, 1, \ldots, J_\alpha - 1\}$, indicates the nominal event times.

There exists a function $\delta_0 : (\tau_1, \infty) \to \mathbb{R}_{>0}$ such that, for any $T > \tau_1$, a trajectory of the *cl*-NSTHS (6) with initial condition $x(t_0, 0) = x_0$ satisfying

$$\|x_0 - \alpha_0\| < \delta_0(T) \quad \text{with} \quad B_{\delta_0(T)}(\alpha_0) \subset C(t_0, 0) \tag{33}$$

is defined at least up to time T and jumps at least $j_\alpha(T) - 1$ times in the interval $t \in [t_0, T]$. Furthermore, when $j_\alpha(T) \ge 2$, the function $\delta_0(\cdot)$ can be chosen such that, in addition, every jump time t_j except the last is bracketed by the nominal jump times τ_{j-1} and τ_{j+1}, i.e.,

$$\tau_{j-1} \le t_j \le \tau_{j+1}, \qquad \text{for } j \in \{1, 2, \ldots, j_\alpha(T) - 1\}. \tag{34}$$

Proof: The proof of Lemma 1 is presented in Appendix A. ■

Stability of the reference trajectory α for the *cl*-NSTHS (6), (32) can now be related to the stability of the *cl*-LTTHS (21)–(31), resulting in the main result of this article as given below. The result is similar in spirit as [1, Th. 9.11] and [33] (covering the inherently different problem of stability of points).

Theorem 1: Adopt Assumptions 1–6. Let a state-input trajectory (α, μ) be a solution to the *cl*-NSTHS (6), (32). If the origin of the associated LTTHS (21)–(30) in closed loop with control law (31) is uniformly asymptotically stable, then, $\alpha(t,j)$ is a (locally) asymptotically stable trajectory of the *cl*-NSTHS (6), (32) in the sense of Definition 3.

Proof: The proof of Theorem 1 is given in Appendix B. ■

To bridge the gap between this work and other approaches in literature for analyzing stability of state-triggered hybrid systems, in Appendix C, we show that uniform asymptotic stability of the *cl*-LTTHS (21)–(31) also implies asymptotic stability of the reference trajectory α for (a variant of the) *cl*-NSTHS (6), (32) in terms of the distance function defined in [8] and [15]. Mind that, in this implication, the hybrid system class that is considered is reduced to one where the jump map g, and the flow and jump sets C and D, respectively, do not depend on time explicitly as the distance function in [8] and [15] does not accommodate such time-varying nature.

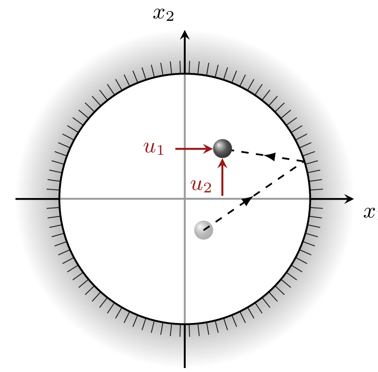

Fig. 3. Schematic representation of a circular billiard.

IV. Illustrative Example

In this section, a trajectory tracking example for mechanical systems with a unilateral constraint is presented. A circular billiard with periodic reference trajectory is considered. In this, the number of jumps $J_\alpha - 1$ becomes infinite as $t \to \infty$. We only consider a periodic reference here for simplicity, but for the theory presented in this work no periodicity is required.

Consider the time-invariant system depicted in Fig. 3 (that is also considered in [11] for fully elastic restitution) consisting of an actuated point mass moving in a plane that is confined by a circular boundary. Such a system is commonly referred to in literature as a billiard.

The position of the point mass in the plane, at a particular time, is given by the coordinates x_1 and x_2 (see Fig. 3). The velocity components of the mass in the x_1 and x_2 directions are denoted by $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$, respectively. The state of the point mass, thus, is $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ and accelerations can be imposed in x_1 and x_2 directions (denoted u_1, respectively u_2, such that $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$). A rigid object confines the space in which the mass can move such that the jump set is defined as $D = \{x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = 1,\ x_1 x_3 + x_2 x_4 > 0\}$. The flow set, therefore, becomes $C = \{x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 \le 1\}$ and a suitable guard function for the system is $\gamma_\alpha = 1 - x_1^2 - x_2^2$ (satisfying Assumption 2). Whenever the mass impacts the boundary, partially elastic restitution occurs with a coefficient of restitution e. The system can be described by (2), with

$$f(x,u,t,j) = \begin{bmatrix} x_3 & x_4 & u_1 & u_2 \end{bmatrix}^T =: Ax + Bu$$
$$A = \begin{bmatrix} 0_{2\times 2} & I_2 \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}, \qquad B = \begin{bmatrix} 0_{2\times 2} \\ I_2 \end{bmatrix} \tag{35}$$

and

$$g(x,t,j) = \begin{bmatrix} x_1 \\ x_2 \\ (x_2^2 - e x_1^2) x_3 - (1+e) x_1 x_2 x_4 \\ (x_1^2 - e x_2^2) x_4 - (1+e) x_1 x_2 x_3 \end{bmatrix} \tag{36}$$

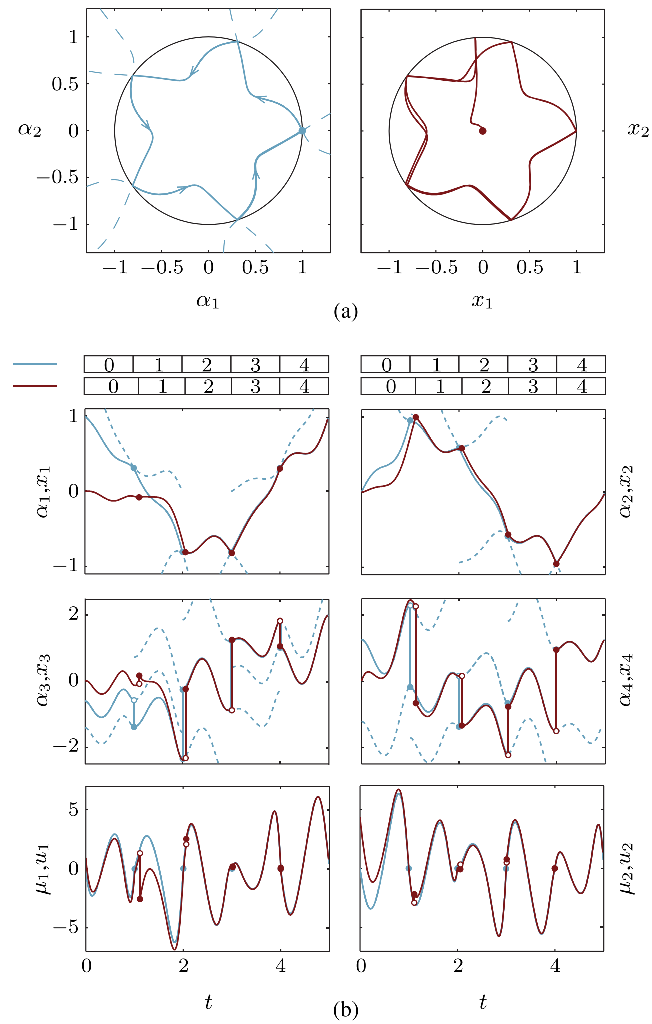

Fig. 4. (a). Reference trajectory $\alpha(t, j)$ (light blue) with extensions (dashed) and a tracking solution $x(t, j)$ (dark red) to the *cl*-NSTHS in Fig. 3 in the x_1, x_2-plane (where the dot indicates the initial position) (b) and the evolution of state and input over time (where the open and closed dots indicate the left, respectively, the right limit).

cf., [11]. In this example, it is assumed that the interaction between actuated mass and obstacle can be modeled fully using the impact law, that is, periods of persistent unilateral contact do not occur and (finite) contact forces, thus, need not be included in the vector field f. Furthermore, the pair (α, μ) is considered to be such that grazing incidence of the point mass on the obstacle is avoided (cf., Assumption 3). Note that, in this example, the flow set C, jump set D, vector field f, and jump map g do not depend explicitly on the hybrid time (t, j). This time-invariance is chosen here for the sake of simplicity of the analysis only, and is not required for the applicability of the theory in this article.

Consider the reference trajectory shown in Fig. 4(a). It is a periodic solution to the system description above where the state returns back to its initial condition after five impacts with coefficient of restitution $e = 0.3$. The impacts are separated in time by a period $\tau = 1$ (cf., Assumption 1) and since the trajectory starts at the boundary it follows that $\tau_j = j\tau$ for $j = 0, 1, \ldots$ (taking $t_0 = 0$). As can be discerned from the fact that the trajectory segments between impacts are curved, the input $\mu(t, j)$ is not zero for all time. As a consequence, constructing the extended trajectory $\overline{\alpha}$ as described in Section III will be different from the strategy of "mirroring" as is used for billiards in [14], [24], and [25]. Our approach and the mirroring approach would provide the same result just if the input would be zero and $e = 1$ (in the simulations, we took $e = 0.3$). Due to the different error definition, our strategy also does not suffer from the peaking phenomenon induced by impact time mismatch as seen in the standard-error-based PD control in [11].

It is straightforward to show that, for the specific example, Assumptions 1–6 are satisfied. Due to space restrictions, this proof is not included here but it is available upon request.

To design the stabilizing control law, by using Theorem 1, we first consider stability of the LTTHS. The desired trajectory is periodic with a period of 5τ, but due to its point symmetry with respect to the origin, it is possible to assess stability of the LTTHS corresponding to the considered system by only looking at the evolution from one postimpact position and velocity to the next. When applying a feedback input of the form (31) (with constant gain $K = K(t, j)$) to this linearized system, the state after the jth impact and that just after the next are related to each other by $z(\tau_{j+1}, j+1) = G(j)\exp((A + BK)\tau)z(\tau_j, j)$, with $G(j)$ given by (24), combined with $\dot{\gamma}_\alpha^- = \mathbf{D}_1\gamma_\alpha^- \cdot f^-$, $\dot{g}^- = \mathbf{D}_1\, g^- \cdot f^-$

$$f^+ = \begin{bmatrix}\alpha_3^+ & \alpha_4^+ & \mu_1^+ & \mu_2^+\end{bmatrix}^T$$

$$f^- = \begin{bmatrix}\alpha_3^- & \alpha_4^- & \mu_1^- & \mu_2^-\end{bmatrix}^T$$

$$\mathbf{D}_1\gamma_\alpha^- = \begin{bmatrix}-2\alpha_1^- & -2\alpha_2^- & 0 & 0\end{bmatrix}$$

$$\mathbf{D}_1 g^- = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ dg_{31} & dg_{32} & dg_{33} & dg_{34}\\ dg_{41} & dg_{42} & dg_{43} & dg_{44}\end{bmatrix}$$

where

$$dg_{31} = -2e\alpha_1^-\alpha_3^- - (1+e)\alpha_2^-\alpha_4^-$$
$$dg_{32} = 2\alpha_2^-\alpha_3^- - (1+e)\alpha_1^-\alpha_4^-$$
$$dg_{33} = (\alpha_2^-)^2 - e(\alpha_1^-)^2$$
$$dg_{34} = -(1+e)\alpha_1^-\alpha_2^-$$
$$dg_{41} = 2\alpha_1^-\alpha_4^- - (1+e)\alpha_2^-\alpha_3^-$$
$$dg_{42} = -(1+e)\alpha_1^-\alpha_3^- - 2e\alpha_2^-\alpha_4^-$$
$$dg_{43} = -(1+e)\alpha_1^-\alpha_2^-$$
$$dg_{44} = (\alpha_1^-)^2 - e(\alpha_2^-)^2$$

and in which $\alpha_s^+ = \alpha_s((j+1)\tau, j+1)$ and $\alpha_s^- = \alpha_s((j+1)\tau, j)$ denote the right, respectively, left limit of the sth state of the reference trajectory at time $t = (j+1)\tau$. Similarly, μ_s^+ and μ_s^- denote the right and left limits (that are, in this case, the same), respectively, of the sth reference input at that time. It follows that the LTTHS is asymptotically stable if all eigenvalues of the matrix $G(j)\exp((A+BK)\tau)$ are within the unit circle in the complex plane. When taking a feedback gain of the form $K = \begin{bmatrix}\beta & 0 & 2\sqrt{\beta} & 0; & 0 & \beta & 0 & 2\sqrt{\beta}\end{bmatrix}$, it is found that the eigenvalues are within the unit circle when $\beta > 0.393$. Applying Theorem 1, we, therefore, conclude that α is asymptotically stable for the *cl*-NSTHS (2), (32), (35), (36) for such choice of feedback gain. A solution to the *cl*-NSTHS (with constant gain extensions) for $\beta = 3$ and initial condition $x_0 = 0$ is depicted in Fig. 4. The figure shows that the solution indeed converges toward the reference state-input trajectory $(\alpha(t, j), \mu(t, j))$, even for this large initial error.

V. Conclusion

In this article, a notion of asymptotic stability and an associated stability analysis for discontinuous trajectories of hybrid systems with state-triggered jumps are detailed. The results have also a direct applicability to the related problem of trajectory tracking.

Asymptotic stability is defined by making use of a notion of error that allows for the comparison of two nearby discontinuous trajectories, even when there is a time mismatch between the jumps of both trajectories. It is shown that asymptotic stability of a discontinuous trajectory of the hybrid system with state-triggered jumps is guaranteed when an associate time-triggered linear system is uniformly asymptotically stable. As this linear system jumps at the same times as the reference trajectory, the design of a stabilizing feedback and stability analysis is greatly simplified. A study of (numerical) methods to estimate the associated region of attraction is left for future research. The results of this article are illustrated by means of a tracking example for a mechanical system with unilateral constraint and partially elastic impacts.

In a series of related publications, the tracking strategy based on reference extensions has been given the name of reference spreading control and has been applied to more complex systems (such as, e.g., a multibody humanoid model) and even for the case where the constrained state space after each jump has a different dimension as, e.g., with inelastic impacts and multidomain hybrid systems. Investigation of the stability of these more challenging cases will be presented in future publications.

Appendix A
Proof of Lemma 1

The proof of Lemma 1 is split in two parts. First, we show that the solution $x(t, j)$ to the NSTHS (6) from initial condition $x(t_0, 0) = x_0$ is defined for all $t \in [t_0, T]$ as long as x_0 is sufficiently close to α_0 (cf., [2, Sec. 1.3.2.3] for a similar proof). Then, we show that the bracketing condition (34) is satisfied as long as this neighborhood is chosen small enough, due to a continuity argument. This sufficiently small neighborhood is, in essence, what defines the function δ_0 at T.

Due to Assumption 1, $j_\alpha(T)$ is finite for any $T > t_0$. The existence of a solution to (6) up to time T is straightforwardly guaranteed if, for every $j \in \{1, 2, \ldots, j_\alpha(T)\}$ and in a neighborhood of α_0, we can define an event-time function $x_0 \mapsto t_j$ that is continuously differentiable, where $t_j(x_0) = \tau_j$ whenever $x_0 = \alpha_0$. Indeed, if these functions $x_0 \mapsto t_j$, $j \in \{1, 2, \ldots, j_\alpha(T)\}$ exist, the flow of the NSTHS (6) is a composition of continuously differentiable jump maps (due to Assumption 4) with continuously differentiable continuous-time flows on the time intervals $[t_{j-1}(x_0), t_j(x_0)]$, $j \in \{1, 2, \ldots, j_\alpha(T) - 1\}$, terminated by a continuous-time flow over $[t_{j_\alpha(T)-1}(x_0), T]$, when $t_{j_\alpha(T)}(x_0) \geq T$, or over $[t_{j_\alpha(T)-1}(x_0), t_{j_\alpha(T)}(x_0)]$ followed by a jump and another flow phase over $[t_{j_\alpha(T)}(x_0), T]$, when $t_{j_\alpha(T)}(x_0) < T$.

Due to Assumption 2, there exists a guard function γ_α that implicitly defines the jump set in a neighborhood of α. Therefore, if the event time functions $x_0 \mapsto t_j$ exist, they have to satisfy the implicit conditions

$$\gamma_\alpha(x(t_{j+1}(x_0), j), t_{j+1}(x_0), j) = 0 \tag{37}$$

for $j \in \{0, 1, \ldots, j_\alpha(T) - 1\}$, where

$$\begin{aligned} x(t_{j+1}(x_0), j) &:= \varphi_j(t_{j+1}(x_0), t_j(x_0), \phi_{\mathcal{H}}(t_j(x_0), t_0, x_0)) \\ &= \varphi_j(t_{j+1}(x_0), t_j(x_0), x(t_j(x_0), j)). \end{aligned} \tag{38}$$

Note that Assumption 6 guarantees that the flow φ_j as used above is defined whenever x_0 is sufficiently close to α_0. By definition of α and γ_α, we know that (37) holds at least when we pose $x_0 = \alpha_0$ and $t_j(\alpha_0) = \tau_j$. Using (38), the implicit condition (37) can equivalently be rewritten as

$$M_{j+1}(x_0, t) := \gamma_\alpha(\varphi_j(t, t_j(x_0), x(t_j(x_0), j)), t, j) = 0. \tag{39}$$

We aim to prove that, for each j, t in (39) is a function of x_0, i.e., that $t = t_{j+1}(x_0)$. The transversality condition provided in Assumption 3 guarantees that one can apply the implicit function theorem for each of the implicit conditions in (39) and conclude that all the functions $x_0 \mapsto t_j$ are continuously differentiable for a sufficiently small neighborhood of α_0.

More precisely, one can employ a proof by induction showing that $x_0 \mapsto t_1$ is continuously differentiable (base induction) and that $x_0 \mapsto t_j$ being continuously differentiable implies $x_0 \mapsto t_{j+1}$ to be continuously differentiable (induction step). The base induction has been proven in [19], while it is straightforward to show that $x_0 \mapsto t_j$ being continuously differentiable implies that M_{j+1} is continuously differentiable in a neighborhood of (α_0, τ_{j+1}) being the composition of continuously differentiable functions. Furthermore, as the partial derivative of M_{j+1} in (39) with respect to t evaluated at $(x_0, t) = (\alpha_0, \tau_{j+1})$ is equivalent to the left-hand side of (16) and, therefore, nonzero by Assumption 3, the implicit function theorem can be applied to conclude that t in (39) is indeed a function of x_0. There is, however, a fundamental limitation in carrying out this induction reasoning for an infinite number of jumps. In the induction step mentioned above, the neighborhood of α_0 for which t_j is defined can, in principle, become smaller and smaller as j is increased and, in the worst case, ceases to exist if no other conditions are imposed (this corresponds to a situation where the intersection of an infinite number of open sets containing α_0 just ends up in the closed set containing just the point itself). This is why, with the given assumptions, the statement of this lemma holds just for any finite value of T, but not for $T = \infty$. We will explain in the proof of Theorem 1 how to overcome this limitation by adding an extra condition on the solutions of the *cl*-LTTHS. This concludes the first part of the proof.

We now prove the existence of $\delta_0(T) > 0$ to satisfy the bracketing condition (34). As the functions $x_0 \mapsto t_j$ are continuous and we are considering the *finite* set $j \in \{1, \ldots, j_\alpha(T) - 1\}$, we can find a closed ball around α_0 with radius $\delta_0(T)$ that is

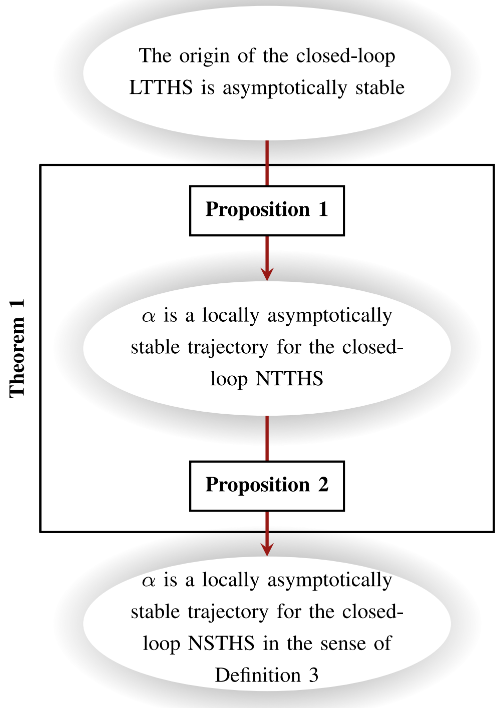

Fig. 5. Subdivision of Theorem 1 in two implication steps, by introducing the NTTHS and Propositions 1 and 2.

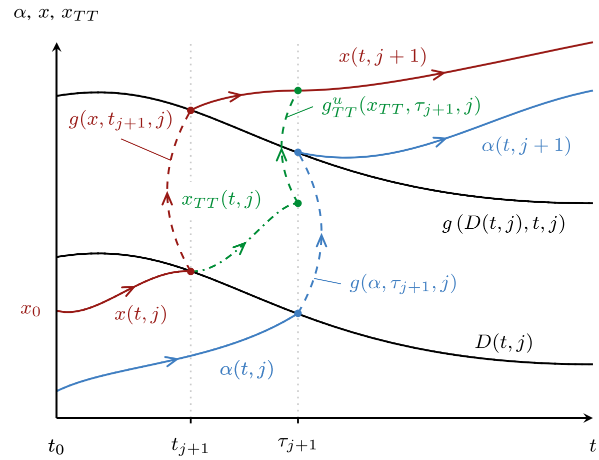

Fig. 6. Relation between trajectory $x(t, j)$ of the NSTHS, corresponding reference $\alpha(t, j)$, and trajectory $x_{\mathrm{TT}}(t, j)$ of the NTTHS for jump $(j + 1)$.

contained in each domain of definition of the functions $x_0 \mapsto t_j$ and such that each function $x_0 \mapsto t_j$ is contained in the interval $[\tau_{j-1}, \tau_{j+1}]$.

Appendix B
Proof of Theorem 1

The proof of Theorem 1 is divided in two steps. First, a *nonlinear time-triggered hybrid system* (NTTHS) related to the NSTHS will be introduced. The NTTHS has $\alpha(t, j)$ as a solution and the LTTHS is its linearization as shown in Appendix D. Fig. 5 schematically depicts the structure of the proof and the role of the NTTHS therein.

A. Nonlinear Time-Triggered Hybrid System

The NTTHS, introduced below, always jumps at the times τ_j of the reference α and, as will be clarified below, its solutions coincide with solutions of the NSTHS for continuous-times sufficiently far away from the reference jump times.

When we start with a state-input trajectory $(\alpha(t, j), \mu(t, j))$ of (2) and slightly change its initial condition or input, we typically obtain a trajectory $x(t, j)$ that jumps at different times than the reference jump times τ_j. This has been shown already schematically in Fig. 2. We illustrate this phenomenon in more detail in Fig. 6 focussing on a single jump. In constructing the NTTHS, the procedure is to replace the trajectory of (2) from initial condition $x(t_0, 0) = x_0$ between the time instances t_j and τ_j with a new trajectory that always jumps at time τ_j, as illustrated in Fig. 6, where the trajectory of the NTTHS is denoted as x_{TT}. This construction is related to the concept of zero-time discontinuity mapping and Poincaré discontinuity mapping in [29, Sec. 6.2]. The trajectory x_{TT} near τ_{j+1} is attained by flowing according to the vector field $f(x, u, t, j)$ similarly as the reference trajectory up to τ_{j+1} and after the time τ_{j+1} by flowing according to the vector field $f(x, u, t, j + 1)$. A suitable jump map is applied at the nominal event time τ_{j+1} such that it maps the trajectory $x_{\mathrm{TT}}(t, j)$ back to the trajectory $x(t, j)$ at the end of this time mismatch period ($[t_{j+1}, \tau_{j+1}]$ or $[\tau_{j+1}, t_{j+1}]$).

In order to define this jump map, we denote by $\varphi^u_j(t, \tau, x)$ the state evolution according to vector field f for jump counter j at time t with initial condition x at time τ and a given input curve $u(t, j)$. Note that $t \leq \tau$ implies integration backwards in time and that this operator is different for different input curves. The NTTHS, with state x_{TT}, is defined as follows.

Definition 7 (NTTHS): The NTTHS is given by

$$\begin{aligned} \dot{x}_{\mathrm{TT}} &= f(x_{\mathrm{TT}}, u(t, j), t, j), && (t, j) \in \operatorname{dom} \alpha \\ x^+_{\mathrm{TT}} &= g^u_{\mathrm{TT}}(x^-_{\mathrm{TT}}, t, j), && (t, j) \in E_\alpha \end{aligned} \tag{40}$$

with initial condition $x_{\mathrm{TT}}(t_0, 0) = x_0$, where $x^+_{\mathrm{TT}} = x_{\mathrm{TT}}(t, j + 1)$, $x^-_{\mathrm{TT}} = x_{\mathrm{TT}}(t, j)$, and the jump map $g^u_{\mathrm{TT}}(x_{\mathrm{TT}}, t, j)$, with $(t, j) \in E_\alpha$, is given by

$$\varphi^u_{j+1}\left(t, t_{j+1}, g(\varphi^u_j(t_{j+1}, t, x_{\mathrm{TT}}), t_{j+1}, j)\right) \tag{41}$$

where t_{j+1} is the $(j + 1)$th jump time of the solution $x(t, j)$ of the NSTHS (2) starting from the initial condition x_0. ∎

The jump map $g^u_{\mathrm{TT}}(x_{\mathrm{TT}}, \tau_{j+1}, j)$ can be defined whenever t_{j+1} is defined, i.e., when the trajectory x of the NSTHS with initial condition $x(t_0) = x_0$ and chosen input $u(t, j)$ will experience the $(j + 1)$th jump (see Fig. 6). This property is satisfied if x_0 is close enough to α_0 and follows from Lemma 1. More details are provided in the proof of Proposition 1.

Furthermore, since the NTTHS jumps at the same times as the reference, the time domain of its solution $x_{\mathrm{TT}}(t, j)$ is the same as that of the reference trajectory, i.e., dom $x_{\mathrm{TT}} =$ dom α, as illustrated in Fig. 7. This figure also illustrates the time-triggered

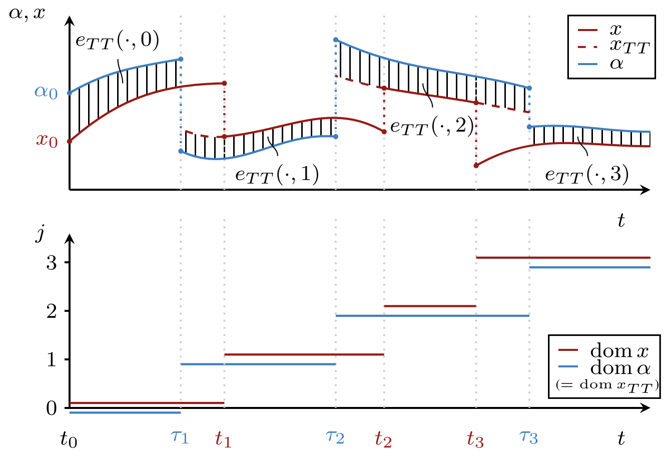

Fig. 7. Classical error for the time-triggered hybrid system ($e_{\mathrm{TT}}(t,j) = x_{\mathrm{TT}}(t,j) - \alpha(t,j)$) and the hybrid time domains of the corresponding required trajectories.

error $e_{\mathrm{TT}}(t,j) := x_{\mathrm{TT}}(t,j) - \alpha(t,j)$ (compare with Fig. 2) and their hybrid time domains, e_{TT} being the error for trajectories with jumps at fixed time instants.

When the input u is given by

$$u = \kappa(x_{\mathrm{TT}}, t, j) \tag{42}$$

as in (5), we obtain the closed-loop NTTHS (or *cl*-NTTHS)

$$\begin{aligned} \dot{x}_{\mathrm{TT}} &= f_{\mathrm{cl}}(x_{\mathrm{TT}}, t, j), && (t,j) \in \mathrm{dom}\,\alpha \\ x^+_{\mathrm{TT}} &= g_{\mathrm{TT}}(x^-_{\mathrm{TT}}, t, j), && (t,j) \in E_\alpha \end{aligned} \tag{43}$$

with $x_{\mathrm{TT}}(t_0, 0) = x_0$ and where the jump map $g_{\mathrm{TT}}(x_{\mathrm{TT}}, t, j)$ is given by

$$\varphi_{j+1}\left(t, t_{j+1}, g(\varphi_j(t_{j+1}, t, x_{\mathrm{TT}}(t,j)), t_{j+1}, j)\right) \tag{44}$$

with φ_j the closed-loop flow as in (41) with u as in (42). More specifically, the *cl*-NTTHS that we will use in the proof of Theorem 1 is the one employing the affine feedback law

$$u(t,j) = \mu(t,j) + K(t,j)(x_{\mathrm{TT}}(t,j) - \alpha(t,j)). \tag{45}$$

Note that (45) is the same as (32), but that in the latter no "bars" are needed on top of K and α as the controller will only be used for the hybrid times $(t,j) \in \mathrm{dom}\, x_{\mathrm{TT}} = \mathrm{dom}\,\alpha$.

B. Proposition 1 and Its Proof

As mentioned at the beginning of this appendix, a key property in the proof of Theorem 1 is that the linearization of the NTTHS in Definition 7 is the LTTHS provided by Definition 4. Given α, solutions to the NTTHS and LTTHS clearly have the same hybrid time domain. Here, we prove that uniform asymptotic stability of the closed-loop linearization implies that the reference trajectory $\alpha(t,j)$ is a locally asymptotically stable solution to the *cl*-NTTHS.

Proposition 1: A trajectory $(\alpha(t,j), \mu(t,j))$ of the NSTHS (6), (32) satisfying Assumptions 1, 2, 3, 4, 5, and 6, is an asymptotically stable trajectory of the *cl*-NTTHS (43), (45), if the associated *cl*-LTTHS (21)–(31) is uniformly asymptotically stable.

Proof: As both *cl*-LTTHS and *cl*-NTTHS are time-triggered and each jump event corresponds to a jump event of α, we can employ the jump counter $j_\alpha : \mathbb{R} \to \mathbb{N}$ of Definition 6 to simplify the notation within this proof. To this end, with a slight abuse of notation, we will write $\alpha(t)$, $z(t)$, $x_{\mathrm{TT}}(t)$, etc., to mean $\alpha(t, j_\alpha(t))$, $z(t, j_\alpha(t))$, $x_{\mathrm{TT}}(t, j_\alpha(t))$, etc. At the event times τ_j, we will write $\alpha^+(\tau_j)$ and $\alpha^-(\tau_j)$ to indicate $\alpha(\tau_j, j_\alpha(\tau_j))$ and $\alpha(\tau_j, j_\alpha(\tau_j) - 1)$, respectively. Similarly, we will employ the $+$ and $-$ notation for other signals.

Our goal is to conclude local asymptotic stability of α for the *cl*-NTTHS. To this end, let us consider the time-triggered error $e_{\mathrm{TT}}(t)$ between the state of the *cl*-NTTHS and the reference α. The error e_{TT} satisfies the hybrid dynamics

$$\begin{aligned} \dot{e}_{\mathrm{TT}} &= A_{\mathrm{cl}}(t)\, e_{\mathrm{TT}} + r_1(e_{\mathrm{TT}}, t) \\ e^+_{\mathrm{TT}} &= G(j)\, e^-_{\mathrm{TT}} + r_2(e^-_{\mathrm{TT}}, j), \qquad \text{for } t = \tau_1, \tau_2, \ldots \end{aligned} \tag{46}$$

with $j = j_\alpha(t) - 1$, $e_{\mathrm{TT}}(t_0) = z_0$, and where the matrices A_{cl} and G and the residuals r_1 and r_2 are defined below. In (46), e_{TT} is obtained by alternating state resets according to the jump map with integrations of the ODE until t equals τ_{j+1}. In (46), $A_{\mathrm{cl}}(t) := A(t, j_\alpha(t)) + B(t, j_\alpha(t))K(t, j_\alpha(t))$ and G is given by (24), therefore corresponding to the *cl*-LTTHS. The residuals r_1 and r_2 are the higher-order terms of the vector field and jump map of the time-triggered error dynamics associated with the *cl*-NTTHS, namely

$$\begin{aligned} r_1(e_{\mathrm{TT}}, t) &:= f_{\mathrm{cl}}(\alpha(t) + e_{\mathrm{TT}}, t, j_\alpha(t)) \\ &\quad - f_{\mathrm{cl}}(\alpha(t), t, j_\alpha(t)) - A_{\mathrm{cl}}(t) e_{\mathrm{TT}} \end{aligned} \tag{47}$$

$$\begin{aligned} r_2(e_{\mathrm{TT}}, j) &:= g_{\mathrm{TT}}(\alpha^-(\tau_{j+1}) + e^-_{\mathrm{TT}}, \tau_{j+1}, j) \\ &\quad - g_{\mathrm{TT}}(\alpha^-(\tau_{j+1}), \tau_{j+1}, j) - G(j) e^-_{\mathrm{TT}}. \end{aligned} \tag{48}$$

The origin $e_{\mathrm{TT}} = 0$ is an equilibrium point for (46) and for the *cl*-LTTHS. Both these hybrid systems jump at the same fixed time instants τ_j, known in advance. For such class of systems, denoting generically the system state with y, uniform asymptotic stability implies that for an arbitrary $\varepsilon_{\mathrm{TT}} > 0$ and $t_S \geq t_0$ there exists a δ_{TT} such that $\|y(t_S)\| < \delta_{\mathrm{TT}}$ implies $\|y(t)\| < \varepsilon_{\mathrm{TT}}$ for all $t \geq t_S$ and that, furthermore, $\lim_{t\to\infty} |y(t)| = 0$ (see, e.g., [31, Sec. 3.1]). We aim to show that uniform asymptotic stability of the *cl*-LTTHS implies that the origin $e_{\mathrm{TT}} = 0$ is uniformly locally asymptotically stable for (46). Let us now consider two cases.

Case 1: Consider first the case where the number of events $J_\alpha - 1$ is infinite. Denote with $L_\kappa > 0$ the Lipschitz constant for which κ in (32) satisfies $\|\kappa(x,t,j) - \kappa(y,t,j)\| < L_\kappa \|x - y\|$, $\forall (t,j) \in \overline{I}_\alpha$, that is $L_\kappa = \sup_{(t,j)\in\overline{I}_\alpha} \|\overline{K}(t,j)\|$. From Assumption 5, for each time interval $I^j_\alpha = [\tau_j, \tau_{j+1}]$, $j = 0, 1, 2, \ldots$, the growth of the solution can be bounded (see, e.g., [34, Cor. 6.4]) according to

$$\|y(t)\| \leq \exp(L_{\mathrm{cl}}(t - \tau_j))\|y^+(\tau_j)\|, \quad \tau_j \leq t \leq \tau_{j+1} \tag{49}$$

with $L_{\mathrm{cl}} = L(1 + L_\kappa)$. Clearly, we require $\|y^+(\tau_j)\|$ to be also sufficiently small to keep $\|y(t)\|$ within the region where the vector field f is uniformly Lipschitz. When the continuous dynamics satisfies an exponential bound as in (49), asymptotic

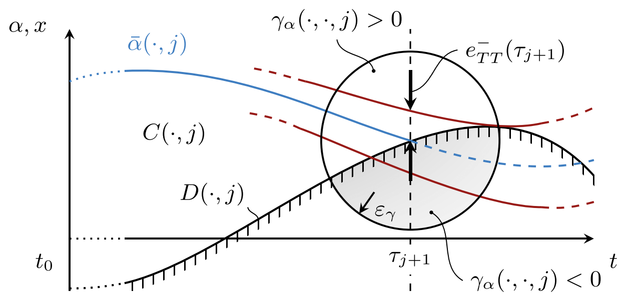

Fig. 8. Illustration of the fact that the transversality assumption (see Assumption 3) implies a relation between state error and event time mismatch.

stability of a time-triggered hybrid system is equivalent to the asymptotic stability of the discrete time system that originates from only considering the state at the event times τ_j [35]. Recall that $\tau_{j+1} - \tau_j$ is uniformly bounded from below and from above (bounded interjump time) due to Assumption 1.

In our case, the local asymptotic stability of the *cl*-NTTHS follows directly from uniform asymptotic stability (for linear systems, equivalent to exponential stability) of the *cl*-LTTHS, by considering their respective associated discrete time systems at the nominal event times. This statement is based on the discrete time version of [30, Th. 4.13]. This concludes the proof for the case where $J_\alpha = \infty$.

Case 2: When the number of events $J_\alpha - 1$ is finite, it suffices to consider the stability of the continuous time dynamics after the last event. This is valid because, from (49) and continuity of g_{TT} with respect to x_{TT}, an exponential bound on the error $\|e_{\mathrm{TT}}\|$ of the *cl*-NTTHS in the finite time interval $[t_0, \tau_{J_\alpha - 1}]$ can be found (in particular, for every $\varepsilon > 0$ and $t_0 \leq t_S \leq \tau_{J_\alpha-1}$ there is a $\delta > 0$ such that $\|e_{\mathrm{TT}}(t_S)\| < \delta$ implies $\|e_{\mathrm{TT}}(\tau_{J_\alpha-1})\| < \varepsilon$). It, thus, suffices to consider the last flow phase when $t \geq \tau_{J_\alpha-1}$.

As the *cl*-LTTHS is, by assumption, exponentially stable and it is the linearization of the error system (46), specifically in the time interval $[\tau_{J_\alpha-1}, \infty)$, the origin of the error system is (locally) asymptotically stable [30, Th. 4.13] and, equivalently, α is an asymptotically stable trajectory of the *cl*-NTTHS. This concludes the proof for finite J_α.

t-completeness: So far, it has been tacitly assumed that the solution of the *cl*-NTTHS exists for all $t > t_0$. However, one has actually to prove this result based on the assumptions in the proposition statement. For the *cl*-NTTHS to be defined, the jump map g_{TT} needs to be defined for all jumps, which requires that the *cl*-NSTHS corresponding to the *cl*-NTTHS experiences those jumps as well as discussed after (41).

From the proof of Lemma 1 and, in particular, from the continuity of the map $t_j(x_0)$, the jth jump of the *cl*-NSTHS occurs close to the jth jump time of α as long as $\|e_{\mathrm{TT}}(t_0)\|$ is chosen sufficiently small (equivalently, $x_{\mathrm{TT}}(t_0, 0)$ is chosen sufficiently close to α_0). This property is illustrated in Fig. 8. However, a uniform (i.e., independent of j) neighborhood is not guaranteed to exist when α has an infinite number of jumps (cf., proof of Lemma 1). Assuming the exponential stability of the *cl*-LTTHS, though, guarantees the existence of such a uniform neighborhood.

We start by noting that the jth time-triggered jump map of (46) is defined as long as $e^-_{\mathrm{TT}}(\tau_j)$ is sufficiently small, following from the fact that the jth jump occurs for the original *cl*-NSTHS if $x(t, j)$ is sufficiently close to $\alpha(t, j)$ as discussed in the proof of Lemma 1. $\|e^-_{\mathrm{TT}}(\tau_j)\|$ is required to be smaller than ε_γ, for example (cf., Assumption 2). Equivalently, the jth time-triggered jump map $g_{\mathrm{TT}}(\cdot, j)$ of the *cl*-NTTHS is defined as long as $x^-_{\mathrm{TT}}(\tau_j)$ is sufficiently close to $\alpha^-(\tau_j) := \alpha(\tau_j, j-1)$. The proof for the existence of a neighborhood about the origin for $e^-_{\mathrm{TT}}(\tau_j)$ where the time-triggered jump map is defined follows again from a straightforward application of the implicit function theorem employing the guard function $\gamma_\alpha(\cdot,\cdot,j)$. In particular, one can show that when $\|e^-_{\mathrm{TT}}(\tau_j)\| = \|x^-_{\mathrm{TT}}(\tau_j) - \alpha^-(\tau_j)\|$ is sufficiently small, then $|t_{j,\mathrm{TT}} - \tau_j| < S_j\|e^-_{\mathrm{TT}}(\tau_j)\|$ where the function $t_{j,\mathrm{TT}} := t_{j,\mathrm{TT}}(e)$ represents the event time of the *cl*-NSTHS corresponding to the time-triggered error trajectory with value $e = e^-_{\mathrm{TT}}(\tau_j)$ at τ_j and S_j a constant that depends on the continuous-time closed-loop vector field f_{cl}, guard function γ_α, and the reference trajectory α.

In details, the function $t_{j,\mathrm{TT}}(e)$ is implicitly defined as $H_j(t, e) := \gamma_\alpha(\varphi_{j-1}(t, \tau_j, \alpha(\tau_j, j-1) + e), t, j-1) = 0$ and Assumption 3 provides the sufficient condition for the validity of the application of the implicit function theorem. Again, note that due to Assumption 6, the flow φ_{j-1} in H_j is defined as long as it is sufficiently close to the reference trajectory. Finally, because $t_{j,\mathrm{TT}}(e) = \tau_j + \mathbf{D}t_{j,\mathrm{TT}}(0) \cdot e + o(|e|)$, by choosing $S_j > \|\mathbf{D}t_{j,\mathrm{TT}}(0)\| > 0$, one obtains $|t_{j,\mathrm{TT}} - \tau_j| < S_j\|e\|$ for sufficiently small e. A uniform bound on the maximum time shift of the form $|t_{j,\mathrm{TT}} - \tau_j| < S\|e\|$, valid for all (possibly infinite) j, follows from the uniform bound on the derivative of γ_α in Assumption 2. It is straightforward to show that $\|\mathbf{D}_1 t_{j,\mathrm{TT}}(0)\| = \|\mathbf{D}_1 H_j(\tau_j, 0)\|^{-1}\|\mathbf{D}_2 H_j(\tau_j, 0)\| \leq c_1/c_2$ (see Assumptions 2 and 3).

From the above, one concludes that as long as $e_{\mathrm{TT}}(t)$ remains within a predetermined sufficiently small neighborhood of the origin, then the (possibly infinite number of) jump maps g_{TT} are all defined (implying $J_x \geq J_\alpha$), with the difference between the jump times t_j of the *cl*-NSTHS and the nominal jump times τ_j becoming smaller as this neighborhood is chosen smaller. Uniform asymptotic stability of the *cl*-LTTHS allows then to conclude that $e_{\mathrm{TT}}(t)$, will remain within such a predetermined, sufficiently small, neighborhood where all jump maps g_{TT} are defined as long as the initial condition $e_{\mathrm{TT}}(t_0)$ is chosen sufficiently close to zero, ensuring the existence of $e_{\mathrm{TT}}(t)$ for $t \to \infty$. This concludes the proof of the proposition. □

C. Proposition 2 and Its Proof

Proposition 2: Let $(\alpha(t, j), \mu(t, j))$ be a trajectory of the NSTHS (6) satisfying Assumptions 1, 2, 3, 4, 5, and 6. If $\alpha(t, j)$ is an asymptotically stable solution of the *cl*-NTTHS (43), (45), then it is also an asymptotically stable solution of the *cl*-NSTHS (6), (32), in the sense of Definition 3.

Proof: By construction, except in suitable neighborhoods of the event times τ_j, the solution of the *cl*-NTTHS is identical, when it exists, to the solution of the *cl*-NSTHS. As α is assumed to be asymptotically stable for the *cl*-NTTHS, the solution of the *cl*-NTTHS exists for all $t \geq t_0$ as long as the corresponding initial condition is taken sufficiently close to $\alpha(t_0, 0)$. This follows, in particular, from the uniform bound:

$$|t_j - \tau_j| < S \|e_{\mathrm{TT}}(\tau_j, j-1)\| = S \|x_{\mathrm{TT}}(\tau_j, j-1) - \alpha(\tau_j, j-1)\| \quad (50)$$

obtained in the proof of Proposition 1, with S a suitable strictly positive constant. From the proof of Proposition 1 and Assumption 6 furthermore follows that, if the initial condition of the *cl*-NSTHS is taken sufficiently close to $\alpha(t_0, 0)$, it is guaranteed that the number of events $J_x - 1$ of the *cl*-NSTHS (equivalently, of the *cl*-NTTHS) equals the number of events $J_\alpha - 1$ of the reference. The equality $J_x = J_\alpha$ is the first condition that α has to satisfy for being locally asymptotically stable for the *cl*-NSTHS, in the sense of Definition 3.

What is left to be shown in this proof is, therefore, that the other two conditions of Definition 3 are fulfilled. Namely, guaranteeing that for an arbitrary $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$ such that $\|x(t_0, 0) - \alpha(t_0, 0)\| < \delta$ implies $|t_j - \tau_j| < \varepsilon$ and $\|x(t, j) - \overline{\alpha}(t, j)\| < \varepsilon$ for all $(t, j) \in \mathrm{dom}\ x$ and that both quantities converge to zero as $t \to \infty$. We will show that taking $\varepsilon_{\mathrm{TT}} = \varepsilon_{\mathrm{TT}}(\varepsilon)$ such that

$$\max\left(\varepsilon_{\mathrm{TT}}, S\varepsilon_{\mathrm{TT}}, \exp(L_{\mathrm{cl}} S \varepsilon_{\mathrm{TT}})\varepsilon_{\mathrm{TT}}\right) \leq \varepsilon \quad (51)$$

with $S > 0$ as in (50) and $L_{\mathrm{cl}} = L(1 + L_\kappa) > 0$ as in (49) (cf., Assumption 5) and $\delta := \delta_{\mathrm{TT}}(\varepsilon_{\mathrm{TT}}(\varepsilon))$ will indeed satisfy the above stability conditions.

First, for a given ε, if $\varepsilon_{\mathrm{TT}}$ is chosen according to (51), then local asymptotic stability of α for the *cl*-NTTHS ensures the existence of a neighborhood of $\alpha(t_0, 0)$ containing a ball of radius $\delta = \delta(\varepsilon)$ such that the corresponding trajectories of the *cl*-NSTHS satisfy the bound $|t_j - \tau_j| < S\,\varepsilon_{\mathrm{TT}} \leq \varepsilon$, as requested by Definition 3.

Second, from (50) and (51), and local asymptotic stability of the *cl*-NTTHS, one concludes immediately that $\|x_{\mathrm{TT}}(t, j) - \alpha(t, j)\| < \varepsilon_{\mathrm{TT}} \leq \varepsilon$ for every time interval I_j^α. However, in fact, we need to prove something stronger, namely that $\|x(t, j) - \overline{\alpha}(t, j)\| < \varepsilon$, for $t \in I_x^j = [t_{j-1}, t_j]$. In the simplest case, namely when $\tau_j \leq t_j$ and $t_{j+1} \leq \tau_{j+1}$, the condition is trivially satisfied on the jth time interval I_j^x, because there $x(t, j) = x_{\mathrm{TT}}(t, j)$ and $\overline{\alpha}(t, j) = \alpha(t, j)$. In general, one will have $t_j \leq \tau_j$ and/or $\tau_{j+1} \leq t_{j+1}$: indeed, if the simplest case occurs in jth interval, by construction, it will not occur in the $(j-1)$th and $(j+1)$th intervals. Fortunately, as in the proof of Proposition 1, we can make use of a uniform exponential bound on the growth of the solutions of locally Lipschitz vector fields, cf., (49). For a given initial condition $x(t_0, 0)$ within the ball of radius $\delta = \delta_{\mathrm{TT}}(\varepsilon_{\mathrm{TT}}(\varepsilon))$ centered at $\alpha(t_0, 0)$, if $\tau_{j+1} \leq t_{j+1}$, then we can bound the solutions of the *cl*-NSTHS for $t \in [\tau_{j+1}, t_{j+1}]$ as

$$\|x(t, j) - \overline{\alpha}(t, j)\| \leq \exp\left(L_{\mathrm{cl}}(t - \tau_{j+1})\right) \|e_{\mathrm{TT}}(\tau_{j+1}, j)\| \leq \exp\left(L_{\mathrm{cl}}(t - \tau_{j+1})\right) \varepsilon_{\mathrm{TT}} \quad (52)$$

where $L_{\mathrm{cl}} = L(1 + L_\kappa)$ follows from the upper bound L on the Lipschitz constants as in Assumption 5 and the upper bound L_κ on Lipschitz continuity of (32). Similarly, if $t_j \leq \tau_j$, then we can bound the solutions of *cl*-NSTHS for $t \in [t_j, \tau_j]$ as

$$\|x(t, j) - \overline{\alpha}(t, j)\| \leq \exp\left(L_{\mathrm{cl}}(\tau_j - t)\right) \|e_{\mathrm{TT}}(\tau_j, j)\| \leq \exp\left(L_{\mathrm{cl}}(\tau_j - t)\right) \varepsilon_{\mathrm{TT}}. \quad (53)$$

Recalling (50) and for each j, the two equations above, due to (51), finally lead to $\|x(t, j) - \overline{\alpha}(t, j)\| < \exp(L\,S\,\varepsilon_{\mathrm{TT}})\varepsilon_{\mathrm{TT}} \leq \varepsilon$ for $t \in I_x^j = [t_{j-1}, t_j]$, as required in Definition 3.

Convergence to zero of $\|t_j - \tau_j\|$ and $\|x(t, j) - \overline{\alpha}(t, j)\|$ for $t \to \infty$ follows from the convergence to zero of $\|x_{\mathrm{TT}}(\tau_j, j) - \alpha(\tau_j, j)\|$ as $j = j_x(t) \to \infty$. This concludes the proof. □

It can be concluded that the trajectory $\alpha(t, j)$ is an asymptotically stable solution of the *cl*-NSTHS (6), (32) if it is an asymptotically stable trajectory for the *cl*-NTTHS (43) (Prop. 2), which is the case if the origin of the closed-loop linearization (21)–(31) about that trajectory, also referred to as the *cl*-LTTHS, is uniformly asymptotically stable (Prop. 1). These two steps together form the proof of Theorem 1.

Appendix C
Relation to Existing Definition of Stability

In this appendix, we will present a link with existing literature on hybrid system stability analysis. More precisely, we will show that uniform asymptotic stability of the origin of the *cl*-LTTHS (21)–(31) implies asymptotic stability of the reference trajectory α with respect to a suitable distance function (cf., [8, Def. 1 and 2] and below), As this distance function is not designed to accommodate nonautonomous jump maps, flow sets, and jump sets, in this appendix, we consider state-triggered hybrid systems of the following form:

$$\dot{x} = f(x, u, t, j), \qquad x \in C \quad (54a)$$

$$x^+ = g(x^-), \qquad x^- \in D \quad (54b)$$

with vector field $f(x, u, t, j) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}^n$, jump map $g : D \to \mathbb{R}^n$, and constant flow and jump sets $C \subseteq \mathbb{R}^n$ and $D \subseteq \partial C$, respectively. Moreover, for the sake of brevity, when we refer to $x(t)$ and $\alpha(t)$ in this appendix, one should read $x(t, j_x(t))$ and $\alpha(t, j_\alpha(t))$, respectively, where j_x and j_α are given by Definition 6 in Section III-D for the trajectories x and α. Asymptotic stability in terms of the distance function d is defined as follows.

Definition 8 ([8, Definition 2]): The trajectory α is stable with respect to the distance d if for all $\varepsilon_d > 0$, there exists a $\delta_d(\varepsilon_d) > 0$ such that for every trajectory x of (32), (54) satisfying $d(x_0, \alpha_0) < \delta_d(\varepsilon_d)$, it holds that $d(x(t), \alpha(t)) < \varepsilon_d$ for all $t \geq t_0$. If it moreover holds that $d(x(t), \alpha(t)) \to 0$ for $t \to \infty$, then the trajectory α is asymptotically stable with respect to d. ■

We can now formulate the following result.

Corollary 1: Adopt Assumptions 1–6. Let a state-input trajectory (α, μ) be a solution to the *cl*-NSTHS (32), (54), starting from the initial condition $\alpha(t_0) = \alpha_0 \in \mathrm{int}\ C \setminus g(D)$. Let the closed-loop vector field furthermore satisfy $\|f_{\mathrm{cl}}(x, t)\| < F$ for some $F > 0$ and for all (x, t) in a neighborhood of the graph of α. If the origin of the associated *cl*-LTTHS (21)–(31) is uniformly asymptotically stable, then, given any uniformly continuous distance function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is compatible with (32), (54) in the sense of [8, Def. 1], the trajectory α of (32), (54) is asymptotically stable with respect to the distance d as given in Definition 8.

Proof: The proof of Corollary 1 can be found in [36, Appendix C.3]. □

Appendix D
Derivation of the LTTHS From the NTTHS

In this appendix, we show that the LTTHS (21)–(30) is the linearization of the NTTHS (40), (41). The hybrid dynamics of the NTTHS are described in terms of the vector field f of the original NSTHS (2) and the input dependent reset map g_{TT}^u given in (41), also graphically represented in Fig. 6. We now perturb the initial condition and input slightly from the reference state-input trajectory $(\alpha(t,j), \mu(t,j))$, that is, we take $x_{\mathrm{TT},\epsilon}(t_0, 0) = \alpha_0 + \epsilon z_0$ and $u_\epsilon(t,j) = \mu(t,j) + \epsilon v(t,j)$ for some initial state perturbation z_0, input perturbation $v(t,j)$ (that may be different for different j) and scalar perturbation parameter ϵ. Standard results can be used [30] to show that the trajectory of the NTTHS can be expanded in series with respect to ϵ as $x_{\mathrm{TT},\epsilon}(t,j) = \alpha(t,j) + \epsilon z(t,j) + o(\epsilon)$. It follows that $\dot{x}_{\mathrm{TT},\epsilon} = f(\alpha + \epsilon z + o(\epsilon), \mu + \epsilon v, t, j)$, for $(t,j) \in$ dom α, which can be expanded, in each of the intervals $[\tau_j, \tau_{j+1}]$, as

$$\begin{aligned}\dot{\alpha} + \epsilon \dot{z} + o(\epsilon) &= f(\alpha(t,j), \mu(t,j), t, j) \\ &\quad + \epsilon\, [\mathbf{D}_1 f(\alpha(t,j), \mu(t,j), t, j)\; z \\ &\quad + \mathbf{D}_2 f(\alpha(t,j), \mu(t,j), t, j)\; v] + o(\epsilon).\end{aligned}$$

Matching terms in the expansion allows to conclude that the flow dynamics of the linear approximation about the state-input trajectory $(\alpha(t,j), \mu(t,j))$ for each continuous time interval $[\tau_j, \tau_{j+1}]$ is given by

$$\begin{aligned}\dot{z} &= \mathbf{D}_1 f(\alpha(t,j), \mu(t,j), t, j)\; z \\ &\quad + \mathbf{D}_2 f(\alpha(t,j), \mu(t,j), t, j) v \\ &=: A(t,j) z + B(t,j) v\end{aligned}$$

that matches the expressions given for A and B in (22) and (23), respectively.

Next, for each jump time τ_{j+1}, we seek a relation between the states $z^-(\tau_{j+1}) := z(\tau_{j+1}, j)$ and $z^+(\tau_{j+1}) := z(\tau_{j+1}, j+1)$ that will eventually be equal to (24). Roughly speaking, this corresponds to the linearization of the jump map g_{TT}^u in (41). We follow a similar strategy in deriving the linear jump map as detailed in [19], but now fitted in the framework of hybrid time.

Consider the $(j+1)$th event with reference event time τ_{j+1} and corresponding event time $t_{j+1,\epsilon}$ of the NSTHS. Define $\Delta_\epsilon = t_{j+1,\epsilon} - \tau_{j+1}$, which is assumed to be small based on the fact that $x_{\mathrm{TT},\epsilon}$ and α are close to each other and Assumption 3. To make the derivation of (24) concise, below we denote the event time τ_{j+1} simply by τ. Furthermore, when we refer to $x_{\mathrm{TT},\epsilon}^-$, one should read $x_{\mathrm{TT},\epsilon}(\tau_{j+1}, j)$ and similarly $\alpha^- = \alpha(\tau_{j+1}, j)$, $\mu^- = \mu(\tau_{j+1}, j)$, and $z^- = z(\tau_{j+1}, j)$. Analogously, for the right limits at the time τ, we employ the notation $x_{\mathrm{TT},\epsilon}^+ = x_{\mathrm{TT},\epsilon}(\tau_{j+1}, j+1)$, $\alpha^+ = \alpha(\tau_{j+1}, j+1)$, $\mu^+ = \mu(\tau_{j+1}, j+1)$, and $z^+ = z(\tau_{j+1}, j+1)$.

The jump map g_{TT}^u in (41) for the event $j+1$ constitutes phases of flow to and from the event time $t_{j+1,\epsilon}$ that depends on ϵ and the choice of $v(t,j)$. These flows can be captured by extending the solution z, using the continuous time part of (21), to form $\overline{z}$ (where states for consecutive counter are related via a "to be defined" jump map) in a similar fashion as done for the reference trajectory α as explained in Section III-A. This allows us to form the extended trajectory of the NTTHS, which, due to the locally Lipschitz property of the vector field f, is the same as that of the original NSTHS, i.e., $\overline{x}_\epsilon(t,j) := \overline{x}_{\mathrm{TT},\epsilon}(t,j) = \overline{\alpha}(t,j) + \epsilon \overline{z}(t,j) + o(\epsilon)$. It follows that, instead of the jump map $x_{\mathrm{TT},\epsilon}^+ = g_{\mathrm{TT}}^u(x_{\mathrm{TT},\epsilon}^-, \tau, j)$, we can consider

$$\overline{x}_\epsilon(\tau + \Delta_\epsilon, j+1) = g(\overline{x}_\epsilon(\tau + \Delta_\epsilon, j), \tau + \Delta_\epsilon, j) \tag{55}$$

and use a series expansion to account for the difference in time. Again to keep notation concise, we append the states $\overline{\alpha}$, $\overline{x}_\epsilon$, and $\overline{z}$ with a superscript, i.e., $(\cdot)^{\lhd}$ to denote that it is the left limit for the event time $t_{j+1,\epsilon} = \tau_{j+1} + \Delta_\epsilon$, e.g., $\overline{z}^{\lhd} = \overline{z}(t_{j+1,\epsilon}, j)$. For the right limit, we employ the superscript $(\cdot)^{\rhd}$, e.g., $\overline{z}^{\rhd} = \overline{z}(t_{j+1,\epsilon}, j+1)$. For the state $\overline{x}_\epsilon^{\lhd}$, we find

$$\begin{aligned}\overline{x}_\epsilon^{\lhd} &= \overline{\alpha}^{\lhd} + \epsilon \overline{z}^{\lhd} + o(\epsilon) \\ &= \alpha^- + \dot{\alpha}^- \Delta_0' \epsilon + (z^- + \dot{z}^- \Delta_0' \epsilon)\epsilon + o(\epsilon) \\ &= \alpha^- + (\dot{\alpha}^- \Delta_0' + z^-)\epsilon + o(\epsilon)\end{aligned} \tag{56}$$

where $\Delta_0' = \frac{\partial \Delta_\epsilon}{\partial \epsilon}|_{\epsilon=0}$ and $\dot{\alpha}^- = f(\alpha^-, \mu^-, \tau, j)$ and similarly that

$$\begin{aligned}\overline{x}_\epsilon^{\rhd} &= \overline{\alpha}^{\rhd} + \epsilon \overline{z}^{\rhd} + o(\epsilon) \\ &= \alpha^+ + (\dot{\alpha}^+ \Delta_0' + z^+)\epsilon + o(\epsilon)\end{aligned} \tag{57}$$

where $\dot{\alpha}^+ = f(\alpha^+, \mu^+, \tau, j+1)$. These expansions make use of the dependence of the jump time difference Δ_ϵ on ϵ. More precisely, they contain the derivative of Δ_ϵ with respect to ϵ evaluated at zero. From Assumption 2, it follows that for the reference jump at time τ holds that $\gamma_\alpha(\alpha^-, \tau, j) = 0$ since, by definition, the state will be in the jump set D. Similarly, the definition of $t_{j+1,\epsilon}$ implies $\gamma_\alpha(\overline{x}_\epsilon(t_{j+1,\epsilon}, j), t_{j+1,\epsilon}, j) = 0$, or formulated differently that $\gamma_\alpha(x_\epsilon^{\lhd}, \tau + \Delta_\epsilon, j) = 0$. Using (56), this can be rewritten as

$$\begin{aligned}&\gamma_\alpha\left(\alpha^-, \tau, j\right) + \mathbf{D}_1 \gamma_\alpha(\alpha^-, \tau, j)\left(\dot{\alpha}^- \Delta_0' + z^-\right)\epsilon \\ &\quad + \mathbf{D}_2 \gamma_\alpha(\alpha^-, \tau, j) \Delta_0' \epsilon + o(\epsilon) = 0.\end{aligned}$$

Since this needs to hold for all ϵ and since $\gamma_\alpha(\alpha^-, \tau, j) = 0$, it follows that

$$\Delta_0' = -\frac{\mathbf{D}_1 \gamma_\alpha(\alpha^-, \tau, j)}{\mathbf{D}_1 \gamma_\alpha(\alpha^-, \tau, j)\dot{\alpha}^- + \mathbf{D}_2 \gamma_\alpha(\alpha^-, \tau, j)} z^-. \tag{58}$$

Note that the transversality assumption (see Assumption 3) implies that the denominator in (58) is nonzero.

Incorporating (56) and (57) in (55) and expanding in series with respect to ϵ gives

$$\begin{aligned}\overline{x}_{\epsilon}^{\triangleright} &= \alpha^{+} + (\dot{\alpha}^{+}\Delta_0' + z^{+})\epsilon + o(\epsilon)\\ &= g(\alpha^{-} + (\dot{\alpha}^{-}\Delta_0' + z^{-})\epsilon + o(\epsilon), \tau + \Delta_t(\epsilon), j)\\ &= g(\alpha^{-}, \tau, j) + \mathbf{D}_1 g(\alpha^{-}, \tau, j)(\dot{\alpha}^{-}\Delta_0' + z^{-})\epsilon\\ &\quad + \mathbf{D}_2 g(\alpha^{-}, \tau, j)\Delta_0'\epsilon + o(\epsilon).\end{aligned}$$

Now we recall that $\alpha^{+} = g(\alpha^{-}, \tau, j)$ and match terms of ϵ to obtain $z^{+} = (-\dot{\alpha}^{+} + \dot{g}^{-})\Delta_0' + \mathbf{D}_1 g(\alpha^{-}, \tau, j)z^{-}$, where $\dot{g}^{-} = \mathbf{D}_1 g(\alpha^{-}, \tau, j)\dot{\alpha}^{-} + \mathbf{D}_2 g(\alpha^{-}, \tau, j)$. After incorporating (58), it follows that the linearized jump map satisfies

$$z^{+} = \left[(\dot{\alpha}^{+} - \dot{g}^{-}) \frac{\mathbf{D}_1 \gamma_\alpha(\alpha^{-}, \tau, j)}{\dot{\gamma}_\alpha^{-}} + \mathbf{D}_1 g(\alpha^{-}, \tau, j) \right] z^{-}$$

with $\dot{\gamma}_\alpha^{-} = \mathbf{D}_1 \gamma_\alpha(\alpha^{-}, \tau, j) \cdot \dot{\alpha}^{-} + \mathbf{D}_2 \gamma_\alpha(\alpha^{-}, \tau, j) \cdot 1$, which is equivalent to (24).

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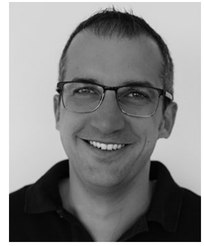

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