Modular Model Reduction of Interconnected Systems: A Top-Down Approach ⋆

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Abstract: Complex systems often involve interconnected subsystem models developed by multi-disciplinary teams. To simulate and control such systems, a reduced-order model of the interconnected system is required. In the scope of this paper, we pursue this goal by subsystem reduction to warrant modularity of the reduction approach. However, reducing subsystem models affects not only the subsystems accuracy, but also the interconnected model accuracy, making it difficult to predict a priori the accuracy impact of subsystem reduction. To address this challenge, we introduce a top-down approach using mathematical tools from robust performance analysis, enabling the translation of accuracy requirements from the interconnected model to the subsystem level. This enables independent subsystem reduction while ensuring the desired accuracy of the interconnected model. We demonstrate the effectiveness of our approach through a structural dynamics case study.

Keywords: Complex systems; Model reduction; Control of interconnected systems; Robust performance; Top-down approach; Error bounds.

1. INTRODUCTION

Many complex dynamical systems consist of multiple interconnected subsystems, which are often designed, manufactured, and tested independently before they are integrated into the complete interconnected system. Typically, high-fidelity subsystem models are created that accurately describe the dynamic behavior of each subsystem. However, to use these models for the analysis of the dynamic behavior of the interconnected system, often, model order reduction (MOR) is required.

In this work, we consider MOR of linear time-invariant (LTI) systems (Antoulas, 2005; Besselink et al., 2013). Examples of commonly used projection-based methods used for MOR are the proper orthogonal decomposition method (Kerschen et al., 2005), reduced basis methods (Boyaval et al., 2010), balancing methods (Gugercin and Antoulas, 2004; Moore, 1981; Glover, 1984) and Krylov methods (Grimme, 1997). All of these MOR methods have in common that they aim to compute a reduced-order model (ROM) that still provides an accurate description of the system dynamics but is significantly reduced in complexity in comparison to the high-order model.

The main goal of this work is to construct a model of the interconnected system that 1) satisfies given accuracy requirements and 2) is of a suitable complexity such that it can be used for the application of the model, e.g., for controller design or diagnostics. The accuracy of the ROM is determined by the difference between the input-to-output behavior of the high-order and the interconnected ROMs.

There are several approaches to reduce the complexity of such interconnected models. Accurate ROMs can be obtained with direct reduction of the entire interconnected model as a whole. However, this completely destroys the interconnection structure (Lutowska, 2012). To avoid this problem, there are several structure-preserving reduction methods available for interconnected systems (Sandberg and Murray, 2009; Vandendorpe and Dooren, 2008) Unfortunately, these methods still require knowledge of the entire interconnected system when computing subsystem ROMs.

Since the subsystem models are developed individually and, often, in parallel, we aim to reduce the complexity of the subsystem *modularly*, i.e., on an individual basis. Such a modular approach has the additional advantages that the computational cost of computing the ROM is significantly reduced (Vaz and Davison, 1990) and different reduction methods can be applied for each subsystem individually (Reis and Stykel, 2008). In the structural dynamics field, component mode synthesis (CMS) methods are also modular (de Klerk et al., 2008).

However, with modular MOR of interconnected systems, we reduce the complexity of subsystem models, which generally leads to an error of the subsystem ROM in compar-

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ison to the high-order subsystem model. If the subsystem ROMs are then interconnected, these errors will propagate to the interconnected system ROM, potentially exceeding the given accuracy requirements on the interconnected system model. Therefore, the need arises for methods to relate interconnected model accuracy requirements to accuracy requirements on the level of individual subsystem models.

The main contribution of this work is a top-down approach that allows us to translate frequency-dependent accuracy requirements on the interconnected model to frequency-dependent accuracy requirements on the input-to-output behavior at a subsystem level. Then, if the subsystem models are reduced (individually) using any reduction method that can satisfy these subsystem accuracy requirements, the accuracy requirements on the interconnected model are also guaranteed to be satisfied. We use methods from robust performance analysis to establish this relation.

In Janssen et al. (2022b), the mathematical foundation of this method is established, including stability guarantees and a top-down approach that allows for the computation of accuracy requirement on *one of the subsystem models* based on requirements on the *largest singular value error of the interconnected model*. In the current paper, we extend this approach in two ways. Namely, we now allow for the computation of accuracy requirements for

(1) *all subsystems simultaneously*, and for
(2) *all input-output pairs* for each of these subsystems,

and the subsequent *modular* reduction of all subsystems by solving a single optimization problem. Furthermore, we show on the illustrative example as used in Janssen et al. (2022b) how these extensions can be used to significantly reduce the system using the top-down, modular MOR approach. Note that in Janssen et al. (2022a), it is shown with a preliminary version of the relation established in Janssen et al. (2022b) that this relation can be used to determine a priori error bounds to the error of the interconnected ROM based on error bounds of ROMs of the subsystems (following a bottom-up approach, i.e., considering the inverse problem of translating the accuracy of subsystem models to that of the interconnected system).

The paper is organized as follows. Section 2 gives the problem statement including the modelling framework. In Section 3, we show how the problem can be reformulated into a robust performance problem and consequently how it can be solved. The top-down approach is demonstrated on an illustrative structural dynamics example system in Section 4. Finally, the conclusions are given in Section 5.

Notation. The set of real numbers is denoted by $\mathbb{R}$, of positive real numbers by $\mathbb{R}_{>0}$, and of complex numbers by $\mathbb{C}$. Given a transfer function (matrix) $G(s)$, s denotes the Laplace variable and $\|G\|_\infty$ its $\mathcal{H}_\infty$-norm. The real rational subspace of $\mathcal{H}_\infty$ is denoted by $\mathcal{RH}_\infty$. Given complex matrix A, A^H denotes its conjugate transpose, $\bar{\sigma}(A)$ its largest singular value, $\rho(A)$ its spectral radius, $A = \text{diag}(A_1, A_2)$ a block-diagonal matrix with submatrices A_1 and A_2, and $A \succ 0$ denotes that A is positive definite. The identity matrix of size n is denoted by I_n.

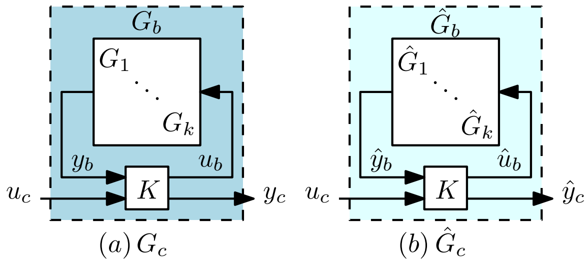

Fig. 1. Block diagram representation of the (a) high-order and (b) reduced-order interconnected systems. K represents a static interconnection.

2. PROBLEM STATEMENT

Consider k high-order, LTI subsystems $j \in \mathbf{k} := \{1, \ldots, k\}$ with transfer functions $G_j(s)$, inputs u_j and outputs y_j of dimensions m_j and p_j, respectively, and McMillan degree n_j. We collect the subsystem transfer functions in the block-diagonal transfer function

$$G_b(s) := \text{diag}\left(G_1(s), \ldots, G_k(s)\right), \tag{1}$$

for which the total number of inputs and outputs are then given by $m_b := \sum_{j=1}^{k} m_j$ and $p_b := \sum_{j=1}^{k} p_j$, respectively. We define inputs $u_b^\top := \left[u_1^\top, \ldots, u_k^\top\right]$ and outputs $y_b^\top := \left[y_1^\top, \ldots, y_k^\top\right]$.

In this paper, we compute the ROM of the system modularly, i.e., we reduce each subsystem model independently. Therefore, consider subsystem ROMs $j \in \mathbf{k}$ and their transfer functions $\hat{G}_j(s)$, each with inputs $\hat{u}_j$ and outputs $\hat{y}_j$ with dimensions m_j and p_j, respectively, and McMillan degree r_j. Let the reduced-order block-diagonal transfer function be given as

$$\hat{G}_b(s) := \text{diag}(\hat{G}_1(s), \ldots, \hat{G}_k(s)). \tag{2}$$

Then, we define inputs $\hat{u}_b^\top := \left[\hat{u}_1^\top, \ldots, \hat{u}_k^\top\right]$ and outputs $\hat{y}_b^\top := \left[\hat{y}_1^\top, \ldots, \hat{y}_k^\top\right]$ with dimensions m_b and p_b, respectively. Both the high-order and reduced-order subsystem models are interconnected according to

$$\begin{bmatrix} u_b \\ y_c \end{bmatrix} = K \begin{bmatrix} y_b \\ u_c \end{bmatrix}, \begin{bmatrix} \hat{u}_b \\ \hat{y}_c \end{bmatrix} = K \begin{bmatrix} \hat{y}_b \\ u_c \end{bmatrix}, K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}. \tag{3}$$

Here, we have also introduced external inputs u_c, high-order external outputs y_c and reduced-order external outputs $\hat{y}_c$. The number of external inputs and outputs is given by m_c and p_c, respectively. Then, the transfer function from u_c to y_c is given by the upper linear fractional transformation (LFT) of $G_b(s)$ and K, which yields

$$G_c(s) := K_{21} G_b(s)(I - K_{11} G_b(s))^{-1} K_{12} + K_{22}. \tag{4}$$

Since we only reduce the subsystem models, the interconnection structure is preserved. Therefore, the reduced-order interconnected system transfer function from u_c to $\hat{y}_c$ is, similar to (4), given by

$$\hat{G}_c(s) := K_{21} \hat{G}_b(s)(I - K_{11} \hat{G}_b(s))^{-1} K_{12} + K_{22}. \tag{5}$$

This model framework is illustrated in Figure 1.

The approach developed in this paper is completely frequency-dependent. Therefore, we analyze the transfer functions for $s = i\omega$ for some $\omega \in \mathbb{R}$. Furthermore, we assume that we can define frequency-dependent requirements on the reduction error dynamics

$$E_c(i\omega) := \hat{G}_c(i\omega) - G_c(i\omega). \tag{6}$$

Specifically, we consider the requirement that $E_c(i\omega)$ is contained in the set

$$\mathcal{E}_c(\omega) := \left\{E_c(i\omega) \mid \bar{\sigma}\big(V_c(\omega)E_c(i\omega)W_c(\omega)\big) < 1\right\}, \quad (7)$$

where diagonal scaling matrices $V_c(\omega) \in \mathbb{R}^{p_c \times p_c}_{>0}$ and $W_c(\omega) \in \mathbb{R}^{m_c \times m_c}_{>0}$ can be used to scale the input-output pairs of $E_c(i\omega)$ to fit the requirements.

The main goal of this work is to find specifications to the subsystem reduction error dynamics

$$E_j(i\omega) := \hat{G}_j(i\omega) - G_j(i\omega) \quad (8)$$

for each subsystem $j \in \mathbf{k}$ based on $\mathcal{E}_c(\omega)$, i.e., using a top-down approach. Specifically, we aim to find some sets

$$\mathcal{E}_j(\omega) := \left\{E_j(i\omega) \mid \bar{\sigma}\big(W_j^{-1}(\omega)E_j(i\omega)V_j^{-1}(\omega)\big) \leq 1\right\} \quad (9)$$

such that $E_j(i\omega) \in \mathcal{E}_j(\omega)$ for all $j \in \mathbf{k}$ implies $E_c(i\omega) \in \mathcal{E}_c(i\omega)$. In (9), $V_j(\omega)$ and $W_j(\omega)$ are diagonal scaling matrices.

Note that for any matrix $A \in \mathbb{C}^{m\times n}$, $\bar{\sigma}(A) < 1$ is only satisfied if the magnitude of all elements in A are less than one. Therefore, both for the interconnected system in (7) and the subsystems in (9), the error requirements implicitly provide a bound on each input-output pair individually, which is a relevant extension to the scaling matrices of Janssen et al. (2022b), where each scaling matrix is defined as a scaled identity matrix, generally resulting in more conservative subsystem requirements.

Once the sets $\mathcal{E}_j(\omega)$ have been determined, it becomes possible to compute subsystem ROMs independently. Namely, if each subsystem $j \in \mathbf{k}$ is reduced such that it satisfies $E_j(i\omega) \in \mathcal{E}_j(\omega)$, it is guaranteed that the reduction error dynamics of the interconnected system satisfy $E_c(i\omega) \in \mathcal{E}_c(\omega)$.

3. METHODOLOGY

To find the subsystem accuracy specifications characterized by $(\mathcal{E}_1(\omega), \ldots, \mathcal{E}_k(\omega))$ based on the requirement $\mathcal{E}_c(\omega)$ for $\omega \in \mathbb{R}$, we reformulate the problem as a robust performance problem. First, we define weighting functions $V_j(\omega) \in \mathbb{R}^{m_j\times m_j}_{>0}$ and $W_j(\omega) \in \mathbb{R}^{p_j \times p_j}_{>0}$ such that $E_j(i\omega)$ can be written as

$$E_j(i\omega) = W_j(\omega)\Delta_j(i\omega)V_j(\omega), \quad (10)$$

for some $\Delta_j(s) \in \mathcal{RH}_\infty$ satisfying $\|\Delta_j\|_\infty \leq 1$. Then, we rewrite the subsystem ROM as

$$\hat{G}_j(i\omega) = G_j(i\omega) + W_j(\omega)\Delta_j(i\omega)V_j(\omega). \quad (11)$$

By replacing $\hat{G}_j(i\omega)$ with $G_j(i\omega) + W_j(\omega)\Delta_j(i\omega)V_j(\omega)$ for all $j \in \mathbf{k}$ in Figure 1(b) and comparing it with the high-order system $G_c(i\omega)$ in Figure 1(a), we obtain the block diagram in Figure 2. Additionally, we define the nominal transfer function $N(s)$, i.e., the grey block in Figure 2, which is given by

$$N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & 0 \end{bmatrix}, \text{where} \quad (12)$$

$$N_{11}(s) = K_{11}(I - G_b(s)K_{11})^{-1},$$
$$N_{12}(s) = (I - K_{11}G_b(s))^{-1}K_{12}, \text{ and}$$
$$N_{21}(s) = K_{21}(I - G_b(s)K_{11})^{-1}.$$

The interconnected system error dynamics E_c, as shown in Figure 2, for $\omega \in \mathbb{R}$, can then be given by

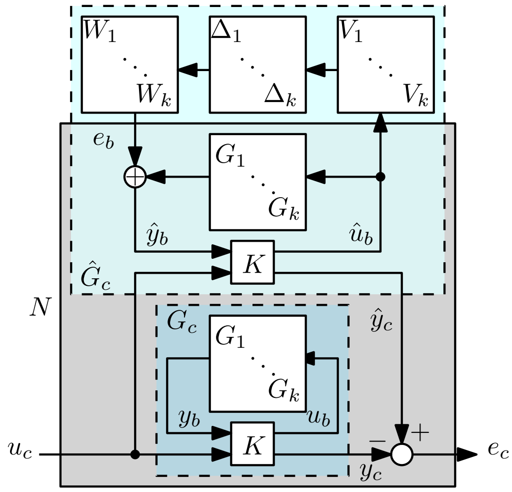

Fig. 2. Block diagram representation of the error dynamics of the interconnected system, $E_c = \hat{G}_c - G_c$, as a function of V_b, W_b, Δ and the nominal system N.

$$E_c(i\omega) = N_{21}(i\omega)W_b(\omega)\Delta_b(i\omega)V_b(\omega)\big(I - N_{11}(i\omega)W_b(\omega)\Delta_b(i\omega)V_b(\omega)\big)^{-1}N_{12}(i\omega), \quad (13)$$

where $\Delta_b := \operatorname{diag}(\Delta_1, \ldots, \Delta_k)$, $V_b := \operatorname{diag}(V_1, \ldots, V_k)$, and $W_b := \operatorname{diag}(W_1, \ldots, W_k)$.

To relate the requirements $\mathcal{E}_c$ on the accuracy of the interconnected ROM to accuracy requirements $(\mathcal{E}_1, \ldots, \mathcal{E}_k)$ on the subsystem ROMs, consider the following sets of matrices.

$$\mathbf{V} := \{\operatorname{diag}(V_1, \ldots, V_k, V_c) \mid V_c = \operatorname{diag}(v_c \in \mathbb{R}^{p_c}_{>0}), V_j = \operatorname{diag}(v_j \in \mathbb{R}^{m_j}_{>0}) \;\forall\; j \in \mathbf{k}\}, \quad (14)$$

$$\mathbf{W} := \{\operatorname{diag}(W_1, \ldots, W_k, W_c) \mid W_c = \operatorname{diag}(w_c \in \mathbb{R}^{m_c}_{>0}), W_j = \operatorname{diag}(w_j \in \mathbb{R}^{p_j}_{>0}) \;\forall\; j \in \mathbf{k}\}, \quad (15)$$

$$\mathbf{D} := \{(D_\ell, D_r) \mid d_1, \ldots, d_k, d_c \in \mathbb{R}_{>0}, D_\ell = \operatorname{diag}(d_1 I_{p_1}, \ldots, d_k I_{p_k}, d_c I_{m_c}), D_r = \operatorname{diag}(d_1 I_{m_1}, \ldots, d_k I_{m_k}, d_c I_{p_c})\}. \quad (16)$$

Then, we can pose the following theorem.

Theorem 1. Let $\omega \in \mathbb{R}$, $V(\omega) \in \mathbf{V}$, and $W(\omega) \in \mathbf{W}$, with $\mathbf{V}$ and $\mathbf{W}$ as in (14) and (15), respectively. Consider the system (4), error dynamics (13) as in Figure 2, and requirements $\mathcal{E}_c(\omega)$ as in (7) and $\mathcal{E}_j(\omega)$, $j \in \mathbf{k}$, as in (9). If there exists a $(D_\ell, D_r) \in \mathbf{D}$, with $\mathbf{D}$ as in (16), such that

$$\begin{bmatrix} W^{-2}(\omega)D_r^{-1} & N^H(i\omega) \\ N(i\omega) & V^{-2}(\omega)D_\ell \end{bmatrix} \succ 0, \quad (17)$$

with N as in (12), then, it holds that if all subsystem ROMs satisfy their respective error requirements, i.e., $E_j(i\omega) \in \mathcal{E}_j(\omega)$ for all $j \in \mathbf{k}$, then the interconnected ROM will satisfy the interconnected system requirements, i.e., $E_c(i\omega) \in \mathcal{E}_c(\omega)$.

Proof. We prove the theorem with the following remarks.

(1) From Janssen et al. (2022b), Theorem 3.5 follows that any $E_j(i\omega)$ satisfying $E_j(i\omega) \in \mathcal{E}_j(\omega)$ for all $j \in \mathbf{k}$, it holds that $E_c(i\omega) \in \mathcal{E}_c(\omega)$, which is equivalent to

$$\mu_{\mathbf{\Delta}}\big(V(\omega)N(i\omega)W(\omega)\big) < 1. \quad (18)$$

Here, $\mu_{\mathbf{\Delta}}$ denotes the structured singular value (Packard and Doyle, 1993, Definition 3.1) defined by

$$\mu_{\mathbf{\Delta}}(M) := \min\{\bar{\sigma}(\Delta) \mid \det(I - M\Delta) = 0, \Delta \in \mathbf{\Delta}\}^{-1}$$

for $M \in \mathbb{C}^{(m_b+p_c)\times(p_b+m_c)}$ and $\mathbf{\Delta}$ given by

$$\mathbf{\Delta} := \{\text{diag}(\Delta_1, \ldots, \Delta_k, \Delta_c) \mid \Delta_c \in \mathbb{C}^{m_c\times p_c}, \Delta_j \in \mathbb{C}^{p_j \times m_j}, j \in \mathbf{k}\}. \quad (19)$$

(2) In Janssen et al. (2022b), Theorem 3.6, it is proven that for any $M \in \mathbb{C}^{(m_b+p_c)\times(p_b+m_c)}$, if there exists a $(D_\ell, D_r) \in \mathbf{D}$ such that $MD_rM^H \prec D_\ell$, then, given $\Delta \in \mathbf{\Delta}$, $\mu_{\mathbf{\Delta}}(M) < 1$.

(3) Since $V(\omega)N(i\omega)W(\omega) \in \mathbb{C}^{(m_b+p_c)\times(p_b+m_c)}$, we have that (18) is satisfied if

$$V(\omega)N(i\omega)W(\omega)D_rW^H(\omega)N^H(i\omega)V^H(\omega) \prec D_\ell. \quad (20)$$

(4) Since $W(\omega)\in\mathbf{W}$ and $(D_\ell, D_r)\in\mathbf{D}$ are diagonal, real, and positive definite, $W(\omega)D_rW^H(\omega) = W^2(\omega)D_r \succ 0$.

(5) Since $V(\omega)\in\mathbf{V}$ is diagonal, real, and positive definite, after pre- and post-multiplying both sides of the inequality (20) by $V^{-1}(\omega)$, we obtain

$$N(i\omega)W^2(\omega)D_rN^H(i\omega) \prec V^{-2}(\omega)D_\ell. \quad (21)$$

(6) We obtain (17) as the equivalent Schur's complement of (21). □

With Theorem 1, for $\omega \in \mathbb{R}$ any combination of requirements on $E_c(i\omega)$ and $E_j(i\omega)$, $j \in \mathbf{k}$, can be validated with (17). We will now show how this can be used directly to find subsystem specifications $(\mathcal{E}_1(\omega), \ldots, \mathcal{E}_k(\omega))$ as in (9) for which it is guaranteed that the requirement $\mathcal{E}_c(\omega)$ as in (7) is satisfied.

Consider the system (4) and error dynamics (13) as in Figure 2. Let $\omega \in \mathbb{R}$ and assume that the requirement $\mathcal{E}_c(\omega)$ as in (7) is given. Consider the optimization problem

$$\begin{aligned} \text{given} \quad & V_c(\omega), W_c(\omega) \\ \text{minimize} \quad & \text{tr}(V^{-2}(\omega)) + \text{tr}(W^{-2}(\omega)) \\ \text{subject to} \quad & \begin{bmatrix} W^{-2}(\omega)D_r^{-1} & N^H(i\omega) \\ N(i\omega) & V^{-2}(\omega)D_\ell \end{bmatrix} \succ 0, \\ & V(\omega) \in \mathbf{V}, W(\omega) \in \mathbf{W}, (D_\ell, D_r) \in \mathbf{D}. \end{aligned} \quad (22)$$

It follows from Theorem 1, that for any feasible solution to (22), we have that our requirement on the interconnected system $E_c(i\omega) \in \mathcal{E}_c(\omega)$ is satisfied if $E_j(i\omega) \in \mathcal{E}_j(\omega)$ as in (9) for all $j \in \mathbf{k}$.

With (22), an optimization problem to find a solution to the problem as stated in Section 2 is given. Namely, we can compute local error requirements, *for all subsystems* $(\mathcal{E}_1(\omega), \ldots, \mathcal{E}_k(\omega))$ *simultaneously*, given the global error requirement $\mathcal{E}_c(\omega)$. Note that in Janssen et al. (2022b), it is shown how these error requirements can be computed for a *single* subsystem, whereas the other subsystems remain unreduced. Solving the optimization problem, i.e., minimizing $\text{tr}(V^{-2}(\omega)) + \text{tr}(W^{-2}(\omega))$, is relatively trivial by iteratively solving for V, W, and D, similar to D-K iteration (see Zhou and Doyle (1998)):

(1) Initially, set $d_j = d_c = 1$ for all $j \in \mathbf{k}$.

(2) Relax $V^{-2}(\omega)$ to diagonal matrix $\mathcal{V} := V^{-2}(\omega)$ and $W^{-2}(\omega)$ to diagonal matrix $\mathcal{W} := W^{-2}(\omega)$ and fix D_r and D_ℓ; the optimization problem (22) is then linear and $\text{tr}(\mathcal{V})+\text{tr}(\mathcal{W})$ can be minimized with semidefinite programming (SDP) tools.

(3) Fix $V(\omega)$, $W(\omega)$ at the solutions of step (2) and keep $d_c = 1$ fixed. Find the scaling matrices D_r and D_ℓ that maximizes γ while satisfying the inequality

$$V^{-2}D_\ell - N(i\omega)W^2D_rN^H(i\omega) \succ \gamma \quad (23)$$

with SDP tools. Note that for $\gamma = 0$, the matrix inequality (23) is equivalent to (17), as proven in the proof of Theorem 1. By maximizing γ, the cost function $\text{tr}(V^{-2}(\omega)) + \text{tr}(W^{-2}(\omega))$ can be minimized further in the next iteration.

(4) Repeat step (2) and (3) until sufficient convergence in $V(\omega)$, $W(\omega)$ is reached, i.e., $\text{tr}(V^{-2}(\omega))+\text{tr}(W^{-2}(\omega))$ is no longer decreasing (significantly).

In general, we aim to find a solution in which $V_j(\omega)$ and $W_j(\omega)$ are as "large" as possible, which allows for more error in the subsystems, which in turn allows for further reduction of the system as a whole. The choice of cost function $\text{tr}(V^{-2}(\omega)) + \text{tr}(W^{-2}(\omega))$ allows to relax the optimization problem (22) to be easily solved iteratively with SDP solvers. Within (17), there is an infinite number of possible combinations $(\mathcal{E}_1(\omega), \ldots, \mathcal{E}_k(\omega))$ that guarantee the satisfaction of the requirement $\mathcal{E}_c(\omega)$. By choosing the cost function $\text{tr}(V^{-2}(\omega)) + \text{tr}(W^{-2}(\omega))$ in (22), given $\mathcal{E}_c(\omega)$, the solution converges to a single distribution of subsystem accuracy requirements $(\mathcal{E}_1(\omega), \ldots, \mathcal{E}_k(\omega))$.

Remark 2. The advantage of this cost function is that it automatically penalizes individual elements in $V(\omega)$ and $W(\omega)$ that are important for the accuracy of the interconnected system and allows for more error on inputs-to-outputs pairs of the subsystem transfer functions that are less important for the overall accuracy of the interconnected system. Moreover, if additional knowledge on subsystems is available, e.g., we know that one of the subsystems is more difficult to reduce than the others, the cost function can be trivially extended to $\sum_{j=1}^{k} \alpha_j \left(\text{tr}(V_j^{-2}(\omega)) + \text{tr}(W_j^{-2}(\omega))\right)$, where α_j is a weighting variable used to provide some control over the distribution of subsystem requirements in the solution to the optimization problem (22).

However, specifying the exact definition of an "optimal" distribution of $(\mathcal{E}_1(\omega), \ldots, \mathcal{E}_k(\omega))$, and, furthermore, finding this distribution, are still open problems. There are various arguments explaining why these problems are not trivial, one of which is the fact that reducing the order of a subsystem generally leads to discrete steps in which the error increases. We expect that to find an optimal distribution of requirements, a heuristic approach, in which communication between subsystems takes place, is required.

In the next section, we will show using an illustrative example that minimizing the cost function $\text{tr}(V^{-2}(\omega)) + \text{tr}(W^{-2}(\omega))$ is sufficient to compute a distribution of subsystem error requirements that allows for significant reduction of each of the subsystems given a required $\mathcal{E}_c(\omega)$.

4. EXAMPLE

In this section, we show on a mechanical system consisting of three interconnected beams, as illustrated schematically in Figure 3, that the top-down approach can be used to determine frequency-dependent accuracy requirements $\mathcal{E}_1(\omega), \mathcal{E}_2(\omega), \mathcal{E}_3(\omega)$ for the ROMs of the three beams based on given requirements $\mathcal{E}_c(\omega)$ for the accuracy of the interconnected ROM, allowing for the independent reduction of these subsystems.

Subsystems 1 and 3 are cantilever beams which are connected at their free ends to free-free beam 2 with transla-

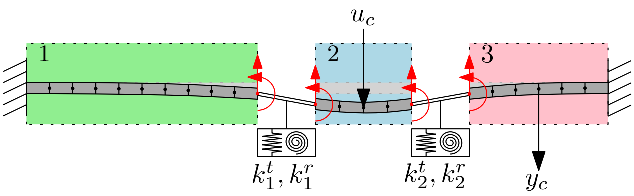

Fig. 3. Example system: A free-free beam (Subsystem 2) connected to cantilever beams (Subsystems 1 and 3) with translational and rotational springs.

Table 1. Parameter values of each subsystem.

Parameter	Subsys. 1	Subsys. 2	Subsys. 3
Cross-sect. area [m^2]	1×10^{-6}	1×10^{-6}	1×10^{-6}
2nd area moment [m^4]	1×10^{-9}	1×10^{-9}	1×10^{-9}
Young's modulus [Pa]	2×10^{11}	2×10^{11}	2×10^{11}
Mass density [kg/m^3]	8×10^3	8×10^3	8×10^3
Length [m]	1	0.4	0.6
# of elements [-]	50	20	30
# of inputs m_j [-]	2	5	2
# of outputs p_j [-]	2	4	3
# of states n_j [-]	200	84	120

tional and rotational springs. The stiffness of both translational interconnecting springs is $k_1^t = k_2^t = 1 \times 10^5$ N/m. The stiffness of both rotational interconnecting springs is $k_1^r = k_2^r = 1 \times 10^3$ Nm/rad. The external input force u_c [N] is applied to the middle of subsystem 2 in the transversal direction. The external output displacement y_c [m] is measured at the middle of subsystem 3 in the transversal direction.

Each beam/subsystem is discretized by linear two-node Euler beam elements (only bending, no shear, see Craig Jr and Kurdila (2006)) of equal length. Per node we have one rotational and one translational degree of freedom. For each beam, viscous damping is modelled using 1% modal damping. Physical and geometrical parameter values of the three beams and information about finite element discretization, the number of states, and the number of subsystem inputs and outputs are given in Table 1. The Bode plot of the unreduced system G_c is given by the black line in Figure 4. For this system, the top-down approach is applied with the following steps:

(1) All frequencies ω over a grid of 1000 logarithmically equally spaced points in the interval $[10^{2.5}, 10^5]$ rad/s are evaluated. For these frequencies, a frequency-dependent accuracy requirement $\mathcal{E}_c(\omega)$ as in (7) is provided by the user. In this example, $V_c^{-1}(\omega)$ is given as some fraction β_1 of $|G_c(i\omega)|$, which is bounded below by β_2, $V_c^{-1}(\omega) = \max\{\beta_1|G_c(i\omega)|, \beta_2\}$, i.e., where $\beta_1 = 0.1$ and $\beta_2 = 5 \times 10^{-7}$ m/N and $W_c(\omega) = 1$. The resulting accuracy requirement is indicated by the grey areas in Figure 4. Any error $E_c(i\omega)$ satisfies $E_c(i\omega) \in \mathcal{E}_c(\omega)$ for the given frequencies ω if and only if $\bar{\sigma}(W_c(\omega)E_c(i\omega)V_c(\omega)) < 1$, i.e., is entirely in the grey area in the top figure of Figure 5.

(2) The optimization problem (22) is solved, which results in subsystem requirements $(\mathcal{E}_1(\omega), \mathcal{E}_2(\omega), \mathcal{E}_3(\omega))$, for the given frequencies ω. These requirements consist, for each subsystem $j = 1, 2, 3$, of diagonal scaling matrices $W_j(\omega)$ and $V_j(\omega)$ that describe the scaling of individual input-output pairs in the requirements. Any error $E_j(i\omega)$ satisfies $E_j(i\omega) \in \mathcal{E}_j(\omega)$ for the given frequencies ω if and only if $\bar{\sigma}(W_j^{-1}(\omega)E_c(i\omega)V_j^{-1}(\omega)) \leq 1$, i.e., is entirely in the grey area in the bottom figure of Figure 5.

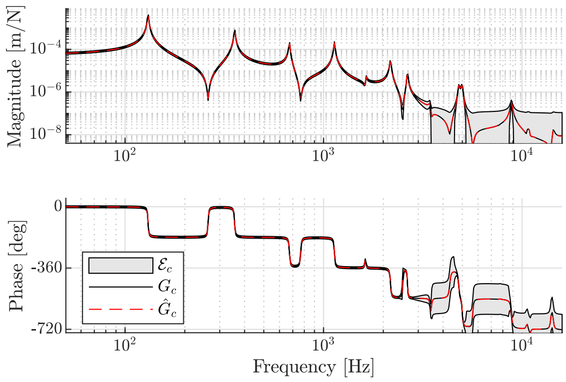

Fig. 4. The Bode plots of the high-order interconnected model G_c with $n_1 = 200$, $n_2 = 84$ and $n_3 = 120$, the areas in which a system satisfies error requirements $E_c(i\omega) \in \mathcal{E}_c(\omega)$, and of the reduced-order interconnected model $\hat{G}_c$ with $r_1 = 19$, $n_2 = 17$ and $n_3 = 13$.

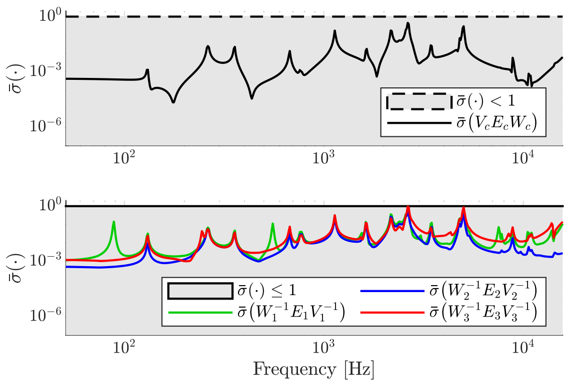

Fig. 5. Largest singular value plots, indicating that the subsystem ROMs satisfy the required $E_j(i\omega) \in \mathcal{E}_j(\omega)$ for $j = 1, 2, 3$, and, consequently, that the interconnected ROM satisfies $E_c(i\omega) \in \mathcal{E}_c(\omega)$.

With the proposed approach, in principle, any MOR method can be used for which it is possible to find a subsystem ROM such that the computed requirements are satisfied, i.e., $E_j(i\omega) \in \mathcal{E}_j(\omega)$. It is even possible to use different MOR methods for each subsystem. However, as the purpose of this work is to show how subsystem error requirements can be determined from the top down, we apply a standard MOR method to all of the subsystems.

(3) Using the computed $(\mathcal{E}_1(\omega), \mathcal{E}_2(\omega), \mathcal{E}_3(\omega))$, model reduction techniques can be used to construct subsystem ROMs satisfying $E_j(i\omega) \in \mathcal{E}_j(\omega), j = 1, 2, 3$, for all frequencies ω. In this example, we use frequency-weighted balanced truncation (FWBT) (Enns, 1984) to reduce the individual subsystems. In FWBT, we can minimize $\|\hat{W}_j(G_j - \hat{G}_j)\hat{V}_j\|_\infty$, where $\hat{V}_j$ and $\hat{W}_j$ are transfer function estimates fitted with a minimum-phase transfer function (Boyd et al., 2004, Chapter 6.5), of the computed weighting functions V_j and W_j, respectively. In Figure 5, for each subsystem, we show that $E_j(i\omega) \in \mathcal{E}_j(\omega)$ with the green, blue and red lines, respectively. With FWBT, the ROMs can be reduced to $r_1 = 19$, $r_2 = 17$ and $r_3 = 13$ while satisfying the given subsystem error requirements.

(4) To validate the approach, we show that $E_c(i\omega) \in \mathcal{E}_c(\omega)$ is indeed satisfied for the interconnected system, as indicated by the black line in Figure 5. Additionally, we show the interconnected ROM in the top figure in Figure 4 with a dashed red line, and see that it indeed satisfies the requirement.

With these steps, we show that, for this system, it is possible to determine error requirements at a subsystem level that 1) guarantee that the overall requirements on the accuracy of the ROM for the interconnected system are satisfied and 2) allow for enough "room" for the significant reduction of the subsystem models. Even with the standard MOR technique we apply on a subsystem level, i.e., FWBT, it is already possible to reduce the number of states of the interconnected system from $\sum n_j = 404$ to $\sum r_j = 49$ within the given requirements. If more involved MOR methods are applied, the subsystem models can potentially be reduced even further within the computed accuracy specifications $\mathcal{E}_j(\omega)$.

5. CONCLUSIONS

In this paper, we demonstrate how, for models of interconnected LTI subsystems, accuracy requirements on the interconnected system can be translated to accuracy requirements of subsystems. With these requirements, modular model reduction can be applied while guaranteeing the required accuracy of the overall interconnected system. The approach is based on the reformulation of subsystem reduction errors to weighted uncertainties. This allows for mathematical tools from the field of robust performance analysis to be applied. We show that with this reformulation, a single matrix inequality can be used to analyze if accuracy requirements at the subsystem can guarantee that given accuracy requirements at the interconnected system level are satisfied. Moreover, we propose an optimization problem that can be used to compute these subsystem accuracy requirements and we show how this problem is solved. Finally, the approach is illustrated with a structural dynamics example, for which the complexity in terms of the number of states in the overall system can be reduced by at least 87% for the given requirements.

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REFERENCES

Antoulas, A.C. (2005). *Approximation of large-scale dynamical systems*. SIAM, Philadelphia.

Besselink, B., Tabak, U., Lutowska, A., Van de Wouw, N., Nijmeijer, H., Rixen, D.J., Hochstenbach, M.E., and Schilders, W.H.A. (2013). A comparison of model reduction techniques from structural dynamics, numerical mathematics and systems and control. *Journal of Sound and Vibration*, 332(19), 4403–4422.

Boyaval, S., Le Bris, C., Lelievre, T., Maday, Y., Nguyen, N.C., and Patera, A.T. (2010). Reduced basis techniques for stochastic problems. *Archives of Computational methods in Engineering*, 17(4), 435–454.

Boyd, S., Boyd, S.P., and Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

Craig Jr, R.R. and Kurdila, A.J. (2006). *Fundamentals of structural dynamics*. John Wiley, Hoboken, N.J.

de Klerk, D., Rixen, D.J., and Voormeeren, S.N. (2008). General framework for dynamic substructuring: history, review and classification of techniques. *AIAA journal*, 46(5), 1169–1181.

Enns, D.F. (1984). Model reduction with balanced realizations: An error bound and a frequency weighted generalization. In *Proceedings of the 23rd IEEE Conference on Decision and Control*, 127–132.

Glover, K. (1984). All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds. *International Journal of Control*, 39(6), 1115–1193.

Grimme, E. (1997). *Krylov projection methods for model reduction*. Ph.D. thesis, University of Illinois at Urbana-Champaign, Urbana-Champaign, USA.

Gugercin, S. and Antoulas, A.C. (2004). A survey of model reduction by balanced truncation and some new results. *International Journal of Control*, 77(8), 748–766.

Janssen, L.A.L., Besselink, B., Fey, R.H.B., Hossein Abbasi, M., and van de Wouw, N. (2022a). A priori error bounds for model reduction of interconnected linear systems using robust performance analysis. In *2022 American Control Conference (ACC)*, 1867–1872.

Janssen, L.A.L., Besselink, B., Fey, R.H.B., and van de Wouw, N. (2022b). Modular model reduction of interconnected systems: A robust performance analysis perspective. URL `https://arxiv.org/abs/2210.15958`.

Kerschen, G., Golinval, J., Vakakis, A.F., and Bergman, L.A. (2005). The method of proper orthogonal decomposition for dynamical characterization and order reduction of mechanical systems: an overview. *Nonlinear Dynamics*, 41(1), 147–169.

Lutowska, A. (2012). *Model order reduction for coupled systems using low-rank approximations*. Ph.D. thesis, Eindhoven University of Technology.

Moore, B.C. (1981). Principal component analysis in linear systems - controllability, observability, and model reduction. *IEEE Transactions on Automatic Control*, AC-26(1), 17–32.

Packard, A. and Doyle, J. (1993). The complex structured singular value. *Automatica*, 29(1), 71–109.

Reis, T. and Stykel, T. (2008). A survey on model reduction of coupled systems. In *Model Order Reduction: Theory, Research Aspects and Applications*, 133–155. Springer.

Sandberg, H. and Murray, R.M. (2009). Model reduction of interconnected linear systems. *Optimal Control Applications and Methods*, 30(3), 225–245.

Vandendorpe, A. and Dooren, P.V. (2008). Model reduction of interconnected systems. In *Model Order Reduction: Theory, Research Aspects and Applications*, 305–321. Springer.

Vaz, A.F. and Davison, E.J. (1990). Modular model reduction for interconnected systems. *Automatica*, 26(2), 251–261.

Zhou, K. and Doyle, J.C. (1998). *Essentials of robust control*. Prentice hall Upper Saddle River, NJ.