ACTIVE DISTURBANCE ATTENUATION FOR AN EXPERIMENTAL PIECEWISE LINEAR BEAM SYSTEM [1]

A. Doris, C.G.M. de Bont, R. Wouters, N. van de Wouw, H. Nijmeijer

Department of Mechanical Engineering
Eindhoven University of Technology
P.O. Box 513, NL 5600 MB Eindhoven
The Netherlands
Email: a.doris@tue.nl, N.v.d.Wouw@tue.nl, h.nijmeijer@tue.nl

Abstract: This paper presents an experimental implementation of an observer-based controller design strategy on a piecewise linear beam system comprising a flexible steel beam with a one-sided support. The observer-based controller design strategy guarantees a unique globally asymptotically stable steady-state solution of the closed-loop system, which allows for unique performance evaluation in terms of disturbance attenuation. Experimental and simulation results are presented to demonstrate the effectiveness of the strategy. *Copyright©2006 IFAC.*

1. INTRODUCTION

This paper presents an experimental study of an observer-based controller design strategy for a continuous piecewise linear (PWL) system by application to an elastic beam with a one-sided support. The experimental beam system consists of a flexible steel beam, which is clamped on two sides and is locally supported by a one-sided linear spring. Due to the one-sided spring the beam has two different dynamical regimes, which can both be well described as linear. As such the beam system can be represented by a bi-modal PWL system. The experimental beam system is periodically excited by a rotating mass unbalance.

The mechanical motivation to study such a PWL beam system is the analysis and control of the dynamics of complicated engineering constructions including structural elements with PWL restoring characteristics, such as tower cranes, suspension bridges, solar panels on satellites, offshore oil production facilities and many more.

For this experimental beam we will design an observer-based controller using the strategy from (Pavlov, 2004; Doris *et al.*, 2005*b*) in order to ultimately achieve performance of the closed-loop system in terms of disturbance attenuation.

Results related to performance of control designs of PWL/PWA systems, in terms of disturbance attenuation, were given in (Rantzer and Johansson, 2000), (Hassibi and Boyd, 2005) and (Feng *et al.*, 2002). The performance results of these papers are based on quadratic or piecewise quadratic Lyapunov functions and provide an upper bound for the system output (given bounds on the input) by bounding the L_2 gain from the system input to the system output. These results have been derived under the assumption of zero initial conditions.

In (Doris *et al.*, 2005*b*) output-feedback controllers are considered for PWL systems, such as the examined PWL beam system. The applied strategy is based on extended notions of convergence ((Demidovich, 1967), (Pavlov *et al.*, 2004)) presented in (Pavlov, 2004). These extended notions of convergence are the notions uniform convergence and input-to-state convergence. A uniformly convergent system has a unique globally asymptotically stable steady-state so-

[1] Supported by European project SICONOS ($IST-2001-37172$) and HYCON (511368)

lution which is determined only by the system input and does not depend on the initial conditions.

The implemented controller is a combination of a bimodal PWL observer, see (Juloski *et al.*, 2002) and (Doris *et al.*, 2005*a*) and a state-feedback controller that uses the estimated states of the system. A distinguishing feature of the observer is that it does not necessary need to know in which 'mode' the system is, in contrast to observers proposed in (Alessandri and Coletta, 2001*a*), (Alessandri and Coletta, 2001*b*) and (Alessandri and Coletta, 2003).

To the best of the authors knowledge, this work presents for the first time experimental results for an observer-based controller design for a real PWL system. These results show the performance of the closed-loop system in terms of disturbance attenuation.

The controller design strategy focuses on the attenuation of exogenous periodic input disturbances on the system state by rendering the closed-loop system convergent. The theory related to this strategy gives conditions under which global exponential convergence and input-to-state-convergence (Pavlov, 2004) of the interconnected system (observer-controller-PWL beam) is achieved. These conditions imply that the closed-loop system at hand exhibits a unique steady-state solution. The latter feature allows for a unique performance evaluation in terms of disturbance attenuation.

The focus of the paper is on the implementation of this observer-based controller design strategy for the experimental beam system and on the evaluation of the obtained results using experimental measurements and simulations.

The paper is structured as follows. In section 2, a description of the experimental set-up is given and the modeling of the beam system is discussed. The observer-based controller design strategy is introduced in section 3. In section 4, simulation and experimental results related to the controller performance in terms of disturbance attenuation are presented. Conclusions are given in section 5.

2. EXPERIMENTAL SET-UP AND MODELING OF THE BEAM SYSTEM

The experimental set-up, in which we apply the controller design proposed in (Doris *et al.*, 2005*b*), consists of a steel beam which is mounted at both ends by two leaf springs see Figures 1 and 2. The steel beam is excited by a force generated by a rotating mass-unbalance. The mass-unbalance is mounted in the middle of the beam. A tacho-controlled motor, that enables a constant rotation speed, drives the mass-unbalance. An actuator is mounted on the beam in order to control the beam dynamics. A second beam is placed parallel with respect to the first beam and represents a one-sided spring. This one-sided spring represents a non-smooth nonlinearity in the system. Therefore, the system has non-smooth and nonlinear dynamics. Assuming that the restoring characteristics of the spring are linear, the beam-spring system can be described as a PWL system.

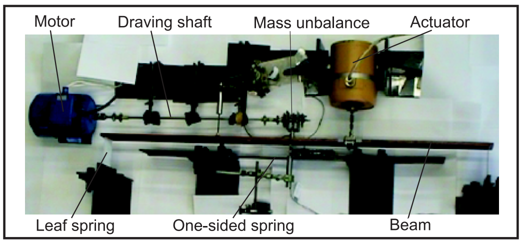

Fig. 1. Photo view of the PWL beam system.

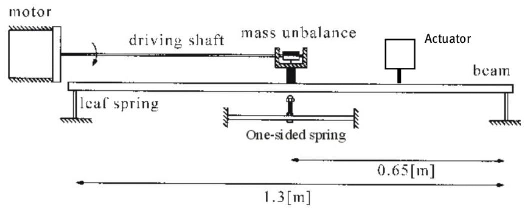

Fig. 2. Schematic view of the PWL beam system.

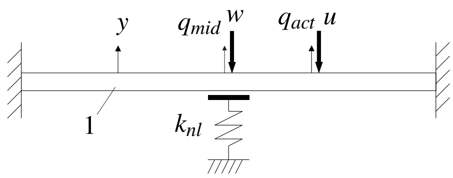

Fig. 3. Elastic beam with one-sided support.

The dynamics of the PWL beam system can be expressed by a three-degree-of-freedom (3DOF) model (Doris *et al.*, 2005*b*):

$$M\ddot{q} + B_s\dot{q} + K_s q + f_{nl}(q) = h_1\, w(t) + h_2\, u(t), \quad (1)$$

where $h_1 = [1 \quad 0 \quad 0]^T$, $h_2 = [0 \quad 1 \quad 0]^T$ and $q = [q_{mid} \quad q_{act} \quad q_\xi]^T$. Herein, q_{mid} is the displacement of the middle of the beam and q_{act} is the displacement of the point of the beam at which the actuator is mounted, see Figure 3. Moreover, q_ξ reflects the contribution of the first eigenmode of the beam and M, B_s and K_s are the mass, damping and stiffness matrices of the 3DOF model, respectively. We apply a periodic (harmonic) excitation force

$$w(t) = A \sin \omega t, \quad (2)$$

which is generated by the rotating mass-unbalance at the middle of the beam. Herein, ω is the excitation frequency and A the amplitude of the excitation force. Moreover, $u(t) \in \mathbb{R}$ is the control force applied by the actuator to the beam and f_{nl} is the restoring force of the one-sided spring:

$$f_{nl}(q) = k_{nl}\, h_1\, min(0, h_1^T q) = k_{nl}\, h_1\, min(0, q_{mid}), \quad (3)$$

where k_{nl} is the stiffness of the spring. In other words, the force f_{nl} acts when there is contact between the middle of the beam and the one-sided spring.

For the 3DOF model, three types of steady-state behavior are known to exist, namely: periodic, quasi-periodic and chaotic solutions (Fey *et al.*, 1996). Herein, it is also shown that such steady-state solution

can coexist (both periodic and sub-harmonic).
In a state-space formulation, the model takes the following form

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Bw(t) + B_1 u(t) \text{ for } H^T x(t) \leq 0 \\ A_2 x(t) + Bw(t) + B_1 u(t) \text{ for } H^T x(t) > 0 \end{cases} \tag{4a}$$

$$y(t) = Cx(t), \tag{4b}$$

where $x = [q^T \quad \dot{q}^T]^T \in \mathbb{R}^6$ is the state and $H = [h_1^T \quad 0^T]^T$. Furthermore, $y(t) \in \mathbb{R}$ is the system output and

$A_1 = \begin{bmatrix} 0 & I \\ -M^{-1}(K_s + k_{nl}\, h_1\, h_1^T) & -M^{-1}B_s \end{bmatrix}$,

$A_2 = \begin{bmatrix} 0 & I \\ -M^{-1}K_s & -M^{-1}B_s \end{bmatrix}$,

$B = \begin{bmatrix} 0 \\ M^{-1}h_1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ M^{-1}h_2 \end{bmatrix}$.

Note that the vectorfields in (4a) coincide on the switching boundary $H^T x = 0$ and $y(t)$ describes a transversal displacement of a point on the beam, depicted in Figure 3 (point 1). The numerical values of M, B_s, K_s, k_{nl} and C are given in the Appendix.

3. OBSERVER-BASED CONTROLLER DESIGN

For the system (4), we consider a switching observer of the following structure (Juloski *et al.*, 2002):

$$\dot{\hat{x}}(t) = \begin{cases} A_1\hat{x}(t) + Bw(t) + B_1 u(t) + L_1 \Delta y(t), \text{ if } H^T \hat{x} \leq 0 \\ A_2\hat{x}(t) + Bw(t) + B_1 u(t) + L_2 \Delta y(t), \text{ if } H^T \hat{x} > 0, \end{cases} \tag{5}$$

with $L_1, L_2 \in \mathbb{R}^6$ and $\hat{x}(t) \in \mathbb{R}^6$. The observer output is $\hat{y}(t) = C\,\hat{x}(t)$ and $\Delta y(t) = y(t) - \hat{y}(t)$. The system output y is used as observer output injection.
The dynamics of the observer error $\Delta x(t) = x(t) - \hat{x}(t)$ is described by

$$\Delta\dot{x}(t) = \begin{cases} (A_1 - L_1 C)\Delta x, & \text{if } H^T x \leq 0 \wedge H^T \hat{x} \leq 0 \\ (A_2 - L_2 C)\Delta x + \Delta A x, & \text{if } H^T x \leq 0 \wedge H^T \hat{x} > 0 \\ (A_1 - L_1 C)\Delta x - \Delta A x, & \text{if } H^T x > 0 \wedge H^T \hat{x} \leq 0 \\ (A_2 - L_2 C)\Delta x, & \text{if } H^T x > 0 \wedge H^T \hat{x} > 0, \end{cases} \tag{6}$$

where $\Delta A = A_1 - A_2$.
Using the control law

$$u(t) = -K\hat{x}(t), \tag{7}$$

in (4a) yields the closed-loop system:

$$\dot{x}(t) = \begin{cases} A_a x(t) + Bw(t) - B_1 K \Delta x(t), \text{ if } H^T x \leq 0 \\ A_b x(t) + Bw(t) - B_1 K \Delta x(t), \text{ if } H^T x > 0. \end{cases} \tag{8}$$

with $A_a = A_1 - B_1 K$ and $A_b = A_2 - B_1 K$. The interconnected system consist of the equations (6) and (8).
In order to use the controller (7) for disturbance attenuation, we show first that the interconnected system is uniformly convergent. Hereto, we use the notion of uniform convergence and input-to-state convergence (Pavlov, 2004).
Consider the system

$$\dot{z} = F(z, w, t), \tag{9}$$

$t \in \mathbb{R}$, $z \in \mathbb{R}^d$, $w \in \mathbb{R}^m$, where $F(z, w, t)$ is piecewise continuous in t, continuous in w and locally Lipschitz in z. The input $w(t)$ is a piecewise continuous function of t.

Definition 1. System (9) with given input $w(t)$ is said to be (uniformly, exponentially) convergent if

(1) all solutions $z(t)$ are well defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}$, $z(t_0) \in \mathbb{R}^m$;
(2) there exists a unique solution $\bar{z}_w(t)$ defined and bounded for all $t \in (-\infty, +\infty)$;
(3) the solution $\bar{z}_w(t)$ is globally (uniformly, exponentially) asymptotically stable.

Definition 2. (Pavlov, 2004) System (9) is said to be input-to-state convergent if it is uniformly convergent for a class of piecewise continuous inputs and, for every input $w(t)$ taken from this class, the system is input-to-state stable with respect to the system's solution $\bar{z}_w(t)$, i.e. there exist a KL-function $\beta(r,s)$ and a class K_∞-function $\gamma(r)$ such that any solution of this system corresponding to some input $\tilde{w}(t) := w(t) + \Delta w(t)$ satisfies

$$|z(t) - \bar{z}_w(t)| \leq \beta(|z(t_0) - \bar{z}_w(t_0)|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} |\Delta w(\tau)|). \tag{10}$$

The problem at hand now can formally be stated as:
Problem: *Determine, if possible, the controller gain K in* (7) *and the observer gains* L_1, L_2 *in* (5) *such that 1) the interconnected system is uniformly convergent for a class of piecewise continuous inputs* $w : \mathbb{R}^+ \longrightarrow \mathbb{R}^m$ *and 2) for a given disturbance* $w(t)$ *the maximum absolute value of the state components of* (8)*,* $max(|x_i|)$*,* $i = 1, \ldots, n$*, is smaller than the maximum absolute value of the uncontrolled state components* $max(|x_i|)$*,* $i = 1, \ldots, n$ *.*
Note that here we consider a class of bounded periodic disturbances $w(t)$ and that the uncontrolled system can be derived from (4) when u=0.
In order to render the interconnected system uniformly convergent we use a property presented in (Pavlov, 2004):

Property 1. Consider the system

$$\begin{cases} \dot{z} = F(z, y, w), \; z \in \mathbb{R}^d \\ \dot{y} = G(z, y, w), \; y \in \mathbb{R}^q. \end{cases} \tag{11}$$

Suppose that the z-subsystem is input-to-state convergent with respect to y and w. Assume that there exists a class KL-function $\beta_y(r,s)$ such that for any piecewise continuous input $(w(\cdot),\ z(\cdot))$, any solution of the y-subsystem satisfies

$$|y(t)| \leq \beta_y(|y(t_0)|, t - t_0). \tag{12}$$

Then the interconnected system (11) is uniformly convergent.
In Figure 4, a schematic representation of the interconnected system (11) is depicted.

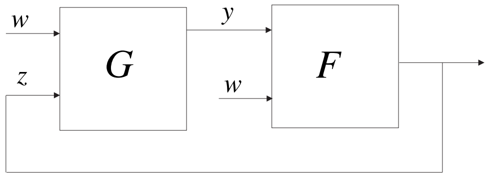

Fig. 4. Schematic representation of the interconnected system (11).

In the present case system (6) is the y-subsystem and system (8) is the z-subsystem in (11).

In order to render the closed-loop system (8) input-to-state convergent we use the results in (Pavlov, 2004). In this paper conditions under which system (8) is uniformly convergent and input-to-state convergent for all piecewise continuous bounded disturbances are given:

Theorem 1. (Pavlov, 2004) Consider the state-space $\mathbb{R}^d$ which is divided into regions Λ_i, $i = 1, ..., l$, by hyperplanes given by equations of the form $H_j^T z + h_j = 0$, for some $H_j \in \mathbb{R}^d$ and $h_j \in \mathbb{R}$, $j = 1, ...k$. Consider the piece-wise affine system

$$\dot{z} = A_i z + b_i + Dw(t), \; for \; z \in \Lambda_i, \; i = 1, ..., l. \quad (13)$$

Suppose that the right-hand side of (13) is continuous and there exists $Q = Q^T > 0$ such that

$$QA_i + A_i^T Q < 0, \; i = 1, ..., l. \quad (14)$$

Then system (13) is exponentially convergent and input-to-state convergent for any piecewise continuous bounded input $w(t)$.

Using *Theorem 1* for (8), the following LMI constraints are derived to guarantee uniform convergence and input-to-state convergence:

$$Q = Q^T > 0, \quad (15a)$$

$$A_a^T Q + QA_a < 0, \quad (15b)$$

$$A_b^T Q + QA_b < 0. \quad (15c)$$

The inequalities (15a)-(15c) are nonlinear matrix inequalities in $\{Q, K\}$ but are linear in $\{Q, K^T Q\}$ and thus can be efficiently solved using standard LMI solvers (such as the LMITOOL in MATLAB).

In order to prove that there exists a class KL function $\beta_y(r,s)$ such that for any piecewise continuous input $(w(\cdot),\ z(\cdot))$, any solution of the observer error dynamics (6) satisfies (12) we show that these are globally exponentially stable. Results for global exponential stability of (6) are given in (Doris *et al.*, 2005*b*), (Juloski *et al.*, 2002):

Theorem 2. The observer error dynamics (6) is globally exponentially stable (GES) for all $x : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ (in the sense of Lyapunov), if there exist matrices $P = P^T > 0, L_1, L_2$ and constants $\tau_1, \tau_2 \geq 0$, $\alpha > 0$ such that the following set of matrix inequalities is satisfied:

$$\begin{bmatrix} (A_2 - L_2 C)^T P + & P\Delta A + \\ +P(A_2 - L_2 C) + \alpha P & +\frac{1}{2}\tau_1 HH^T \\ & \\ \Delta A^T P + & -\tau_1 HH^T \\ +\tau_1 \frac{1}{2} HH^T & \end{bmatrix} \leq 0 \quad (16a)$$

$$\begin{bmatrix} (A_1 - L_1 C)^T P + & -P\Delta A + \\ +P(A_1 - L_1 C) + \alpha P & +\frac{1}{2}\tau_2 HH^T \\ & \\ -\Delta A^T P + & -\tau_2 HH^T \\ +\tau_2 \frac{1}{2} HH^T & \end{bmatrix} \leq 0. \quad (16b)$$

Now, using Property 1 we can conclude that if the LMIs (15) and (16) are satisfied the controlled system (6), (8) is uniformly convergent. Consequently, for any bounded disturbance $w(t)$, with period T there exists a unique globally asymptotically periodic steady-state solution with period T, see (Pavlov, 2004). The latter fact is favorable since it allows to uniquely assess the performance of the controller in terms of disturbance attenuation.

4. EXPERIMENTAL AND SIMULATION RESULTS

In order to validate the applicability and the performance of the presented controller design strategy, simulation and experimental results related to the experimental PWL system are shown for periodic excitations.

More specifically, we show that the PWL beam system in closed-loop with the observer-based controller is uniformly convergent and that the maximum value of the transversal displacement of a point on the beam is significantly smaller when the controller is active than in the open-loop case. The numerical values of the gains L_1, L_2 and K (calculated using the LMIs (15) and (16)), that guarantee global exponential stability of the observer error dynamics, input-to-state convergence of (8) with respect to the input $\Delta x(t)$ and $w(t)$ and, moreover, guarantee disturbance attenuation for the closed-loop system, are shown in the Appendix.

a: open-loop stable steady-state solution (simulation)
b: open-loop unstable steady-state solution (simulation)
c: open-loop steady-state solution (experiment)
d: closed-loop steady-state solution (experiment)
e: closed-loop steady-state solution (simulation)

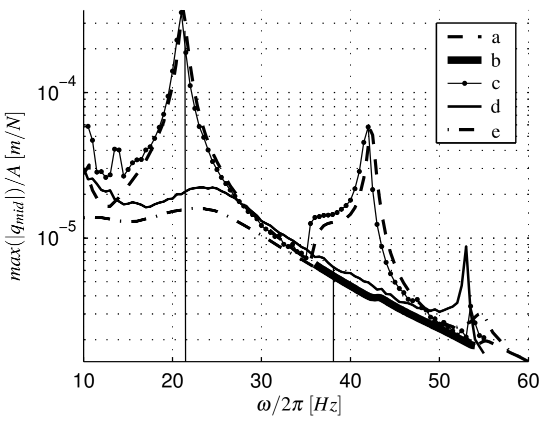

Fig. 5. $max(|q_{mid}|)/A$ for the open-loop system (4) (solid doted line, dashed line) and the controlled system (6) and (8) (dashed-dotted line, thin solid line).

In Figure 5, the scaled maximum absolute value of the transversal displacement of the middle of the beam for a frequency range of periodic disturbances, is shown. More specifically, in this Figure the dashed line (a) represents the simulated stable open-loop ($u = 0$) steady-state response, while the thick solid line (b) represents the simulated unstable open-loop steady-state response. Both responses are based on numerical computations. For the numerical computation of these responses the path following procedure (Ascher *et al.*, 1995) is used. In the same Figure, the solid line with the black points (c) represents the experimental (stable) steady-state response of the open-loop system. These data where taken from a displacement transducer (LVDT) that is mounted on the middle of the beam. Note that by using experimental data we can only show the stable steady-state response of the PWL beam system for obvious reasons. Based on the results related to the open-loop system we conclude that it is not uniformly convergent since it has two steady state-solutions within the frequency range of $35 - 55Hz$. In this frequency range, the unstable steady-state response is a harmonic solution and the stable steady-state response is a $\frac{1}{2}$ subharmonic solution.

Comparing the experimental and the computational results for the open-loop case, we can conclude that the PWL model predicts with high accuracy the responses of the real system. Small differences are due to unavoidable model inaccuracies and noise in the experimental measurement signals.

In the same Figure, the closed-loop response of the PWL beam system is shown. In order to derive this response we use simulation and experimental results. The thin dashed-dotted line (e) depicts the numerically computed closed-loop response (using the same computational methods as in the open-loop case), while the thin solid line (d) depicts the experimentally obtained response (using a LVDT). Based on the experimental and model results, we can conclude that the closed-loop steady-state response is indeed unique in the given frequency range, due to the fact that the closed-loop system (6), (8) is uniformly convergent. Given the fact that the controlled system is uniformly convergent, performance assessment in terms of disturbance attenuation can be performed. Based on the presented results we conclude that there is a nice match between experiments and model results for the closed-loop case. Note that in the frequency range above $50Hz$ there is a significant difference between simulation and experimental results. This difference is due to the fact that resonance frequencies that are insignificant in the open-loop case become significant in the closed-loop case. An explanation of this phenomenon for the experimental system follows. The open-loop system corresponds to the PWL beam without the control action. The 3DOF model that is used to simulate the system dynamics considers only the first eigenmode which occurs at $21Hz$. The second eigenmode, which occurs at $23Hz$, is due to the flexibility of the leaf springs and forces the beam to move in a longitudinal direction. Therefore, it has a minor influence in the transversal displacements of the beam that we want to attenuate. In the third eigenmode ($54Hz$) the middle of the beam stands still. Due to the fact that the excitation mechanism (rotating mass-unbalance) is mounted in the middle of the beam this eigenmode is not excited. By applying a control action on the point depicted in Figure 3, the third eigenmode becomes active and influences the closed-loop response.

For additional insight in the obtained results, we present a time response of $q_{mid}(t)$ for four different initial conditions x_{0i}, $i = 1, \ldots, 4$, in Figure 6. In this Figure, the thin solid line and the dashed (thin and thick) lines represent a time response which has been computed using the $3DOF$ model of the PWL beam. In the same Figure the thick solid line represents a time response which is based on experimental data. Herein, the excitation frequency and the force amplitude are $f = 38\,Hz$ and $A = 58\,N$, respectively. Figure 6 shows that the time response $q_{mid}(t)$ converges to a unique steady-state solution for different initial conditions see small picture in Figure 6.

The comparison of the plot of $max(|q_{mid}|)$ in Figure 5 for the open- and closed-loop system shows that the closed-loop system responses are significantly smaller than those of the open-loop system. Based on this comparison, it is concluded that the effect of the disturbances w to the PWL beam is attenuated due to the control. Note that especially the nonlinear resonances are suppressed. This can also be noticed in Figure 7, where the time response of $q_{mid}(t)$ in steady-state is shown. In this Figure, the thin solid line corresponds to the open-loop solutions of q_{mid} while the thick solid line corresponds to the closed-loop solution. The excitation frequency for this case is $21.5\,Hz$ and the force amplitude is $A = 20.5\,N$ (see also the vertical solid line in Figure 5). It should be noted that we have achieved similar results for every point on the beam.

5. CONCLUSIONS AND FUTURE WORK

An observer-based controller design strategy is applied to a periodically excited beam with a one-sided support in order to achieve disturbance attenuation. A linear output-feedback control law is used for this purpose. A non-smooth yet simple model in combination with a model-based piecewise linear observer estimates the states of the experimental beam system with high accuracy.

The controller design strategy is based on the notion of input-to-state convergence and uniform convergence. By using the input-to-state convergence property we render the observer/controller combination stable. Moreover, the uniform convergence property of the closed-loop guarantees that we can uniquely define the performance of the closed-loop system in terms of disturbance attenuation.

Experimental and simulation results are presented to assess the controller performance. According to these results the controller performs well, since it renders the closed-loop convergent and suppresses all the (nonlinear) resonance peaks of the beam's transversal vibrations considerably in the presence of periodic disturbances. Despite the fact that uncertain factors are present, such as inevitable model mismatch and noise in the output injection term of the observer, the observer-based controller attenuates the periodic disturbances in all given excitation frequencies.

Due to the fact that the proposed low-order observer/controller combination is able to cope with distributed parameter systems, the presented case study can be considered as a benchmark for observer and controller designs for complex non-smooth and hybrid engineering systems.

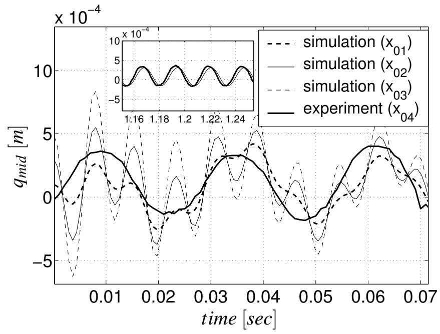

Fig. 6. Steady-state solution for $q_{mid}(t)$; comparison experimental and model results for the closed-loop system ($\omega = 2\pi 38$ rad/s and $A = 58$ N).

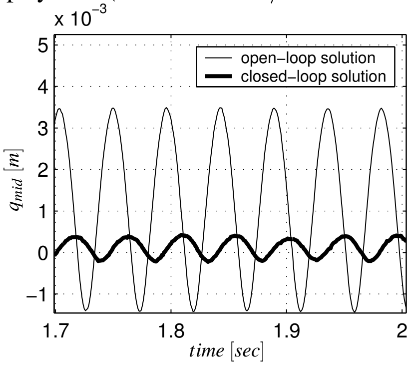

Fig. 7. Experimental steady-state solution $q_{mid}(t)$ of the closed-loop (solid line) and open-loop (dashed line) system for $\omega = 2\pi 21.5$ rad/s and $A = 20.5$ N.

6. APPENDIX

The matrices M, K_s, B_s, L_1, L_2, K, C and k_{nl} are

$$M = \begin{bmatrix} 3.7898 & 0.1626 & 1.6218 \\ 0.1626 & 6.5622 & 2.4830 \\ 1.6218 & 2.4830 & 2.7756 \end{bmatrix},$$

$$K_s = 10^6 \begin{bmatrix} 2.4148 & 0.0061 & 1.1254 \\ 0.0061 & 0.7704 & 0.3082 \\ 1.1254 & 0.3082 & 0.658 \end{bmatrix},$$

$$B_s = 10^2 \begin{bmatrix} 108.9828 & 3.3894 & 52.0716 \\ 3.3894 & 89.0314 & 33.9348 \\ 52.0716 & 33.9348 & 41.7312 \end{bmatrix},$$

$$L_1 = 10^4 \; [\, 0.0101 \; 0.0079 \; -0.0197 \;\; 0.7094 \; 0.5963 \; -1.4450 \,],$$

$$L_2 = 10^4 \; [\, 0.01 \; 0.0078 \; -0.0196 \;\; 0.8823 \; 0.6426 \; -1.5885 \,],$$

$$K = 10^4 \; [\, -0.8796 \; 0.6541 \; -1.6953 \;\; 0.0027 \; 0.0055 \; -0.0236 \,],$$

$$C = [-0.9579 \quad 1.2165 \quad -0.2642 \quad 0 \quad 0 \quad 0]$$

and $k_{nl} = 160000\, N/m$.

REFERENCES

Alessandri, A. and P. Coletta (2001*a*). Design of Luenberger observers for a class of hybrid systems. In: *Proc. of Hybrid Systems: Computation and Control*. Rome, Italy. pp. 7–18.

Alessandri, A. and P. Coletta (2001*b*). Switching observers for continuous-time and discrete-time linear systems. In: *Proc. of the American Control Conference*. Arlington, USA. pp. 2516–2521.

Alessandri, A. and P. Coletta (2003). Design of observers for switched discrete-time linear systems. In: *Proc. of the American Control Conference*. Denver, Colorado, USA. pp. 2785–2790.

Ascher, U.M., R.M.M. Mattheij and R.D. Russel (1995). Numerical solution of boundary value problem for ordinary differential equations. In: *Society for Industrial and Applied Mathematics*. Philadelphia, USA.

Demidovich, B.Pb. (1967). Lectures on stability theory (in Russian). *Nauka, Moscow, Russia.*

Doris, A., A. Juloski, M. Heemels, N. van de Wouw and H. Nijmeijer (2005*a*). Switching observer design for an experimental piece-wise linear beam system. In: *Proc. of the 16th IFAC conf.*. Prague, Czech Republic.

Doris, A., N. van de Wouw and H. Nijmeijer (2005*b*). Disturbance attenuation for a periodically excited piece-wise linear beam system. In: *Proc. of the ENOC Conf.*. Eindhoven, The Netherlands. pp. 301–310.

Feng, G., G. P. Lu and S. S. Zhou (2002). An approach to H_∞ controller synthesis of piece-wise linear systems. *International Press communications in information and systems vol 2 no. 3.*

Fey, R.H.B., D.H. van Campen and A. de Kraker (1996). Longterm structural dynamics of mechanical systems with local nonlinearities,. *ASME Journal of Vibration and Acoustics, 118(2).*

Hassibi, A. and S. Boyd (2005). Quadratic stabilization and control of piece-wise-linear systems. In: *Proc. of the American Control Conf.*. Philadelphia PA., USA. pp. 3659–3664.

Juloski, A.Lj., W.P.M.H. Heemels and S. Weiland (2002). Observer design for a class of piece-wise affine systems. In: *Proc. of the 41st IEEE Conf.: Decision and Control*. Las Vegas, Nevada, USA. pp. 123–126.

Pavlov, A. (2004). The output regulation problem: a convergent dynamics approach. *PhD thesis, Eindhoven University of Technology.*

Pavlov, A., A. Progromsky, N. van de Wouw and H. Nijmeijer (2004). Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Systems & Control Letters 52.* pp. 257–261.

Rantzer, A. and M. Johansson (2000). Piece-wise linear quadratic optimal control. *IEEE transactions on automatic control vol 45 no. 4.* pp. 629–637.