Multiple Faults Estimation in Dynamical Systems: Tractable Design and Performance Bounds

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***Abstract*—In this article, we propose a tractable nonlinear fault estimation filter along with explicit performance bounds for a class of linear dynamical systems in the presence of both additive and nonlinear multiplicative faults. We consider the case, where both faults may occur simultaneously and through an identical dynamical relationship, a setting that is relevant to several application domains, including automotive driving, aviation, and chemical plants. The proposed filter architecture combines tools from model-based approaches in the control literature and regression techniques from machine learning. To this end, we view the regression operator through a system-theoretic perspective to develop operator bounds that are then utilized to derive performance bounds for the proposed estimation filter. In the case of constant, simultaneously, and identically acting additive and multiplicative faults, it can be shown that the estimation error converges to zero with an exponential rate. The performance of the proposed estimation filter in the presence of incipient faults is validated through an application on the lateral safety systems of SAE level 4 automated vehicles. The numerical results show that the theoretical bounds of this study are indeed close to the actual estimation error.**

***Index Terms*—Convex optimization, fault estimation, regression.**

I. Introduction

Fault *detection* and *isolation* in dynamical systems are fundamental problems for safety–critical applications. In the detection task, the objective is to detect the presence of a fault in real-time while being insensitive to natural disturbances [1] and/or model uncertainty [2] to prevent false positives. Considering the occurrence of multiple faults at the same time, we typically refer to isolation as the task to identify, which one of the faults occurs. A classical approach toward isolation is to treat the problem as a special case of detection in which all the possible faults are viewed as natural disturbances. This methodology is found in a great variety of model settings, e.g., from single nonlinear systems [3] toward multiagent, possibly large-scale, systems [4]–[6]. A more integral approach for detection and isolation of faults is an unknown-input type estimator, which decouples the effect of unknown state measurements and disturbances (or faults) from the *residual* through an algebraic approach [7], [8] or approaches using the generalized inverse [9], [10].

A follow-up step to fault detection and isolation is fault-tolerance (or fault-resilience) control in which the objective is to counteract and mitigate the faults in real-time. To this end, *estimation* of the exact value of the fault signal is a vital aspect. When assuming a linear or linearized system description, additive faults can be estimated using standard system-theoretic tools [11]. When the fault is multiplicative, estimation is a more challenging task due to the nonlinear impact of the fault. A possible approach to deal with multiplicative dynamics is to borrow tools from the machine learning literature (e.g., regression [12]), or by reformulating multiplicative faults as additive faults [13]. The combined *estimation* problem of both additive and multiplicative faults, acting on the same system, can be considered a form of simultaneous state and parameter estimation. This problem is relevant in a broad range of application domains (e.g., automotive as illustrated later in this work, aviation [14] and chemical plants [15]), where actuators/sensors, which can inhibit simultaneously a bias (i.e., an additive fault) or loss-of-effectiveness (i.e., a multiplicative fault) [16], are used in safety–critical applications. This problem has been considered in several different settings, an example of which is the simultaneous appearance of multiplicative input faults and additive output faults [17], i.e., the faults are assumed to appear linearly independent. Other works consider additive and multiplicative faults acting through the same dynamical relationship (i.e., linearly dependent) [16]. The problem is, however, largely unexplored when both additive and multiplicative fault types act simultaneously in the system while assuming this linear dependence between the faults.

The central problem, defined and solved in this manuscript, is to *estimate* the fault signals (rather than only acknowledging/detecting their *presence*) in *real-time* when both additive and multiplicative faults are present and act *simultaneously* through *identical* dynamical relationships. Due to the dynamical inseparability of the additive and multiplicative faults, the estimates of such faults will, by definition, be affected by one another. It is, therefore, vital to determine an explicit performance bound that quantifies these effects. In this light, the following problem is the main focus of this study.

Problem: Consider a linear dynamical system with the available measurement signal z and the multivariate signal $f = [f_{\mathrm{a}}, f_{\mathrm{m}}]$ comprising possibly both additive (f_{a}) and multiplicative (f_{m}) faults that are not dynamically separable. We aim to design an estimation filter that turns the signal z to an estimation signal $\widehat{f} = [\widehat{f}_a, \widehat{f}_m]$ (i.e., a causal dynamic mapping $z \mapsto \widehat{f}$) such that the additive and multiplicative faults can be estimated separately, where the combined estimation error $\|f - \widehat{f}\|_2$ is bounded by

$$\|f(k) - \widehat{f}(k)\|_2 \leq \mathcal{C}(C_z, C_f, k - k_0) \tag{1}$$

where the constant $\mathcal{C}$ is an explicit bound depending on the dynamical model, the parameters C_z and C_f representing characteristics of the measurement z and fault signals f, and the time difference $k - k_0$ in which k_0 denotes the discrete starting sample of the fault signal f and k

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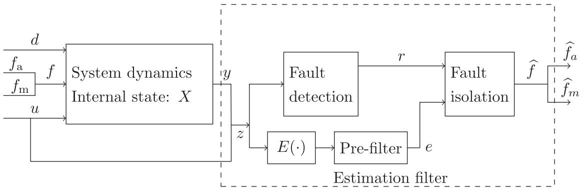

Fig. 1. Block diagram of the proposed estimation filter.

is the current time instance. The signals characteristics can, for instance, include the information of the average and variance of the respective signals.

Let us emphasize that a performance bound in the form of (1) provides a real-time estimation error for every single element of the multivariate signal f.

Our Contributions: The distinct feature of the problem above that makes it particularly challenging is the combination of three aspects: (i) real-time estimation of a multivariate fault signal, (ii) the presence of inseparable[1] additive and multiplicative faults, and hence a (particular) form of nonlinear dynamics, (iii) explicit, rigorous performance bounds for fault detection. To our best of knowledge, no approach in the existing literature addresses all these aspects at the same time. Our proposed solution method leverages concepts from the system theory literature concerning the aspects (i) and (iii) while using tools from the machine learning literature to deal with the aspect (ii). This combination yields an estimation filter with three components as depicted in Fig. 1. More specifically, the technical contributions of this work are summarized as follows.

(i) We develop system-theoretic (error) bounds for the regression operator, a well-known scheme borrowed from the machine learning literature (see Proposition 3.3). These bounds are crucial to quantify the performance of the proposed estimation filter.

(ii) We propose a general estimation architecture as in Fig. 1 that comprises three intertwined components. When the component prefilter is a simple identity operation, we develop an explicit, computable performance bound in terms of the average and variance of the fault signals (see Theorem 3.5). In the special case of constant faults, the proposed performance bound provides insight regarding the convergence and boundedness of the estimation error (see Corollary 3.6).

(iii) Building on the insight obtained from the performance of the static prefilter, we propose an alternative design utilizing a dynamic prefilter and develop the corresponding performance bound (see Theorem 3.7). We further show that in the special case of constant fault the estimation error decays to zero exponentially fast (see Corollary 3.8).

Furthermore, we also develop two technical results concerning the output bounds of linear time-invariant systems with zero steady-state gain (see Lemma 3.4) and the variance of product signals (see Lemma 4.1) that facilitate the proof of the main results highlighted above. While these results admittedly seem standard, we, however, did not find them in the literature in the present form as needed for the main results of this study. The proof of these lemmas is relegated to an extended version of this work [18] due to the space limitation.

The rest of this article is organized as follows. Section II introduces a detailed problem description and challenges of the research topic; furthermore, an outline of the proposed approach is given. Following the problem description, the main theoretical results of the work are given in Section III. The theoretical results are backed by technical proofs, which are given in Section IV. In Section V, the theoretical results are accompanied by numerical simulations, showing the contributions of the set of developed theorems in more practical daylight. Finally, Section VI concludes this article.

Notation: The symbols $\mathbb{N}$ and $\mathbb{R}$ represent the set of integers and real numbers and the symbol $\mathbb{R}_+$ represents the set of nonnegative real numbers. The ones column vector with the length n is denoted by $\mathbb{1}_n := [1, 1, \ldots, 1]^\intercal$. The p-norm of a vector v is denoted by $\|v\|_p$, where $p \in [1, \infty]$. Given a square matrix A with strictly real eigenvalues, we denote by $\overline{\lambda} \in \mathbb{R}$ and $\underline{\lambda} \in \mathbb{R}$ the maximum and minimum eigenvalue values of the matrix, respectively. Given a matrix $A \in \mathbb{R}^{n \times m}$, its transpose is denoted by $A^\intercal \in \mathbb{R}^{m \times n}$, the norm $\|A\|_2 = \overline{\sigma}(A) = \sqrt{\overline{\lambda}(A^\intercal A)}$ is the largest singular value, and $A^\dagger := (A^\intercal A)^{-1} A^\intercal$ is the pseudoinverse. Given two matrices with an equal dimension $A, B \in \mathbb{R}^{m \times n}$, the operator $A \circ B \in \mathbb{R}^{m \times n}$ denotes the element-wise (also known as Hadamard) product of two matrices. The operators $\mu_n[x]$ and $V_n[x]$ map $\mathbb{R}$-valued discrete-time signals to $\mathbb{R}$-valued discrete-time signals, and are defined as the first moment $\mu_n[x](k) := \frac{1}{n}\sum_{i=0}^{n-1} x(k-i)$ and the centered second moment $V_n^2[x](k) := \frac{1}{n}\sum_{i=0}^{n-1} x^2(k-i) - \mu_n^2[x](k)$ of the signal x over the last n time instants. Throughout this study, we reserve the bold subscripted by n $\mathbf{x}_n$ as the concatenated version of the signal x over the last n time instants: $\mathbf{x}_n(k) := \Big[x(k), x(k-1), \ldots, x(k-n+1)\Big]^\intercal$. The symbol $\mathfrak{q}$ represents the shift operator, i.e., $\mathfrak{q}[x(k)] = x(k+1)$.

II. Problem Description and Outline of the Proposed Approach

In this section, a formal description of the generic model class along with the basic principles of existing FDI schemes is given. Using this class of models, a high-level problem can be formulated. We further elaborate on the challenges and shortcomings of the current literature. Finally, an outline of the proposed solution is provided, addressing the challenges in the preceding parts.

A. Model Description

Throughout this study, we consider dynamical systems described via a discrete-time nonlinear differential-algebraic equation (DAE) of the form

$$H(\mathfrak{q})[x] + L(\mathfrak{q})[z] + F(\mathfrak{q})\,[f_{\mathrm{a}} + E(z) f_{\mathrm{m}}] = 0 \tag{2}$$

where $x, z, f_{\mathrm{a}}, f_{\mathrm{m}}$ represent discrete-time signals, indexed by the counter k [e.g., $x(k)$], taking values in $\mathbb{R}^{n_x}, \mathbb{R}^{n_z}, \mathbb{R}^{n_f}$, respectively. The mapping $E : \mathbb{R}^{n_z} \to \mathbb{R}^{n_E}$ is a static algebraic mapping capturing the nonlinearity of the fault dynamics, which is assumed known and, depending on the application, can be obtained through first-principle modeling (see Section V). The dependency on the signal z of the mapping E can be extended to other unknown signals x through the use of additional state estimators. Let n_r represent the number of rows in (2), and the matrices $H(\mathfrak{q}), L(\mathfrak{q}), F(\mathfrak{q})$ are polynomial functions with n_r rows and n_x, n_z, n_f columns in the variable $\mathfrak{q}$, which represents the shift operator. As such, these matrices may be cast as linear operators in the space of discrete-time signals. The signal x contains all unknown signals in the DAE system, typically comprising the internal states and unknown exogenous disturbances. The signal z is composed of all known signals, including the control inputs u and the output measurements y. The signal f_{a} represents an additive fault while the signal f_{m} is considered to be a multiplicative fault or intrusion, which interacts nonlinearly with the signal $E(z)$. The overall contribution of both fault signals can then be seen in the term $f_{\mathrm{a}} + E(z) f_{\mathrm{m}}$, to which we may refer as the "*aggregated fault signal*" hereafter. Note that, for the sake of generality in this work, the location of the faults f_{a} and f_{m} is not restricted to any particular location and hence could represent among others the notions of, e.g., sensor faults or actuator faults as widely adopted in the FDI literature.

The modeling framework (2) encompasses a large class of dynamical systems. A motivating example to show its level of generality is the set

[1] See Section II-B for a more precise definition of this terminology.

of nonlinear ordinary difference equations (ODE) described by

$$\begin{cases} GX(k+1) = AX(k) + B_u u(k) + B_d d(k) \\ \qquad +B_f\left(f_\mathrm{a}(k) + E_X\left(B_X X(k), u(k)\right) f_\mathrm{m}(k)\right) \\ y(k) = CX(k) + D_u u(k) + D_d d(k) \\ \qquad +D_f\left(f_\mathrm{a}(k) + E_Y\left(B_Y X(k), u(k)\right) f_\mathrm{m}(k)\right) \end{cases} \tag{3}$$

where u is the input signal, d the unknown exogenous disturbance, X the internal state of the system, Y the measurable output, f_a the additively acting set of faults or intrusions, and finally f_m the set of faults acting as a multiplication on a nonlinear combination of the internal states and input. The matrices G, A, B_u, B_d, B_f, B_X, B_Y, C, D_u, D_d, and D_f are constant matrices with appropriate dimensions. The following fact provides a simple-to-check condition under which the ODE model (3) falls into the category of our nonlinear DAE model (2).

Fact 2.1 (From ODE to DAE): Consider the ODE (3) and suppose there exist matrices K_X, K_Y such that

$$\begin{cases} B_X = K_X C,\ K_X D_f = 0,\ K_X D_d = 0 \\ B_Y = K_Y C,\ K_Y D_f = 0,\ K_Y D_d = 0. \end{cases} \tag{4}$$

Then, the ODE model can be viewed as a DAE model (2) by introducing

$$x = \begin{bmatrix} X \\ d \end{bmatrix},\ z = \begin{bmatrix} y \\ u \end{bmatrix},\ E(z) = \begin{bmatrix} E_X(K_X(Y - D_u u), u) \\ E_Y(K_Y(Y - D_u u), u) \end{bmatrix}$$

$$H(\mathfrak{q}) = \begin{bmatrix} -\mathfrak{q}G + AB_d \\ C \quad D_d \end{bmatrix},\ L(\mathfrak{q}) = \begin{bmatrix} 0 & B_u \\ -I & D_u \end{bmatrix},\ F(\mathfrak{q}) = \begin{bmatrix} B_f \\ D_f \end{bmatrix}.$$

Note that from a computational point of view checking the existence condition in (4) is a linear programming (LP) problem, which can be certified highly efficiently.

Throughout this study, the following assumption holds, which serves as a necessary and sufficient condition for the detectability of the aggregated fault signal $f_\mathrm{a} + E(z)f_\mathrm{m}$ in (2).

Assumption 2.2 (Detectability): The polynomial matrices $H(\mathfrak{q})$ and $F(\mathfrak{q})$ in (2) satisfy the rank condition $\mathrm{Rank}\left\{[H(\mathfrak{q}), F(\mathfrak{q})]\right\} > \mathrm{Rank}\left\{H(\mathfrak{q})\right\}$. For simplicity of the exposition, we further assume that $F(\mathfrak{q})$ is a polynomial column vector, i.e., $n_{f_\mathrm{a}} = n_{f_\mathrm{m}} = 1$.

Assumption 2.2 paves the way to acknowledge whether the aggregated fault signal is nonzero. However, differentiating the exact contribution between additive fault f_a and the multiplicative fault f_m introduces challenges that we shall discuss in the following section.

B. Current Approach and Open Challenges

In order to design a residual generator for the system (2), fulfilling the desired conditions of fault detection, it suffices to introduce a linear filter polynomial $N(\mathfrak{q})$, which can be characterized through the following polynomial arguments:

$$N(\mathfrak{q})H(\mathfrak{q}) = 0 \tag{5a}$$

$$N(\mathfrak{q})F(\mathfrak{q}) \neq 0. \tag{5b}$$

The first condition (5a) is concerned with the rejection of the natural disturbances and the unknown states, while the second condition (5b) ensures a nonzero response of the residual generator when the fault is nonzero. In the light of Assumption 2.2, we restrict our attention to a proper LTI estimation filter of the form

$$r := a^{-1}(\mathfrak{q})N(\mathfrak{q})L(\mathfrak{q})[z] \tag{6}$$

where the polynomial row vector $N(\mathfrak{q})$ fulfills the requirements (5), and the stable transfer function $a^{-1}(\mathfrak{q})$ is intended to make the residual generator proper [i.e., the degree of $a(\mathfrak{q})$ is not less than the degree of $N(\mathfrak{q})L(\mathfrak{q})$ and stable (i.e., all the zeros of the polynomials $a(\mathfrak{q})$ reside inside in the unit circle]. Following the definition of the residual (6) and the DAE model (2), it holds that the mapping from the signals f_a, f_m to the residual r can be described by

$$r = \mathcal{T}\left[E(z)f_\mathrm{m} + f_\mathrm{a}\right], \quad \text{where} \quad \mathcal{T} := -\frac{N(\mathfrak{q})F(\mathfrak{q})}{a(\mathfrak{q})}. \tag{7}$$

A typical approach to isolate multiple faults (f_a, f_m) from one another is to introduce all the faults but one as natural disturbances encoded in the signal d. However, this technique fails for the DAE systems of the form (2) since Assumption 2.2 does no longer hold in that case. In fact, by virtue of (7), one can see that the residual r is linearly dependent on both fault signals f_a, f_m. Due to this linear dependency, the residual can only be sensitive to the aggregated fault signal $f_\mathrm{a} + E(z)f_\mathrm{m}$ and it is not possible to isolate this combination utilizing linear filters, an important scenario, which we define in this work as *dynamical inseparability*. This is the central fault isolation challenge studied in this work.

C. Outline of the Proposed Approach

As mentioned in the preceding section, the key challenge of fault isolation is to estimate the additive fault f_a and multiplicative fault f_m when their impact on the dynamics [i.e., the corresponding dynamic matrix $F(\mathfrak{q})$] are linearly dependent. In this study, we aim to address this challenge by leveraging tools from the regression theory, a well-known concept from the Machine learning literature [19]. However, to integrate those tools in a dynamical system setting and provide rigorous performance guarantees, it is required to view these tools from a system-theoretic perspective and treat them as a dynamical system. This is the main part of the focus of this study.

More specifically, our proposed "*estimation filter*" comprises three blocks, see Fig. 1. The first block is called "*fault detection*" and its role is to estimate the aggregated signal $f_\mathrm{a} + E(z)f_\mathrm{m}$. This is essentially adopted from the current literature of fault detection with a slight extension that the residual signal r is expected to estimate the behavior of $f_\mathrm{a} + E(z)f_\mathrm{m}$ (rather than only acknowledging the existence of a fault). We call the second block "*fault isolation*" that aims at isolating and estimating the contribution of the additive fault signal f_a and the multiplicative one f_m. This block is essentially a (nonlinear) regression operator that also receives an additional signal e, a required regressor signal containing the information of $E(z)$. As we will discuss in detail later, the dynamics of the system [and as such the dynamics of $E(z)$] have a nontrivial impact on the performance of the fault isolation block. This effect motivates the inclusion of the third block, to which we refer as the "*prefilter*." With regards to the prefilter, we consider two cases in which one is a trivial identity (i.e., $e = E(z)$), and the second case is a linear transfer function with the input $E(z)$, aiming to compensate for the dynamical behavior between the true aggregated signal and the residual.

III. Estimation Filter Design: Main Results

As sketched in Fig. 1, the proposed estimation filter in this study comprises three blocks: 1) fault detection, 2) fault isolation, and 3) prefilter, which will be elaborated in detail in this section. Here, we only discuss the main results and their implications, and we will present the technical preliminaries and proofs in Section IV.

A. Fault Detection: Linear Residual Generators

The following lemma is a slight specialization of [8, Lemma 4.2] that characterizes the class of linear residual generators with a desired asymptotic behavior. In this refined lemma, a steady-state condition on the residual is introduced. This serves as the basis for the detection block whose main objective is to detect and track (i.e., estimate) the aggregated fault signal $f_\mathrm{a} + E(z)f_\mathrm{m}$.

Lemma 3.1 (LP Characterization of Fault Detection): Consider a polynomial row vector $N(\mathfrak{q}) = \sum_{i=0}^{d_N} N_i \mathfrak{q}^i$, and the system (2) with the model polynomial matrices

$$H(\mathfrak{q}) = \sum_{i=0}^{d_H} H_i \mathfrak{q}^i, \quad F(\mathfrak{q}) = \sum_{i=0}^{d_F} F_i \mathfrak{q}^i, \quad a(\mathfrak{q}) = \sum_{i=0}^{d_a} a_i \mathfrak{q}^i$$

where d_H, d_F, d_N, d_a denote the degree of matrices $H(\mathfrak{q})$, $F(\mathfrak{q})$, $N(\mathfrak{q})$, $a(\mathfrak{q})$, respectively. Let us define the matrices

$$\overline{H} := \begin{bmatrix} H_0 & H_1 & \dots & H_{d_H} & 0 & \dots & 0 \\ 0 & H_0 & H_1 & \dots & H_{d_H} & 0 & \vdots \\ \vdots & & \ddots & \ddots & & \ddots & 0 \\ 0 & \dots & 0 & H_0 & H_1 & \dots & H_{d_H} \end{bmatrix}$$

$$\overline{F} := \begin{bmatrix} F_0 & F_1 & \dots & F_{d_F} & 0 & \dots & 0 \\ 0 & F_0 & F_1 & \dots & F_{d_F} & 0 & \vdots \\ \vdots & & \ddots & \ddots & & \ddots & 0 \\ 0 & \dots & 0 & F_0 & F_1 & \dots & F_{d_F} \end{bmatrix}$$

$$\overline{N} := \begin{bmatrix} N_0 & N_1 & \dots & N_{d_N} \end{bmatrix}, \quad \overline{a} := \begin{bmatrix} a_0 & a_1 & \dots & a_{d_a} \end{bmatrix}.$$

Under Assumption 2.2, the linear program

$$\begin{cases} \overline{N}\,\overline{H} = 0 \\ \overline{N}\,\overline{F}\mathbb{1}_{d_N \times d_F} = -\overline{a}\mathbb{1}_{d_a} \end{cases} \tag{8}$$

is feasible and any solution $\overline{N}$ is an admissible fault detector filter with zero-steady state error from the aggregated fault to the residual. For any constant fault signals $(f_{\mathrm{a}}, f_{\mathrm{m}})$ and filter initial conditions, the residual (7) fulfills $\lim_{t\to\infty} f_{\mathrm{a}} + E(z(t))f_{\mathrm{m}} - r(t) = 0$.

The proof is omitted as it is a straightforward adaptation from [20, Lemma 4.6].

B. Fault Isolation: Nonlinear Regression

Next, we present the design of the fault isolation block. A central object of this part is the *regression operator*, a well-known scheme adopted from the machine learning literature [19]. This operator represents the fault isolation block whose domain and range spaces are discrete-time signals with appropriate dimensions.

Definition 3.2 (Regression Operator): Given an integer n and scalar-valued signals e and r, we define

$$\Phi_n[e,r](k) := \phi_n^{\dagger}[e](k)\,\mathbf{r}_n(k)$$
$$\text{where} \quad \phi_n[e](k) := [\mathbf{e}_n(k), \mathbb{1}_n] \in \mathbb{R}^{n\times 2} \tag{9}$$

with the operator † as the pseudoinverse (i.e., $A^\dagger := (A^\intercal A)^{-1}A^\intercal$).

In the context of the fault estimation scheme in Fig. 1, the output $\Phi_n[e,r](k)$ of the nonlinear regression operation in Definition 3.2 is, in fact, equal to the fault estimate $\widehat{f}$. The nonlinear regression operator in Definition 3.2 enjoys certain regularity properties that are key for the results we will develop later. The following proposition provides input–output bounds of the regression operator. These bounds will be utilized later to develop a performance bound for the proposed estimation filter.

Proposition 3.3 (Regression Bounds): Consider the regressor operator in Definition 3.2. For all discrete-time scalar-valued signals r, e, and $y = [y^{(1)}, y^{(2)}]^\intercal$, at each time instant $k \in \mathbb{N}$, where $V_n[e] \neq 0$ it holds that

$$\left\|\Phi_n[e, y^{(1)} + e\,y^{(2)}] - \mu_n[y]\right\|_2 \le \frac{\mathcal{C}_n(\mathbf{e}_n)}{V_n[e]}\left(V_n\left[y^{(1)}\right] + V_n\left[y^{(2)}\right]\|\mathbf{e}_n\|_\infty\right) \tag{10a}$$

$$\left\|\Phi_n[e,r]\right\|_2 \le \frac{\mathcal{C}_n(\mathbf{e}_n)}{\sqrt{n}V_n[e]}\|\mathbf{r}_n\|_2 \tag{10b}$$

where the constant is defined as $\mathcal{C}_n(\mathbf{e}_n) := \sqrt{V_n^2[e] + \mu_n^2[e] + 1}$.

Proof: Due to the space limitation, we relegate the proof to the extended version [18, Proposition III.3 in our arXiv paper]. □

We emphasize that the bounds in (10) hold for each time instant $k \in \mathbb{N}$, but to avoid clutter we drop the time-dependency of the signals (e.g., $\Phi_n[e,r]$ instead of $\Phi_n[e,r](k)$). We also note that the parameter $\mathcal{C}_n$ only depends on the signal e (more precisely, on the last n time instants of the signal e denoted by $\mathbf{e}_n$). In this view, the inequality (10b) indeed represents an operator norm for the linear mapping $r \mapsto \Phi_n[e,r]$. Let us elaborate further on how the bounds as in (10) are the first stepping-stones toward our main goal in this study. Measuring the "aggregated" signal $y^{(1)} + e\,y^{(2)}$, one can utilize (10a) to bound the error on the estimation of the average of the multivariate signal $y = [y^{(1)}, y^{(2)}]^\intercal$ (i.e., $\mu_n[y]$) via the regression operator. It is worthwhile to note that when the signal y is constant, then $y = \mu_n[y]$ and $V_n[y^{(1)}] = V_n[y^{(2)}] = 0$, and that the estimation error reduces to zero provided that $V_n[e] \neq 0$. The second result (10b) allows us to bound the output of the nonlinear regression operator given a bounded input, which can be viewed as a means to bound estimated faults given the dynamically filtered residual r as an input. The bounds (10) offer a rigorous framework to treat the isolation block as a nonlinear dynamical system whose induced gain, and as such the boundedness of its output, is determined by $V_n[e]$, the variance of signal e over a horizon with the length n.

C. Prefilter: Dynamic Compensator

In this section, we focus on the prefilter block in Fig. 1. Before presenting the main results of this article, we first need to proceed with a basic preparatory lemma on the output bound of LTI systems. To improve the flow of this article, we skip the technical proofs of the results in this section and defer them to Section IV.

Lemma 3.4 (Zero Steady-State LTI Output Bound): Consider a proper LTI system with the numerator $b(\mathfrak{q}) = \sum_{i=0}^{d} b_i \mathfrak{q}^i$, and the denominator $a(\mathfrak{q}) = \prod_{i=1}^{d}(\mathfrak{q} - p_i)$, where the poles are distinct and the dominant one (i.e., the one closest to the unit circle) is p with $|p| < 1$. Suppose the steady-state gain of the filter is 0 [i.e., $b(1) = 0$], the internal state (in the Jordan canonical form) is initiated at $X(0)$, and the input signal $u(t)$ is 0 until time k_0 and takes possibly nonzero values for $t \ge k_0$. Then, the output signal $y(t)$ satisfies the bound

$$\|\mathbf{y}_n\|_2 \le \mathcal{C}_0\|X(0)\|_2|p|^{k-n} + \mathcal{C}_1\left\|\mu_{k-k_0}[u]\right\|_2|p|^{k-n-k_0} + \mathcal{C}_2\sqrt{k-k_0}\,V_{k-k_0}[u]$$

where the constants $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ are defined as

$$r_i = \frac{b(-p_i)}{\prod_{j\neq i}(p_j - p_i)}, \qquad \mathcal{C}_0 = \sqrt{n\sum_{i=1}^{d} r_i^2}$$

$$\mathcal{C}_1 = \frac{\sqrt{n\,d\sum_{i=1}^{d} r_i^2}}{1-|p|}, \qquad \mathcal{C}_2 = |b_d| + \sum_{i=1}^{d}\frac{|r_i|}{1-|p_i|}.$$

Proof: Due to the space limitation, we relegate the proof to the extended version [18, Lemma 3.4 in our arXiv paper]. □

The statement of Lemma 3.4 is rather classical and is not unexpected. However, we need such an assertion with explicit, computable bounds for the main results of this study, which to our best knowledge does not exist in this form in the literature.

We further propose two possible designs for the prefilter, each of which comes along with certain pros and cons. The first, and simplest, design option is the static identity block. The next theorem presents a performance bound for this static prefilter design.

Theorem 3.5 (Performance Bound: (I) Static Prefilter): Consider the system (2) and the fault estimation filter in Fig. 1, where the fault detection block is characterized by the linear program (8) and a denominator $a(\mathfrak{q})$ with distinct and real-valued poles. The fault isolation block is the regression operator in (9) with the horizon n. Suppose the prefilter block is identity [i.e., $e = E(z)$], and the fault signal starts at time k_0. Then, at each time instant $k \in \mathbb{N}$, we have

$$\left\|\widehat{f} - \mu_n[f]\right\|_2 \leq \frac{1}{V_n[e]}\left(\alpha_0 |p|^{k-k_0} + \alpha_1 V_{k-k_0}[f_\mathrm{a}] + \alpha_2 V_{k-k_0}[f_\mathrm{m}] + \alpha_3\right) \tag{11a}$$

where the constant $p \in \mathbb{R}$ is the dominant pole of the denominator $a(\mathfrak{q})$ and the involved constants are defined as

$$\alpha_0 = \mathcal{C}_1 \frac{\mathcal{C}_n(\mathbf{e}_n)}{\sqrt{n}}\left(|\mu_{k-k_0}[f_\mathrm{a}]| + |\mu_{k-k_0}[e f_\mathrm{m}]|\right) \tag{11b}$$

$$\alpha_1 = \mathcal{C}_2 \mathcal{C}_n(\mathbf{e}_n)\sqrt{\frac{k-k_0}{n}} \tag{11c}$$

$$\alpha_2 = \mathcal{C}_2 \mathcal{C}_n(\mathbf{e}_n)\sqrt{\frac{k-k_0}{n}}\left(\sqrt{k-k_0} V_{k-k_0}[e] + |\mu_{k-k_0}[e]|\right) \tag{11d}$$

$$\alpha_3 = \mathcal{C}_n(\mathbf{e}_n)\left(V_n[f_\mathrm{a}] + V_n[f_\mathrm{m}]\,\|\mathbf{e}_n\|_\infty + \mathcal{C}_2\sqrt{\frac{k-k_0}{n}}|\mu_{k-k_0}[f_\mathrm{m}]| V_{k-k_0}[e]\right). \tag{11e}$$

in which $\mathcal{C}_n(\mathbf{e}_n)$ is defined in Proposition 3.3 and the constants $\mathcal{C}_1, \mathcal{C}_2$ are defined in Lemma 3.4.

Proof: The proof is provided in Section IV. □

By Theorem 3.5, one can inspect how different aspects of the proposed design contribute to the fault estimation error. The most critical term is $V_n[e]$ in the denominator of the right-hand side of (11a). This challenging dependency is, however, an inherent limitation of the desired isolation task. In fact, one can show that when the signal $E(z)$ is constant (i.e., $V_n[e] \equiv 0$), separation of the two faults $(f_\mathrm{a}, f_\mathrm{m})$ is even theoretically impossible. To reinforce this statement, consider the case $V_n[e] \equiv 0$ with arbitrary faults f_a and f_m. It can be observed that the regression operator (9) contains the inverse of a degenerate component $\phi_n[e]^\intercal \phi_n[e]$ which, by definition, does not exist in such a case. This result shows that the signal e, over horizon n, does then not span the behavior of the aggregated fault over the same horizon, a concept close to the well-known *persistence of excitation* property for LTI systems [21]. The term α_0 in (11b) reflects the contribution of the average behavior of fault signals. In (11b), it can be seen that this term diminishes exponentially fast after the start of the fault signal due to its proportionality with the exponentially decaying term containing the dominant pole p of the stable denominator $a(\mathfrak{q})$. In this light, we can deduce that the impact of these average behaviors on performance is negligible. The terms concerning α_1 and α_2 in (11) are mainly influenced by the variance of the fault since the beginning of the fault. The contribution of these variances in combination with the dynamics constants $\mathcal{C}_1, \mathcal{C}_2$ from Lemma 3.4 is also an inevitable factor in the estimation error, since the regression model in Definition 3.2 assumes constant contributions of the faults f_a and f_m appearing through the transfer function (7) in the residual r over a horizon n. Finally, the last term involving α_3 is a critical and potentially persistent source of error. In particular, the variance signal $V_{k-k_0}[e]$ introduces a nonzero estimation error even in the case of constant fault signals. The next corollary highlights this effect.

Corollary 3.6 (Constant Faults: Part I): Consider the system and the estimation filter as in Theorem 3.5. Suppose the fault signals are constant $f = (\bar{f}_\mathrm{a}, \bar{f}_m)$, starting from the time k_0. Then, for any time instant $k \geq k_0 + n$, we have

$$\left\|\widehat{f} - f\right\|_2 \leq \frac{\mathcal{C}_n(\mathbf{e}_n)}{\sqrt{n} V_n[e]}\left(\mathcal{C}_1\left(|\bar{f}_\mathrm{a}| + |\bar{f}_m| \mu_{k-k_0}[e]\right)|p|^{k-n-k_0} + \mathcal{C}_2\sqrt{k-k_0}|\bar{f}_m| V_{k-k_0}[e]\right). \tag{12}$$

Proof: The proof is a direct application of Theorem 3.5. Under the assumption that the fault signals are constants after time $k \geq k_0$, we know that $V_{k-k_0}[f_\mathrm{a}] = V_{k-k_0}[f_\mathrm{m}] = 0$. Moreover, assuming further that $k \geq n + k_0$, we can also conclude that $V_n[f_\mathrm{a}](k) = V_n[f_\mathrm{m}](k) = 0$. In addition, the average terms of the signal reduces to $\mu_{k-k_0}[f_\mathrm{a}] = \bar{f}_a$ and $\mu_{k-k_0}[f_\mathrm{m}] = \bar{f}_m$. Substituting these quantities in the bound (11) concludes (12). □

As noted above, the variance of the signal e is a persistent factor contributing to the performance bound, which is captured by the last term on the right-hand side of the inequality (12). This is somehow expected due to the causality effect of the system dynamics. More specifically, the residual r, the output of the fault detection block, opts to follow the aggregated fault signal $f_\mathrm{a} + E(z) f_\mathrm{m}$ but it relies on the dynamics $\mathcal{T}(\mathfrak{q})$ [cf., (7)]. However, when the prefilter is set to identity (i.e., $e = E(z)$), the information of the signal is provided instantly for the isolation block (due to the static identity prefilter), rendering some persistent potential error, that is proportional to $V_{k-k_0}[e]$. This error exists because the fault isolation block assumes a static mapping $e \mapsto r$ for the static prefilter case, whereas this mapping is inherently dynamic due to the dynamics of the system and the fault detection block (7). This dynamical misalignment in the fault isolation block manifests itself in the estimation error, even for constant faults, as shown in (12). Next, we aim to address this issue by filtering the information of the signal $E(z)$ through the same dynamics that the residual of the detection filter experiences. This novel viewpoint brings us to the second choice of prefilter next.

Theorem 3.7 (Performance Bound: (II) Dynamic Prefilter): Consider the system (2) and the fault estimation filter in Fig. 1, where the fault detection block is characterized by the linear program (8) and a denominator $a(\mathfrak{q})$ with distinct and real-valued poles. The fault isolation block is the regression operator in (9) with the horizon n. Suppose the prefilter block is the linear system $\mathcal{T}$ as defined in (7) (i.e., $e = \mathcal{T}[E(z)]$) with the internal states denoted by X_p. If the fault signal starts at time k_0, then at each time instant k, we have

$$\left\|\widehat{f} - \mu_n[f]\right\|_2 \leq \frac{1}{V_n[e]}\left(\beta_0 |p|^{k-k_0} + \beta_1 V_{k-k_0}[f_\mathrm{a}] + \beta_2 V_{k-k_0}[f_\mathrm{m}] + \beta_3\right) \tag{13a}$$

where the constant $p \in \mathbb{R}$ is the dominant pole of the denominator $a(\mathfrak{q})$ and the involved constants are defined as

$$\beta_0 = \frac{\mathcal{C}_n(\mathbf{e}_n)}{\sqrt{n}}\left(\mathcal{C}_1\left(|\mu_{k-k_0}[E(z) f_\mathrm{m}] - \mu_{k-k_0}[E(z)]\,\mu_n[f_\mathrm{m}]|\right) + \mathcal{C}_0 |\mu_n[f_\mathrm{m}]|\|X_p(k-k_0)\|_2 + |\mu_n[f_\mathrm{a}]|\right) \tag{13b}$$

$$\beta_1 = \mathcal{C}_2 \mathcal{C}_n(\mathbf{e}_n)\sqrt{\frac{k-k_0}{n}} \tag{13c}$$

$$\beta_2 = \mathcal{C}_2 \mathcal{C}_n(\mathbf{e}_n)\sqrt{\frac{k-k_0}{n}}\left(\sqrt{k-k_0} V_{k-k_0}[e] + |\mu_{k-k_0}[e]|\right) \tag{13d}$$

$$\beta_3 = \mathcal{C}_n(\mathbf{e}_n)\left(V_n[f_\mathrm{a}] + V_n[f_\mathrm{m}]\left(\|\mathbf{e}_n\|_\infty + \|\mathbf{e}_n - E(\mathbf{z}_n)\|_\infty\right) + \mathcal{C}_2\sqrt{\frac{k-k_0}{n}}\left|\mu_{k-k_0}[f_\mathrm{m}] - \mu_n[f_\mathrm{m}]\right| V_{k-k_0}[E(z)]\right). \tag{13e}$$

in which $\mathcal{C}_n(\mathbf{e}_n)$ is defined in Proposition 3.3, the constants $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ are defined in Lemma 3.4, and the vector-valued signal $E(\mathbf{z}_n)$ is understood as the evaluation of the function $E(\cdot)$ on each of the elements of the vector $\mathbf{z}_n$.

Proof: The proof is provided in Section IV. □

In a comparison with Theorem 3.5, one can see that the main difference in the fault estimation error bound appears in the last coefficient of the error bounds [cf, α_3 in (11a) and β_3 in (13a)]. In particular, the idea of an appropriate dynamic prefilter allows us to shift the contribution of the variance signal $V_{k-k_0}[e]$ to the third term related to β_2 in (13d), which is multiplied by the variance of the multiplicative fault $V_{k-k_0}[f_\mathrm{m}]$. This shift has a significant impact on the performance when the fault signals are constant during the activation time (i.e., $k \geq k_0$). Before proceeding with the simplification of the result, in this case, let us note that the dynamic prefilter does not necessarily outperform the static one proposed by Theorem 3.5 due to the difference in the term $V_n[e]$. Indeed, the filtered signal $\mathcal{T}[E(z)]$ may have a lower variance, which has a negative impact on the performance bounds.

Corollary 3.8 (Constant Faults: Part II): Consider the system and the estimation filter as in Theorem 3.7. Suppose the fault signals are constant $(\bar{f}_\mathrm{a}, \bar{f}_m)$ starting at time k_0. Then, for any time instant $k \geq k_0 + n$

$$\|\widehat{f} - f\|_2 \leq \frac{\mathcal{C}_n(\mathbf{e}_n)}{\sqrt{n}V_n[e]}\left(\mathcal{C}_1|\bar{f}_\mathrm{a}| + \mathcal{C}_0|\bar{f}_m|\|X_p(k_0)\|_2\right)|p|^{k-k_0}. \quad (14)$$

Proof: In parallel to Corollary 3.6, the proof is a direct application of Theorem 3.7 when the fault signals are constants after time $k \geq n + k_0$, and as such $V_{k-k_0}[f_\mathrm{a}] = V_{k-k_0}[f_\mathrm{m}] = 0$, $V_n[f_\mathrm{a}](k) = V_n[f_\mathrm{m}](k) = 0$, $\mu_{k-k_0}[f_\mathrm{a}] = \bar{f}_a$ and $\mu_{k-k_0}[f_\mathrm{m}] = \bar{f}_m$. Besides, we also note that the term $\mu_{k-k_0}[f_\mathrm{m}] - \mu_n[f_\mathrm{m}] = 0$ vanishes as well. Substituting these quantities in the bound (13) concludes (14). □

In the case of constant faults, Corollary 3.8 indicates that the fault estimation error goes to zero exponentially fast if the filtered signal e behaves "nicely" (i.e., $V_n[e]$ is uniformly away from zero). In comparison with the assertion of Corollary 3.6, this outcome highlights the role of the dynamic prefilter on the estimation performance.

The following remark provides insight on the computational complexity of the used fault estimation methods.

Remark 3.9 (Computational Complexity): Given the fact that the optimal fault-detection filter (6) and prefilter (7) are computed offline, the computational complexity of the real-time running algorithm, for both Theorems 3.5 and 3.7, is governed by the fault isolation block. Due to the regression operation, as defined in Definition 3.2, this method will have a time computational complexity of $\mathcal{O}(4n + 8)$, where n represents the prediction horizon of the fault isolation filter.

Let us close this section with a summary of the results. In this section, a general estimation architecture has been proposed. System theoretic bounds for the regression operator (see Definition 3.2) and the LTI bound (see Lemma 3.4) have been used for the construction of guaranteed performance bounds for two prefilter variants within this estimation architecture. The insights gained from the first prefilter variant (i.e., the identity block in Theorem 3.5) and its behavior for constant faults (see Corollary 3.6), have been leveraged to propose a second design variant (i.e., the dynamic prefilter in Theorem 3.7), which has been proven to have an exponentially decaying performance bound for constant faults (see Corollary 3.8).

IV. Technical Preliminaries and Proofs of Main Results

This section presents the technical proofs of the theoretical results in Section III. Before proceeding with the proofs of the main theorems, we need first to provide a useful additional lemma concerning the variance of the product of two signals. The results of this section will later facilitate the proofs of the main theorems.

Lemma 4.1 (Variance of Product Signals): Consider the discrete-time signals a, b over time-horizon n. At each time instant, we have

$$\begin{aligned}&|V_n^2[a+b] - V_n^2[a] - V_n^2[b]| \\ &\quad \leq 2\min\left\{\|\mathbf{a}_n\|_2 V_n[b], \|\mathbf{b}_n\|_2 V_n[a]\right\} \quad (15a)\\ &V_n[ab] \leq \sqrt{n}V_n[a]\,V_n[b] + |\mu_n[a]|\,V_n[b] + |\mu_n[b]|\,V_n[a]. \quad (15b)\end{aligned}$$

Proof: Due to the space limitation, we relegate the proof to the extended version [18, Lemma IV.1 in our arXiv paper]. □

Proof of Theorem 3.5: Let us first introduce the shorthand notation

$$\mathcal{G} := \mathcal{T} - \mathcal{I}, \quad \delta(k) := f_\mathrm{a} + ef_\mathrm{m}(k), \quad e = E(z(k)) \quad (16)$$

where the transfer function $\mathcal{T}$ is as defined in (7) and $\mathcal{I}$ is the identity transfer function. Notice that in this part the prefilter is the static gain identity, and that its output signal e is indeed the measurement signal $E(z)$ (cf. Fig. 1). Based on the definition of the estimated fault and the regression operator in Definition 3.2, we have

$$\widehat{f} = \Phi_n[e, r] = \Phi_n[e, r-\delta] + \Phi_n[e, \delta] - \mu_n[f] + \mu_n[f]$$

where the second equality simply follows from the linearity of the regression operator in the second argument. Let moving the term $\mu_n[f]$ to the left-hand side and taking the two-norm on both sides of the above equality. Using the triangle inequality and the regression bounds from Proposition 3.3, we arrive at

$$\begin{aligned}\|\widehat{f} - \mu_n[f]\|_2 &\leq \|\Phi_n[e, r-\delta]\|_2 + \|\Phi_n[e,\delta] - \mu_n[f]\|_2\\ &\leq \frac{\mathcal{C}_n(\mathbf{e}_n)}{\sqrt{n}V_n[e]}\|\mathbf{r}_n - \delta_n\|_2\\ &\quad + \frac{\mathcal{C}_n(\mathbf{e}_n)}{V_n[e]}\left(V_n[f_\mathrm{a}] + V_n[f_\mathrm{m}]\,\|\mathbf{e}_n\|_\infty\right) \quad (17)\end{aligned}$$

where the first and second bounds in (17) follow from (10b) and (10a), respectively. It then remains to bound the term $\|\mathbf{r}_n - \delta_n\|_2$ on the right-hand side of (17). Following the definitions of the residual r in (7), and the signal δ and the transfer function $\mathcal{G}$ in (16), we have

$$r - \delta = \mathcal{T}\left[f_\mathrm{a} + E(z)f_\mathrm{m}\right] - (f_\mathrm{a} + ef_\mathrm{m}) = \mathcal{G}\left[f_\mathrm{a}\right] + \mathcal{G}\left[ef_\mathrm{m}\right].$$

Note that by construction the transfer function $\mathcal{G}$ has a zero steady-state gain since the transfer function $\mathcal{T}$ has *unit* steady-state gain (see Lemma 3.1). As such, we can apply Lemma 3.4 to the right-hand side of the above relation. This leads to

$$\begin{aligned}\|\mathbf{r}_n - \delta_n\|_2 &\leq \mathcal{C}_1\left(|\mu_{k-k_0}[f_\mathrm{a}]| + |\mu_{k-k_0}[ef_\mathrm{m}]|\right)|p|^{k-k_0}\\ &\quad + \mathcal{C}_2\sqrt{k-k_0}\left(V_{k-k_0}[f_\mathrm{a}] + V_{k-k_0}[ef_\mathrm{m}]\right)\\ &\leq \mathcal{C}_1\left(|\mu_{k-k_0}[f_\mathrm{a}]| + |\mu_{k-k_0}[ef_\mathrm{m}]|\right)|p|^{k-k_0}\\ &\quad + \mathcal{C}_2\sqrt{k-k_0}\bigl(V_{k-k_0}[f_\mathrm{a}] + |\mu_{k-k_0}[f_\mathrm{m}]|\,V_{k-k_0}[e]\\ &\quad + \sqrt{k-k_0}V_{k-k_0}[f_\mathrm{m}]\,V_{k-k_0}[e]\\ &\quad + |\mu_{k-k_0}[e]|\,V_{k-k_0}[f_\mathrm{m}]\bigr)\end{aligned}$$

where in the last line we apply (15b) from Lemma 4.1 to the variance of the product signals $V_{k-k_0}[ef_\mathrm{m}]$. Substituting the above bound in (17) results in

$$\begin{aligned}&\|\widehat{f} - \mu_n[f]\|_2\\ &\quad \leq \frac{\mathcal{C}_n(\mathbf{e}_n)}{\sqrt{n}V_n[e]}\Bigl(\mathcal{C}_1\left(|\mu_{k-k_0}[f_\mathrm{a}]| + |\mu_{k-k_0}[ef_\mathrm{m}]|\right)|p|^{k-k_0}\\ &\qquad + \mathcal{C}_2\sqrt{k-k_0}\,\bigl(V_{k-k_0}[f_\mathrm{a}] + |\mu_{k-k_0}[e]|\,V_{k-k_0}[f_\mathrm{m}]\\ &\qquad + V_{k-k_0}[e]\left(\sqrt{k-k_0}V_{k-k_0}[f_\mathrm{m}] + |\mu_{k-k_0}[f_\mathrm{m}]|\right)\bigr)\Bigr)\\ &\qquad + \frac{\mathcal{C}_n(\mathbf{e}_n)}{V_n[e]}\left(V_n[f_\mathrm{a}] + V_n[f_\mathrm{m}]\,\|\mathbf{e}_n\|_\infty\right).\end{aligned}$$

Finally, it suffices to factor out the right-hand side of the above inequality to the exponentially decaying term and the variance terms $V_{k-k_0}[f_\mathrm{a}]$, $V_{k-k_0}[f_\mathrm{m}]$, as well as the remaining parts, including $V_n[f_\mathrm{a}]$, $V_n[f_\mathrm{m}]$, $V_{k-k_0}[e]$. This concludes the desired assertion (11).

Proof of Theorem 3.7: The key difference between the setting of this theorem with Theorem 3.5 is the choice of prefilter, and as such, the definition of the signal e. Consider the same definitions of the transfer function $\mathcal{G}$ and the signal δ as in (16), where the output of the prefilter is defined as

$$e = \mathcal{T}\left[E(z)\right], \qquad \text{with the internal states } X_p. \tag{18}$$

Note that the relation (17) still holds in the setting here as well. We then only need focus on the term $\|\mathbf{r}_n - \delta_n\|_2$. In the rest of the proof, we fix the time instant $k \in \mathbb{N}$ and define the average of the multiplicative fault f_{m} over the horizon $[k-n, k]$ as the constant denoted by

$$\overline{f_{\mathrm{m}}} := \mu_n[f_{\mathrm{m}}]\,(k). \tag{19}$$

Let us emphasize that we view the value $\overline{f_{\mathrm{m}}}$ as constant over the entire time horizon prior to k. We further introduce the shorthand notation of the step function

$$\mathfrak{U}_{k_0}(k) := \begin{cases} 0 & k < k_0 \\ 1 & k \geq k_0\,. \end{cases}$$

With straightforward but tedious algebraic computation, the signal $r - \delta$ can be rewritten as

$$\begin{aligned} r - \delta = {} & \mathcal{G}\left[f_{\mathrm{a}}\right] + \mathcal{G}\left[E(z)(f_{\mathrm{m}} - \overline{f_{\mathrm{m}}}\mathfrak{U}_{k_0})\right] \\ & + \overline{f_{\mathrm{m}}}\mathcal{T}\left[E(z)(\mathfrak{U}_{k_0} - 1)\right]\mathfrak{U}_{k_0} - \mathcal{G}\left[E(z)\right](f_{\mathrm{m}} - \overline{f_{\mathrm{m}}}\mathfrak{U}_{k_0}). \end{aligned} \tag{20a}$$

Recall that $\mathcal{G} = \mathcal{T} - \mathcal{I}$ is a stable transfer function with zero steady-state gain. Also, note that for $k \geq k_0$ the third term $\mathcal{T}[E(z)(\mathfrak{U}_{k_0} - 1)]$ in (20a) is in fact the contribution of the internal states $X_p(k_0)$ of the transfer function $\mathcal{T}$ when the input signal is $E(z)$ ($\mathfrak{U}_{k_0} - 1 = 0$ for all $k \geq 0$). Therefore, we can apply Lemma 3.4 to each term on the right-hand side in (20a) and obtain the bound

$$\begin{aligned} \|\mathbf{r}_n - \delta_n\|_2 \leq {} & \mathcal{C}_1|\overline{f_{\mathrm{a}}}||p|^{k-k_0} + \mathcal{C}_2\sqrt{k-k_0}V_{k-k_0}[f_{\mathrm{a}}] \\ & + \mathcal{C}_1|\mu_{k-k_0}\left[E(z)(f_{\mathrm{m}} - \overline{f_{\mathrm{m}}}\mathfrak{U}_{k_0})\right]||p|^{k-k_0} \\ & + \mathcal{C}_2\sqrt{k-k_0}V_{k-k_0}\left[E(z)(f_{\mathrm{m}} - \overline{f_{\mathrm{m}}}\mathfrak{U}_{k_0})\right] \\ & + |\overline{f_m}|\mathcal{C}_0\|X_p(k_0)\|_2|p|^{k-k_0} \\ & + \|\mathbf{e}_n - E(\mathbf{z}_n)\|_\infty\sqrt{n}V_n[f_{\mathrm{m}}]\,. \end{aligned} \tag{21}$$

We first note that in (21) we can simplify the first term of the second line as $\mu_{k-k_0}[E(z)(f_{\mathrm{m}} - \overline{f_{\mathrm{m}}}\mathfrak{U}_{k_0})] = \mu_{k-k_0}[E(z)f_{\mathrm{m}}] - \mu_{k-k_0}[E(z)]\overline{f_{\mathrm{m}}}$. We further borrow the results of Lemma 4.1 to bound the terms involving the product of two signals in (21). More specifically, we have

$$\begin{aligned} & V_{k-k_0}\left[E(z)(f_{\mathrm{m}} - \overline{f_{\mathrm{m}}}\mathfrak{U}_{k-k_0})\right] \\ & \quad \leq \sqrt{k-k_0}V_{k-k_0}[E(z)]\,V_{k-k_0}[f_{\mathrm{m}}] \\ & \qquad + |\mu_{k-k_0}\left[f_{\mathrm{m}} - \overline{f_{\mathrm{m}}}\mathfrak{U}_{k_0}\right]|V_{k-k_0}[E(z)] \\ & \qquad + |\mu_{k-k_0}[E(z)]\,|V_{k-k_0}[f_{\mathrm{m}}] \\ & \quad = \sqrt{k-k_0}V_{k-k_0}[E(z)]\,V_{k-k_0}[f_{\mathrm{m}}] \\ & \qquad + |\mu_{k-k_0}[f_{\mathrm{m}}] - \overline{f_{\mathrm{m}}}\mathfrak{U}_{k_0}|V_{k-k_0}[E(z)] \\ & \qquad + |\mu_{k-k_0}[E(z)]\,|V_{k-k_0}[f_{\mathrm{m}}]\,. \end{aligned} \tag{22}$$

It now suffices to substitute the upper bounds (22) in (21), and then invoke the resulting bound on $\|\mathbf{r}_n - \delta_n\|_2$ in (17). Finally, it remains to factor out the right-hand side of the inequality to the exponentially decaying term, the variance terms $V_{k-k_0}[f_{\mathrm{a}}]$, $V_{k-k_0}[f_{\mathrm{m}}]$, as well as the remaining parts including $V_n[f_{\mathrm{a}}]$, $V_n[f_{\mathrm{m}}]$, $V_{k-k_0}[E(z)]$. This concludes the desired assertion (13). □

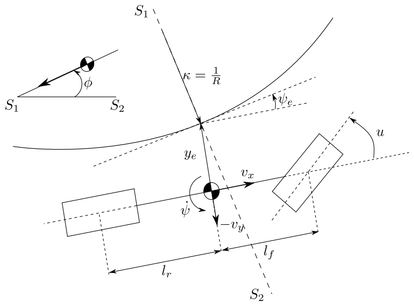

Fig. 2. Visual representation of the bicycle model.

V. Case Study: Lateral Control of Autonomous Vehicles

In this section, the presented theory is illustrated using a fault isolation problem in the scope of the lateral control of autonomous vehicles. In this context, fault detection and isolation are increasingly important in the automotive industry. The lateral dynamics of the vehicle are modeled using the bicycle model [22, eq. (1)] (depicted in Fig. 2). This model represents a linearization of the full nonlinear lateral dynamics of an automated vehicle. The state of the vehicle is chosen as $X = [v_y,\ \dot{\psi},\ y_e,\ \psi_e]^\intercal$, where v_y represents the lateral velocity, $\dot{\psi}$ represents the yaw-rate, y_e represents the lateral error from the lane centre and ψ_e represents the heading error from the lane centre. The disturbance vector is chosen as $d = [\sin(\phi),\ \kappa]$, where ϕ represents the banking angle of the road (as depicted by cross-section $S_1 - S_2$ in Fig. 2) and κ represents the curvature of the road. The input u represents the steering wheel angle of the front wheels of the vehicle. In this case study, we consider additive and multiplicative faults acting on the steering input signal u. These faults could realistically occur as an offset in the steering column, f_{a}, or a loss of actuator efficiency, f_{m}. These faults may result in unexpected transient and steady-state tracking errors and hence result in dangerous situations for the vehicle passengers if not estimated and mitigated independently. The model description with its states, disturbances, faults, and input can be written in the continuous-time equivalent of the linear difference equation from (3), with system matrices [22, eq. (1)]

$$\bar{A} = \begin{bmatrix} \frac{C_f + C_r}{v_x m} & \frac{l_f C_f - l_r C_r}{v_x m} - v_x & 0 & 0 \\ \frac{l_f C_f - l_r C_r}{v_x I} & \frac{l_f^2 C_f + l_r^2 C_r}{v_x I} & 0 & 0 \\ -1 & 0 & 0 & v_x \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\bar{B}_u = \bar{B}_f = \begin{bmatrix} -\frac{C_f}{m} \\ -\frac{l_f C_f}{I} \\ 0 \\ 0 \end{bmatrix}$$

$$G = \mathbb{I}_4, \qquad \bar{B}_d = \begin{bmatrix} g & 0 & 0 & 0 \\ 0 & 0 & 0 & v_x \end{bmatrix}^\intercal, \qquad C = \begin{bmatrix} 0 & \mathbb{I}_3 \end{bmatrix}$$

where $\mathbb{I}_3$ and $\mathbb{I}_4$ represent the identity matrix of size 3 and 4, respectively. Furthermore, $C_f = 1.50 \cdot 10^5\ \mathrm{N} \cdot \mathrm{rad}^{-1}$, $C_r = 1.10 \cdot 10^5\ \mathrm{N} \cdot \mathrm{rad}^{-1}$, $l_f = 1.3\ \mathrm{m}$, $l_r = 1.7\ \mathrm{m}$, $v_x = 19\ \mathrm{m} \cdot \mathrm{s}^{-1}$, $m = 1500\ \mathrm{kg}$, $I = 2600\ \mathrm{kg} \cdot \mathrm{m}^2$, and finally $g = 9.81\ \mathrm{m} \cdot \mathrm{s}^{-2}$. The physical intuition behind these constants is omitted for the sake of brevity and can be found in [22]. In order to fit the discrete-time model setting employed in this article, we first exactly discretize the dynamical system by calculating the discrete-time states-space matrices as $A = e^{\bar{A}h}$ and $B = \int_0^h e^{\bar{A}s}\bar{B}ds$ (for all matrices $\bar{B}_u$, $\bar{B}_f$, and $\bar{B}_d$). For which the sampling interval h is chosen as $h = 0.01s$ and $f = f_{\mathrm{m}}u + f_{\mathrm{a}}$ represents the aggregated fault signal. Note, that the discretized system matrices can be written

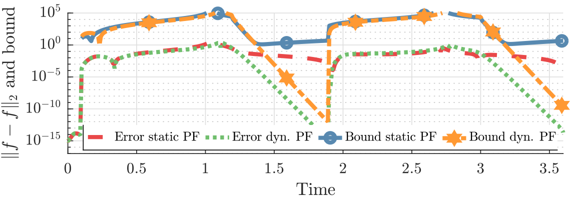

Fig. 3. True estimation error in comparison with the corresponding performance bounds in (12) and (14).

in the DAE framework by virtue of Fact 2.1 when setting $K_X = 0$, $K_Y = 0$, and $E_X = u$. For the synthesis of the fault detection filter, the degree of the filter $N(\mathfrak{q})$ is set to $d_N = 3$. The denominator $a(\mathfrak{q})$ is selected as $a(\mathfrak{q}) = (\mathfrak{q} + 0.60)(\mathfrak{q} + 0.59)(\mathfrak{q} + 0.58)$. The filter $N(\mathfrak{q})$ can be found by solving the linear program in (8). For the synthesis of the fault isolation filter, the time horizon n is initially chosen as $n = 10$. The input signal u to the system dynamics is selected to be a sinusoidal signal with an amplitude of $2.3 \cdot 10^{-3}$ radians at a frequency of 0.3 Hz. The frequency content of the input signal is inspired by experimental data of an automated vehicle, driving in-lane using a PD-type controller, while being excited by natural disturbances [22]. For this simulation study, the additive fault and the multiplicative fault are selected as *incipient* fault functions $f_\mathrm{a} = \pi/1800 \cdot 10^{-2}k$, $f_\mathrm{m} = -0.2 \cdot 10^{-2}k$, reaching their final values after $1s$, starting from time instances $0.1s$ and $1.9s$, respectively.

Fig. 3 depicts the estimation errors [left-hand side of (11a) and (13a) for the static prefilter and the dynamic prefilter, respectively] and their simulated performance bounds [right-hand side of (11a) and (13a) for the static prefilter and the dynamic prefilter, respectively]. As expected (according to Corollary 3.6), the performance bound and estimation error for the static prefilter remain nonzero for as long as the input signal u is excited, which is inherently needed for separation of the fault terms. The dynamic prefilter follows the result from Corollary 3.8, where the performance bound and estimation error converge to zero in finite time. This allows the automated vehicle to act upon two different fault-types accordingly, as opposed to having to conservatively act on the presence of an aggregated fault signal $f_\mathrm{a} + ef_\mathrm{m}$ without knowing the separate contributions of the faults.

VI. Conclusion

In this work, a fault estimation architecture for the estimation of additive and multiplicative faults, acting simultaneously through identical dynamical relationships is presented. Simulation results in the domain of SAE level 4 automated driving show the practical value and the potential of the proposed approach in relevant future-proof applications. Future work includes incorporating model uncertainty, nonlinearities, delays, and closed-loop fault mitigation.

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