

On convergence properties of piecewise affine systems

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In this paper convergence properties of piecewise affine (PWA) systems are studied. In general, a system is called convergent if all its solutions converge to some bounded globally asymptotically stable steady-state solution. The notions of exponential, uniform and quadratic convergence are introduced and studied. It is shown that for non-linear systems with discontinuous right-hand sides, quadratic convergence, i.e., convergence with a quadratic Lyapunov function, implies exponential convergence. For PWA systems with continuous right-hand sides it is shown that quadratic convergence is equivalent to the existence of a common quadratic Lyapunov function for the linear parts of the system dynamics in every mode. For discontinuous bimodal PWA systems it is proved that quadratic convergence is equivalent to the requirements that the system has some special structure and that certain passivity-like condition is satisfied. For a general multimodal PWA system these conditions become sufficient for quadratic convergence. An example illustrating the application of the obtained results to a mechanical system with a one-sided restoring characteristic, which is equivalent to an electric circuit with a switching capacitor, is provided. The obtained results facilitate bifurcation analysis of PWA systems excited by periodic inputs, substantiate numerical methods for computing the corresponding periodic responses and help in controller design for PWA systems.

1. Introduction

In many control problems it is required that controllers are designed in such a way that all solutions of the corresponding closed-loop system “forget” their initial conditions. Actually, one of the main tasks of feedback is to eliminate the dependency of solutions on initial conditions. In this case, all solutions converge to some steady-state solution that is determined only by the input of the closed-loop system. This input can be, for example, a command signal or a signal generated by a feedforward part of the controller or, as in the observer design problem, it can be the measured signal from the observed system. Such a “convergence” property of a system plays an important role in many non-linear control problems including tracking, synchronization, observer design, and the output regulation problem; see,

e.g., Special Issue (1997) and Pavlov *et al.* (2005b) and references therein. From a dynamics point of view, “convergence” is an interesting property because it excludes the possibility of multiple coexisting steady-state solutions: a convergent system excited by a periodic input has a unique globally asymptotically stable periodic solution (Demidovich 1967). Also, the notion of “convergence” is a powerful tool for analysis of time-varying systems. This tool can be used, for example, for performance analysis of non-linear control systems (Heertjes *et al.* 2006, Pavlov *et al.* 2006).

For asymptotically stable linear systems excited by inputs, convergence is a natural property. Indeed, due to linearity of the system, every solution is globally asymptotically stable and, therefore, all solutions of such a system “forget” their initial conditions and converge to each other. After transients, the dynamics of the system are determined only by the input. For non-linear systems, in general, global asymptotic

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stability of a system with the zero input does not guarantee that all solutions of this system with a non-zero input “forget” their initial conditions and converge to each other. There are many examples of non-linear globally asymptotically stable systems that, being excited by a periodic input, have coexisting periodic solutions. Such periodic solutions do not converge to each other. This fact indicates that for non-linear systems the convergent dynamics property requires additional conditions.

The property that all solutions of a system “forget” their initial conditions and converge to some steady-state solution has been addressed in a number of publications. In Pliss (1966) this property was investigated for systems with right-hand sides that are periodic in time. In that work systems with a unique periodic globally asymptotically stable solution were called convergent. Later, the definition of convergent systems given in Pliss (1966) was extended in Demidovich (1967) to the case of systems that are not necessarily periodic in time. According to Demidovich (1967), a system is called convergent if there exists a unique solution that is bounded on the whole time axis and this solution is globally asymptotically stable; see also Pavlov *et al.* (2004). Obviously, if such a solution does exist, all other solutions, regardless of their initial conditions, converge to this solution, which can be considered as a steady-state solution. In Demidovich (1961, 1967) (see also Pavlov *et al.* (2004)) a sufficient condition for such a convergence property for smooth non-linear systems was presented. With the development of absolute stability theory, Yakubovich (1964) showed that for a linear system with one scalar non-linearity satisfying some incremental sector condition, the circle criterion guarantees the convergence property for this system with any non-linearity satisfying this incremental sector condition. The property of solutions converging to each other was also addressed in LaSalle and Lefschetz (1961), Yoshizawa (1966) and Chua and Green (1976).

Several decades after these publications, the interest in stability properties of solutions with respect to one another revived. Incremental stability, incremental input-to-state stability and contraction analysis are some of the terms related to such properties. In the mid-nineties, Lohmiller and Slotine (1998) introduced the notion of contraction and independently extended the result of Demidovich. A different approach was pursued in Fromion *et al.* (1996, 1999). In this approach a dynamical system is considered as an operator that maps some functional space of inputs to a functional space of outputs. If such an operator is Lipschitz continuous (has a finite incremental gain or is incrementally stable), then, under certain observability and reachability conditions, all solutions of a state-space

realization of this system converge to each other. In Angeli (2002) a Lyapunov approach was developed to study both the global uniform asymptotic stability of all solutions of a system (in Angeli (2002), this property is called incremental stability) and the so-called incremental input-to-state stability property, which is compatible with the input-to-state stability approach; see, e.g., Sontag (1995). Problems of analysis and design of convergent systems (in the sense of Demidovich) were studied in Pavlov *et al.* (2005a,b).

It is interesting to note that for a long time it was a common belief that a non-linear system perturbed by a periodic external signal should have a unique periodic steady-state response. Van der Pol and van der Mark (1927) demonstrated that this is not the case even for a simple second order system. Nevertheless, in their mathematical analysis, Cartwright and Littlewood (1945) remarked that their “faith in results was at one time sustained only by the experimental evidence that stable subharmonics of two distinct orders did occur”. With the development of Melnikov’s method (see, e.g., Wiggins (2003)) it is now well known that a periodically driven system can exhibit chaotic behaviour. This situation is impossible for convergent systems, so the convergent systems are those that would not surprise van der Pol, Cartwright and Littlewood. Nevertheless conditions for convergence are important from a practical point of view.

In this paper we study the convergence properties for the class of piecewise affine (PWA) systems. This class of systems attracted a lot of attention over the last years, see, e.g., Johansson and Rantzer (1998), Bemporad *et al.* (2000), Heemels *et al.* (2002), Johansson (2002), Juloski (2004) and references therein. It includes, for example, mechanical systems with piecewise linear restoring characteristics, systems with friction, electrical circuits with diodes and other switching characteristics, and control systems with switching controllers. In this paper we present conditions for convergence for both continuous (but non-smooth) and discontinuous PWA systems. Most of the existing conditions for convergence (or convergence-type properties like incremental stability, contraction) are formulated in terms of the existence of some Lyapunov-type function and for this reason are hardly checkable (Yoshizawa 1966, Chua and Green 1976, Angeli 2002). In this paper we present computationally efficient conditions for convergence of PWA systems. Existing computationally tractable conditions for this property require continuous differentiability of the right-hand side (see, e.g., Lohmiller and Slotine (1998) and Fromion *et al.* (1999)) and therefore they are not directly applicable to PWA systems, which have non-smooth, and, in general, discontinuous right-hand sides. This fact indicates the novelty of the presented results.

Results on convergence can be used in several ways. It is known that a convergent system excited with a periodic input has a unique globally asymptotically stable periodic solution with the same period time as the period time of the input; see, e.g., Demidovich (1967) and Pavlov *et al.* (2005a). In bifurcation analysis such a property allows one to significantly reduce computational efforts for constructing the bifurcation diagram. Namely, if the system is convergent, only period-1 steady-state solutions can exist, while other responses (and thus bifurcations giving rise to such responses), such as period- k , $k=2, 3, \dots$, solutions or quasi-periodic behaviour, cannot occur. In practice these period- k resonances often represent some unwanted dynamics of the system and should be avoided. If a system is designed to be convergent or it is made convergent by means of feedback, it does not have these problematic dynamics. Moreover, the existence and uniqueness of a periodic response of a convergent system to a periodic excitation substantiates many numerical methods for computing periodic solutions of periodically excited systems, see, e.g., Aprille and Trick (1972), Hajj and Skelboe (1981), Parker and Chua (1989) and van den Eijnde and Schoukens (1990).

The paper is organized as follows. In §2 we provide preliminaries on systems with discontinuous right-hand sides. In §3 definitions of (uniformly, exponentially, quadratically) convergent systems are provided and some properties of convergent systems are studied. In §4 necessary and sufficient conditions for quadratic convergence of PWA systems with continuous right-hand sides are provided. In §5 we present necessary and sufficient conditions for quadratic convergence of bimodal PWA systems with possibly discontinuous right-hand sides and extend the sufficient conditions to the case of multi-modal PWA systems. Theoretical results presented in this paper are illustrated in §6 with application to a perturbed mass-spring-damper system with a one-sided spring, which is equivalent to some *RLC* circuit with a diode. Conclusions are presented in §7.

2. Preliminaries

In this paper we consider systems of the form

$$\dot{x} = f(x, t), \quad (1)$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $f(x, t)$ is a possibly discontinuous vector field. It is assumed that $f(x, t)$ satisfies some mild regularity assumptions that guarantee the existence of solutions of the system in the sense of Filippov (1988). According to Filippov (1988), one can construct a

set-valued function $F(x, t)$ such that a solution of the differential inclusion

$$\dot{x} \in F(x, t)$$

is called a solution for system (1). By definition, the solution $x(t, t_0, x_0)$ with the initial condition $x(t_0, t_0, x_0) = x_0$ is an absolutely continuous function of time.

Consider a scalar continuously differentiable function $V(x)$. Define a time derivative of this function along solutions of system (1) as follows

$$\dot{V} := \frac{\partial V(x)}{\partial x} \dot{x}(t, t_0, x_0).$$

Since $V(x)$ is continuously differentiable and the solution $x(t, t_0, x_0)$ is an absolutely continuous function of time, the derivative of $V(x(t, t_0, x_0))$ exists for almost all t in the interval of existence $[t_0, \bar{T})$ of the solution $x(t, t_0, x_0)$.

For the function $V(x)$ we can also define its upper derivative along solutions of system (1) as follows:

$$\dot{V}^*(x, t) = \sup_{\xi \in F(x, t)} \left(\frac{\partial V(x)}{\partial x} \xi \right).$$

Then for almost all $t \in [t_0, \bar{T})$ it follows that

$$\dot{V}(x(t, t_0, x_0), t) \leq \dot{V}^*(x(t, t_0, x_0), t). \quad (2)$$

Remark 1: Notice that in the domains of continuity of the function $f(x, t)$ the derivative of $V(x)$ along solutions of system (1) equals $\dot{V}(x, t) = (\partial V(x)/\partial x)f(x, t)$. According to Filippov (1988, p. 155), for a continuously differentiable function $V(x)$ it holds that if the inequality

$$\frac{\partial V(x)}{\partial x} f(x, t) \leq 0$$

is satisfied in the domains of continuity of the function $f(x, t)$, then the inequality $\dot{V}^*(x, t) \leq 0$ holds for all $(x, t) \in \mathbb{R}^{n+1}$.

For the sake of clarity, we provide definitions of stability of an arbitrary solution.

Definition 1: A solution $\bar{x}(t)$ of system (1) that is defined for $t \in (t_*, +\infty)$ is said to be

- stable if for any $t_0 \in (t_*, +\infty)$ and $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $|x(t_0) - \bar{x}(t_0)| < \delta$ implies $|x(t) - \bar{x}(t)| < \varepsilon$ for all $t \geq t_0$;
- uniformly stable if it is stable and the number δ in the definition of stability is independent of t_0 ;
- globally asymptotically stable if it is stable and any solution of system (1) satisfies $|x(t) - \bar{x}(t)| \rightarrow 0$ as $t \rightarrow +\infty$;
- uniformly globally asymptotically stable if it is uniformly stable and for any $R > 0$ and any $\varepsilon > 0$ there exists $T(\varepsilon, R) > 0$ such that if $|x(t_0)| \leq R$,

- $t_0 \in (t_*, +\infty)$, then $|x(t) - \bar{x}(t)| < \varepsilon$ for all $t \geq t_0 + T(\varepsilon, R)$;
- globally exponentially stable if there exist constants $C > 0$ and $\alpha > 0$ such that any solution of system (1) satisfies

$$|x(t) - \bar{x}(t)| \leq Ce^{-\alpha(t-t_0)} |x(t_0) - \bar{x}(t_0)|. \quad (3)$$

3. Convergent systems

In this section we give definitions of convergent systems. These definitions extend the definition given in (Demidovich 1967).

Definition 2: System (1) is said to be

- convergent if there exists a solution $\bar{x}(t)$ satisfying the following conditions
 - $\bar{x}(t)$ is defined and bounded on \mathbb{R} ,
 - $\bar{x}(t)$ is globally asymptotically stable;
- uniformly convergent if it is convergent and $\bar{x}(t)$ is uniformly globally asymptotically stable;
- exponentially convergent if it is convergent and $x(t)$ is globally exponentially stable.

The solution $\bar{x}(t)$ is called a steady-state solution. As follows from the definition of convergence, any solution of a convergent system “forgets” its initial condition and converges to some steady-state solution. In general, the steady-state solution $\bar{x}(t)$ may be non-unique. But for any two steady-state solutions $\bar{x}_1(t)$ and $\bar{x}_2(t)$ it holds that $|\bar{x}_1(t) - \bar{x}_2(t)| \rightarrow 0$ as $t \rightarrow +\infty$. At the same time, for uniformly convergent systems the steady-state solution is unique, in the sense that it is the only solution that is bounded on \mathbb{R} (Pavlov et al. 2005b).

Remark 2: In the original definition of convergent systems given in Demidovich (1967), the steady-state solution $\bar{x}(t)$ is required to be unique. In Definition 2 this requirement of uniqueness is omitted, since for the practically important case of uniform convergence, uniqueness of the steady-state solution can be proved as a corollary to the definition of the uniform convergence.

The convergence property is an extension of stability properties of asymptotically stable linear time-invariant (LTI) systems. One can easily show that for a piecewise continuous vector-function $g(t)$, which is defined and bounded on \mathbb{R} , the system $\dot{x} = Ax + g(t)$ with a Hurwitz matrix A is exponentially convergent.

In systems and control theory, time dependency of the right-hand side of system (1) is usually due to

some input. This input may represent, for example, a disturbance or a feedforward control signal. Below we will consider convergence properties for systems with inputs. So, instead of systems of the form (1), we consider systems

$$\dot{x} = f(x, w), \quad (4)$$

with state $x \in \mathbb{R}^n$ and input $w \in \mathbb{R}^m$. In the sequel we will consider the class $\overline{\mathbb{PC}}_m$ of piecewise continuous inputs $w(t): \mathbb{R} \rightarrow \mathbb{R}^m$ that are bounded on the whole time axis \mathbb{R} . We assume that the function $f(x, w)$ is bounded on any compact set of (x, w) and the set of discontinuity points of the function $f(x, w)$ has measure zero. Under these assumptions on $f(x, w)$, for any input $w \in \overline{\mathbb{PC}}_m$ the differential equation $\dot{x} = f(x, w(t))$ has well-defined solutions in the sense of Filippov.

Below we define the convergence property for systems with inputs.

Definition 3: System (4) is said to be (uniformly, exponentially) convergent if it is (uniformly, exponentially) convergent for every input $w \in \overline{\mathbb{PC}}_m$. In order to emphasize the dependency on the input $w(t)$, the steady-state solution is denoted by $\bar{x}_w(t)$.

Note that the property of (uniform) convergence is invariant under smooth coordinate transformations. The property of exponential convergence is invariant under coordinate transformations $z = \psi(x)$ with the functions ψ and ψ^{-1} being globally Lipschitz.

Convergent systems enjoy various properties which are encountered in asymptotically stable LTI systems, but which are not usually met in general asymptotically stable non-linear systems; see e.g., Pavlov et al. 2005b. As an illustration, we provide a statement that summarizes some properties of uniformly convergent systems excited by periodic or constant inputs.

Proposition 1 (Demidovich 1967, see also Pavlov et al. 2005b): *Suppose system (4) with a given input $w(t)$ is uniformly convergent. If the input $w(t)$ is constant, the corresponding steady-state solution $\bar{x}_w(t)$ is also constant; if the input $w(t)$ is periodic with period T , then the corresponding steady-state solution $\bar{x}_w(t)$ is also periodic with the same period T .*

Below we give an important technical definition of quadratic convergence.

Definition 4: System (4) is called quadratically convergent if there exists a positive definite matrix $P = P^T > 0$ and a number $\alpha > 0$ such that for any input $w \in \overline{\mathbb{PC}}_m$, the function $V(x_1, x_2) = 1/2(x_1 - x_2)^T \times P(x_1 - x_2)$ satisfies

$$\dot{V}^*(x_1, x_2, t) \leq -2\alpha V(x_1, x_2), \quad (5)$$

where $V^*(x_1, x_2, t)$ is the upper derivative of the function $V(x_1, x_2)$ along any two solutions $x_1(t)$ and $x_2(t)$ of the corresponding differential inclusion $\dot{x} \in F(x, w(t))$, i.e.,

$$\dot{V}^*(x_1, x_2, t) = \sup_{\xi_1 \in F(x_1, w(t))} \left(\frac{\partial V}{\partial x_1}(x_1, x_2) \xi_1 \right) + \sup_{\xi_2 \in F(x_2, w(t))} \left(\frac{\partial V}{\partial x_2}(x_1, x_2) \xi_2 \right).$$

Quadratic convergence is a useful tool for establishing exponential convergence for systems with possibly discontinuous right-hand sides, as follows from the next lemma.

Lemma 1: *If system (4) is quadratically convergent, then it is exponentially convergent.*

Proof: See Appendix.

Although for the case of systems with continuous right-hand sides, statements similar to Lemma 1 have been studied in several publications (see, e.g., Yoshizawa (1966) Chua and Green (1976), Angeli (2002)), the case of systems with discontinuous right-hand sides considered in Lemma 1 is not that straightforward and involves a lot of additional technicalities.

Remark 3: As follows from Remark 1, quadratic convergence (inequality (5)) is equivalent to the inequality

$$(x_1 - x_2)^T P(f(x_1, w) - f(x_2, w)) \leq -\alpha(x_1 - x_2)^T P(x_1 - x_2) \quad (6)$$

being satisfied for all $w \in \mathbb{R}^m$ and all x_1 and x_2 from the continuity domain of the function $f(x, w)$. This fact will be used in the proofs of subsequent results.

For some particular classes of systems, quadratic convergence can be equivalent to certain simple easily verifiable conditions. In the next two sections we present computationally tractable conditions for quadratic convergence for PWA systems with continuous and discontinuous right-hand sides.

4. Convergence for continuous PWA systems

Consider the state space \mathbb{R}^n divided into polyhedral cells Λ_i , $i = 1, \dots, l$, by hyperplanes given by equations of the form $H_j^T x + h_j = 0$, for some $H_j \in \mathbb{R}^n$ and $h_j \in \mathbb{R}$, $j = 1, \dots, k$. We will consider piecewise-affine systems of the form

$$\dot{x} = A_i x + b_i + Dw, \quad \text{for } x \in \Lambda_i, \quad i = 1, \dots, l. \quad (7)$$

Here $A_i \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times m}$ and $b_i \in \mathbb{R}^n$, $i = 1, \dots, l$, are constant matrices and vectors, respectively. The vector

$x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^m$ is the input. The hyperplanes $H_j^T x + h_j = 0$, $j = 1, \dots, k$, are the switching surfaces. The following theorem establishes necessary and sufficient conditions for the quadratic convergence of system (7) with a continuous right-hand side.

Theorem 1: *Consider system (7). Suppose the right-hand side of system (7) is continuous. Then the following statements are equivalent.*

- (i) *System (7) is quadratically convergent.*
- (ii) *There exists a positive definite matrix $P = P^T > 0$ such that*

$$PA_i + A_i^T P < 0, \quad i = 1, \dots, l. \quad (8)$$

Proof: See Appendix.

Remark 4: Condition (8) is a standard condition for exponential stability of the piecewise linear system $\dot{x} = A_i x$, $x \in \Lambda_i$, without inputs. In Theorem 1 we deal with the convergence property of piecewise affine systems (7) with inputs. In general, exponential stability of a system without inputs does not imply the convergence property of this system excited by non-zero inputs, even though the system can be exponentially stable with a quadratic Lyapunov function. For general PWA systems this statement will be illustrated in the next section.

Remark 5: Theorem 1 can be viewed as a counterpart of the following statement for smooth non-linear systems of the form

$$\dot{x} = f(x, w), \quad x \in \mathbb{R}^n, \quad w \in \mathbb{R}^m. \quad (9)$$

It is known—see, e.g., results on convergent systems in Demidovich (1967) and Pavlov *et al.* (2004), results on incremental stability in Fromion *et al.* (1999) and results on contraction analysis in Lohmiller and Slotine (1998)—that if the matrix inequality

$$P \frac{\partial f}{\partial x}(x, w) + \frac{\partial f^T}{\partial x}(x, w) P \leq -Q \quad (10)$$

holds for some $P = P^T > 0$, $Q = Q^T > 0$ and all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, then system (9) is quadratically convergent (contracting or quadratically incrementally stable, as it would be called in Lohmiller and Slotine (1998) and Fromion *et al.* (1999), respectively). In the above mentioned references $f(x, w)$ is required to be continuously differentiable with respect to x . Although there is a clear analogy between condition (10) for smooth systems and condition (8) for PWA systems, the above mentioned result is not directly applicable to PWA systems since the right-hand side of a PWA system is not continuously differentiable. In the proof of Theorem 1 we overcome this technical difficulty of non-differentiability and introduce a technique that

will be used in subsequent results on convergence of PWA systems with discontinuous right-hand sides.

Remark 6: As follows from Remark 3, Theorem 1 shows that for a continuous piecewise-affine vector-field $f(x, w)$ of the form

$$f(x, w) = A_i x + b_i + Dw, \quad \text{for } x \in \Delta_i, \quad i = 1, \dots, l,$$

condition (8) is equivalent to the inequality (6) being satisfied for some $\alpha > 0$ and all $w \in \mathbb{R}^m$, x_1 and $x_2 \in \mathbb{R}^n$. This fact will be used in subsequent results in this paper.

The continuity requirement on the right-hand side of system (7) can be checked with the following lemma, whose proof can be found, for example, in Pavlov *et al.* (2005b).

Lemma 2: Consider system (7). The right-hand side of system (7) is continuous iff the following condition is satisfied: for any two cells Δ_i and Δ_j having a common boundary $H_{ij}^T x + h_{ij} = 0$ the corresponding matrices A_i and A_j and the vectors b_i and b_j satisfy the equalities

$$\begin{aligned} G_{ij} H_{ij}^T &= A_i - A_j \\ G_{ij} h_{ij} &= b_i - b_j, \end{aligned} \quad (11)$$

for some vector $G_{ij} \in \mathbb{R}^n$.

5. Convergence for discontinuous PWA systems

Based on the result of the previous section one can conjecture that an arbitrary piecewise affine system (7) with a possibly discontinuous right-hand side is convergent provided there is a common quadratic Lyapunov function for the state matrices A_i . However this is not the case as one can see from the following simple example. Suppose the system dynamics is governed by the following scalar differential equation with a discontinuous right-hand side

$$\dot{x} = a(x), \quad x \in \mathbb{R}^1,$$

where the function $a(x)$ is depicted schematically in figure 1. It is seen that the system belongs to the class of piecewise affine systems and in each region the dynamics are linear. Moreover, it is not difficult to see that the system is globally exponentially stable with the common quadratic Lyapunov function $V = x^2$.

Now suppose that the dynamics of the system is modified with an additive input signal

$$\dot{x} = a(x) + w(t), \quad x \in \mathbb{R}^1.$$

It is clear from the picture that for some input signals (e.g., constant) the dynamics of the system can depend on the initial conditions (one can take such a constant input signal that the system has two asymptotically

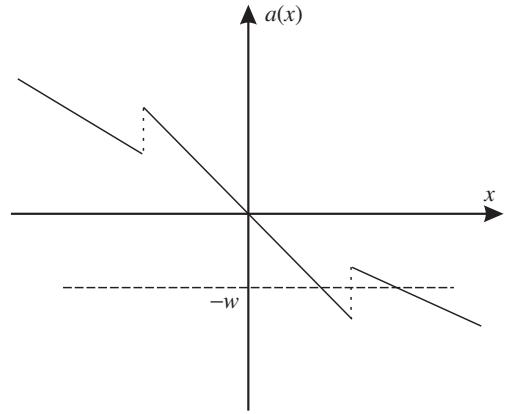


Figure 1. Piecewise affine characteristics $a(x)$.

stable equilibria), or, in other words, the system is not convergent. This simple example illustrates that even the existence of a common Lyapunov function for each mode of a piecewise affine system is not sufficient to guarantee its convergence. Moreover, this example shows that the requirement of continuity of the right-hand side of a PWA systems plays an important role for the convergence property and we have to be careful when analysing convergence for discontinuous PWA systems. In fact, for bimodal piecewise-affine systems the existence of a common Lyapunov function and the requirement similar to the continuity conditions (11), provided that some passivity-like condition is satisfied, are even necessary and sufficient for the quadratic convergence, as follows from the result presented hereafter.

Consider the bimodal system

$$\dot{x} = \begin{cases} A_1 x + b_1 + Dw, & \text{for } H^T x \geq 0 \\ A_2 x + b_2 + Dw, & \text{for } H^T x < 0, \end{cases} \quad (12)$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ and A_i , b_i , $i = 1, 2$, and D are matrices of the appropriate dimensions. The switching plane is determined by the constant vector $H \in \mathbb{R}^n$. Denote $\Delta A := A_1 - A_2$, $\Delta b := b_1 - b_2$.

Theorem 2: Consider system (12). The following statements are equivalent.

- (i) System (12) is quadratically convergent.
- (ii) There exist a positive definite matrix $P = P^T > 0$ and numbers $\beta > 0$ and $\gamma \geq 0$ satisfying the following LMI

$$\begin{pmatrix} PA_1 + A_1^T P + \beta I & P\Delta A - \frac{1}{2}HH^T \\ \Delta A^T P - \frac{1}{2}HH^T & -HH^T \end{pmatrix} \leq 0, \quad (13)$$

$$P\Delta b = -\gamma H. \quad (14)$$

- (iii) There exist a positive definite matrix $P = P^T > 0$, a number $\gamma \in \{0, 1\}$ and a vector $G \in \mathbb{R}^n$ such that

$$PA_i + A_i^T P < 0, \quad i = 1, 2, \quad (15)$$

$$\Delta A = GH^T, \quad (16)$$

$$P\Delta b = -\gamma H. \quad (17)$$

Proof: see Appendix.

Remark 7: In part (iii) of Theorem 2 there are two options: $\gamma=0$ and $\gamma=1$. For the case $\gamma=0$ condition (17) yields $\Delta b=0$. This, together with condition (16), implies that the right-hand side of system (12) is continuous (see Lemma 2). For the case of $\gamma=1$, we see that the discontinuity may occur only due to the shift terms b_i . In this case conditions (15) and (17) express that the two linear systems $(A_1, \Delta b, H^T)$ and $(A_2, \Delta b, H^T)$ with the state matrices A_1, A_2 , input matrix Δb and output matrix H^T are strictly passive with the same matrix P .

The implication (iii) \Rightarrow (i) in Theorem 2 can be extended to the case of PWA systems (7) having more than two modes with the switching surfaces not necessarily going through the origin, i.e., given by equations of the form $H_j^T x + h_j = 0$. This statement is formulated in the next theorem.

Theorem 3: Consider PWA system (7). Suppose there exists a positive definite matrix $P = P^T > 0$ satisfying

$$PA_i + A_i^T P < 0, \quad i = 1, \dots, l, \quad (18)$$

and for any pair of cells Δ_i and Δ_j having a common boundary given by $H_{ij}^T x + h_{ij} = 0$ (such that $\Delta_i \subset \{x \in \mathbb{R}^n: H_{ij}^T x + h_{ij} \geq 0\}$ and $\Delta_j \subset \{x \in \mathbb{R}^n: H_{ij}^T x + h_{ij} < 0\}$) there exist a vector $G_{ij} \in \mathbb{R}^n$ and a number $\gamma_{ij} \in \{0, 1\}$ satisfying

$$A_i - A_j = G_{ij} H_{ij}^T, \quad (19)$$

$$P(b_i - b_j - G_{ij} h_{ij}) = -\gamma_{ij} H_{ij}. \quad (20)$$

Then system (7) is quadratically convergent.

Proof: see Appendix.

Remark 8: In general, the converse statement is not true. There is an example of a quadratically convergent PWA system with 4 switching modes with the system matrices satisfying condition (18), but not satisfying condition (19); see van den Berg *et al.* (2006).

6. Illustrating example

In this section, we illustrate the theory presented in §4 on the convergence condition for continuous PWA systems. An important class of engineering systems, namely mechanical systems with one-sided restoring

characteristics (such systems also have their counterparts in electric circuits), can be described as continuous PWA systems. Many mechanical systems exhibit such one-sided stiffness characteristics. Practical examples are elastic stops in vehicle suspensions, rubber snubbers on solar panels on satellites (van Campen *et al.* 1997), mooring cables securing drilling platforms to the sea bed (Thompson and Stewart 1986) or suspension bridges. From an engineering perspective, the behaviour of such systems under external perturbations is important to ensure performance and/or the avoidance of damage or failure. Often the perspective of periodic disturbances (Fey 1992, Fey *et al.* 1996, Heertjes 1999, Heertjes *et al.* 1999) or stochastic disturbances (van de Wouw *et al.* 2002) is taken to investigate the perturbed non-linear dynamics of these types of systems. Here, we will adopt the perspective of periodic disturbances. In Fey (1992), Fey *et al.* (1996), Heertjes (1999) and Heertjes *et al.* (1999) it is shown that the non-smooth non-linearity induced by a one-sided restoring characteristic causes a multitude of non-linear phenomena, such as period-1 solutions, period- k , $k=2, 3, \dots$, solutions, quasi-periodic behaviour and even chaos. In these references extensive (and computationally expensive) numerical bifurcation analysis are performed. Herein, it is shown that in a wide range of parameters, steady-state solutions, such as period-1 solutions, i.e., periodic with the same period time as the period time of the excitation, and period- k , $k=2, 3, \dots$, solutions, i.e., periodic with the period time k times larger than the period time of the excitation can coexist. From the perspective of mechanical vibrations the period- k solutions lead to additional non-linear resonances (often called subharmonic resonances), which are often considered to be unwanted.

Clearly, the coexistence of period-1 and period- k , $k=2, 3, \dots$, attractors is excluded in uniformly convergent systems, since these systems exhibit, for any bounded input, a unique solution bounded on \mathbb{R} , which is the globally asymptotically stable steady-state solution, see §3. Moreover, for periodic disturbances the steady-state solution is also periodic with the same period time; see Proposition 1. Clearly, the conditions for uniform convergence proposed in this paper can help in identifying areas in parameter space in which no coexisting solutions that are bounded on \mathbb{R} can occur and can thereby support a bifurcation analysis in an efficient manner. Of course it should be noted that the convergence conditions proposed in §§4 and 5 are conservative and can therefore never identify the exact location of a bifurcation point.

Here we will consider a single-degree-of-freedom mass-spring-damper system with a one-sided spring, as depicted in figure 2. An electric circuit equivalent to this mechanical system is depicted in figure 3. In Fey (1992) and Fey *et al.* (1996) it is shown that more complex

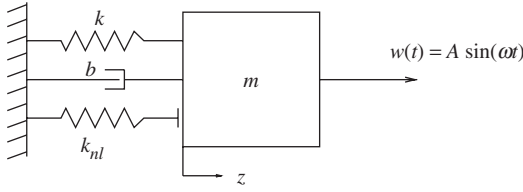


Figure 2. Mass-spring-damper system with a one-sided support.

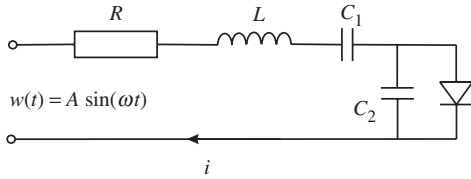


Figure 3. RLC circuit with a switching capacitor.

multi-degree-of-freedom systems exhibit a similar behavioural complexity and therefore the systems in these figures represent a relevant case-study.

The dynamics of the mechanical system shown in figure 2 can be formulated in the form (7), with $l=n=2$, $m=k=1$, $x = [z \dot{z}]^T$, $w(t) = \mathcal{A} \sin(\omega t)$, $b_1 = b_2 = [0 \ 0]^T$, $\Lambda_1 = \{x \mid x_1 > 0\}$, $\Lambda_2 = \{x \mid x_1 < 0\}$ and

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -\frac{k+k_{nl}}{m} & -\frac{b}{m} \end{bmatrix},$$

$$D = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}. \quad (21)$$

The displacement of the mass is denoted by z , its velocity by \dot{z} . The system has mass m , (linear, two-sided) stiffness k , damping constant b and the stiffness of the one-sided spring is k_{nl} . Moreover, the harmonic input $w(t) = \mathcal{A} \sin(\omega t)$ is characterized by an amplitude \mathcal{A} and an angular frequency ω .

The dynamics of the equivalent electric circuit depicted in figure 3 can also be represented in the form (7), with $l=n=2$, $m=k=1$, $x = [\int_0^t i(s) ds \ i]^T$, $w(t) = \mathcal{A} \sin(\omega t)$, $b_1 = b_2 = [0 \ 0]^T$, $\Lambda_1 = \{x \mid x_1 \geq 0\}$, $\Lambda_2 = \{x \mid x_1 < 0\}$ and

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC_1} & -\frac{R}{L} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -\frac{C_1+C_2}{LC_1C_2} & -\frac{R}{L} \end{bmatrix},$$

$$D = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \quad (22)$$

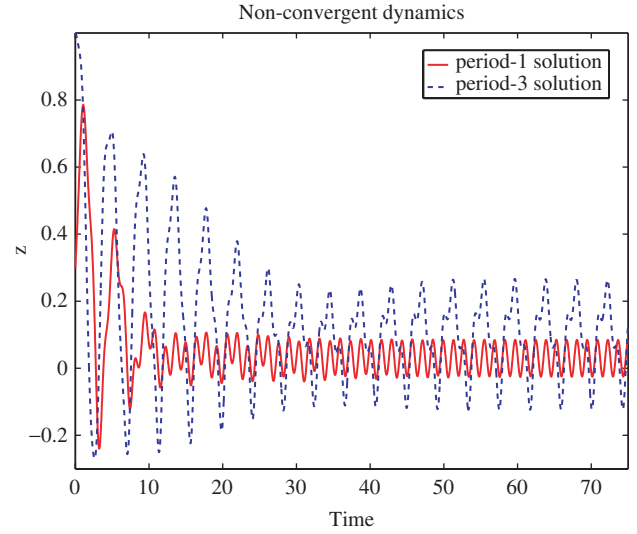


Figure 4. Displacement $z(t)$ for $k_{nl}=6$ (non-convergent dynamics); coexistence of period-1 and period-3 steady-state solutions, for the respective initial conditions $x_0 = [0.3 \ 0.5]^T$ and $x_0 = [1 \ 0]^T$.

Here i is current, w is voltage, C_1 and C_2 are capacities, R is resistance and L is inductance.

Because of the equivalence of the mechanical system shown in figure 2 and the circuit shown in figure 3, we will consider only the dynamics of the mechanical system. We adopt the following parameter setting: $m=k=1$, $b=0.2$, $\mathcal{A}=1$ and $k_{nl}=6$. In van de Wouw (1999), it is shown that for this parameter setting, the system exhibits coexisting period-1 and period- k , $k=2, 3, \dots$, for a wide range of excitation frequencies ω . For an excitation frequency $\omega=4.5$, the coexistence of a period-1 solution (period time $2\pi/\omega$) and a period-3 solution (period time $3 \cdot 2\pi/\omega$) is illustrated in figure 4. It should be noted that the period- k , $k=2, 3, \dots$, solutions are generally born through fold bifurcations or period doubling bifurcations, while taking ω as a bifurcation parameter (Fey 1992). Clearly, the system is not convergent in this case and no solution for the LMIs (8) exists.

In figure 5, results are displayed for $k_{nl}=0.4$. The LMIs (8) are now solvable with a matrix P satisfying the LMIs:

$$P = \begin{bmatrix} 1546.7 & 132.9 \\ 132.9 & 1274.6 \end{bmatrix}. \quad (23)$$

This figure illustrates that all solutions converge to a unique globally asymptotically stable steady-state solution. Moreover, this solution is period-1; i.e., it exhibits the same period time as the input w . The convergence condition implies the absence of a bifurcation, giving rise to period- k , $k=2, 3, \dots$, solutions, for

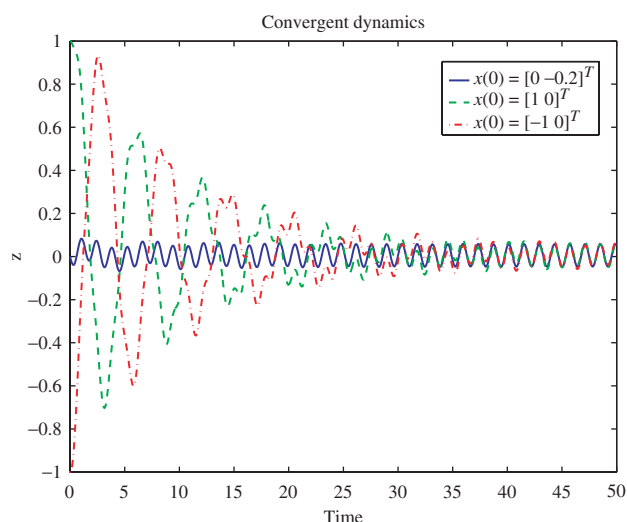


Figure 5. Displacement $z(t)$ for $k_{nl}=0.4$ (convergent dynamics); all solutions converge to the globally asymptotically stable steady-state period-1 solution.

any frequency of the disturbance w (in fact the convergence condition implies the absence of any type of bifurcation). This, in turn, implies the absence of the undesired non-linear resonances due to the existence of period- k , $k=2, 3, \dots$, solutions. Therefore, uniform convergence of a system is a desired property, which should be aimed at already at the stage of system design.

This example also clearly illustrates the difference between the stability of the unperturbed system and convergence. Namely, for any value $k_{nl} > -k$ of the one-sided spring stiffness, the unperturbed system, i.e., with $w=0$, exhibits a globally asymptotically stable equilibrium point $x=0$. This can easily be derived using the dissipative nature of the system. However, the discussion above shows that convergence property encompasses much more than asymptotic stability of the equilibrium point of the unperturbed system (although it implies it) and that the system under consideration is only convergent for small values of k_{nl} .

7. Conclusions

In this paper we have considered convergence properties for piecewise affine (PWA) systems. First, we have introduced the notion of (exponential, uniform) convergence and studied some properties of convergent systems. Secondly, the notion of quadratic convergence has been introduced. Quadratic convergence is a useful tool for establishing the exponential convergence property. It is shown that for a non-linear system with a possibly discontinuous right-hand side, quadratic convergence implies exponential convergence.

Thirdly, for PWA systems with continuous right-hand sides we have shown that quadratic convergence is equivalent to the existence of a common quadratic Lyapunov function for the linear parts of the system dynamics in every mode. As it has been demonstrated with an example, for discontinuous PWA systems the existence of a common quadratic Lyapunov function for the linear parts of the system dynamics does not guarantee convergence. For a discontinuous bimodal PWA system we have proved that for quadratic convergence it is necessary and sufficient that discontinuity occurs only due to the shift terms, while the state matrices of the linear parts of the dynamics, the difference between the shift terms, and the vector of the switching plane satisfy certain passivity condition. Then the sufficient conditions from this statement have been extended to the case of (discontinuous) PWA systems with arbitrary number of modes. The obtained results provide tools for studying convergence properties for non-smooth and discontinuous systems. Application of these results has been illustrated with an example containing analysis of dynamics of a mass-spring-damper system with a one-sided spring, which is equivalent to some electric circuit with a diode.

The presented results on convergence can be used in several ways. A uniformly convergent system excited with a periodic input has a unique periodic solution, which is globally asymptotically stable and has the same period time as the period time of the input. In bifurcation analysis such a property allows one to significantly reduce computational efforts for finding other periodic (period-1 or period- k , $k=2, 3, \dots$) responses to periodic excitations. In practice period- k , $k=2, 3, \dots$, responses represent unwanted non-linear phenomena. They may lead to undesired resonances in the low frequency range and, for this reason, should be avoided. If a system is designed to be convergent or it is made convergent by means of feedback, it does not have these problematic dynamics. Moreover, the existence and uniqueness of the periodic response of a convergent system to a periodic excitation substantiates many numerical methods for computing periodic solutions of periodically excited systems. Further work will be directed towards exploiting the presented results on convergent PWA systems in the scope of tracking, observer design, synchronization and the output regulation problem.

Acknowledgments

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Appendix

7.1 Proof of Lemma 1

Consider the system

$$\dot{x} = f(x, w(t)), \quad (24)$$

where $w(t)$ is some bounded piecewise-continuous input. First, we show the existence of a solution $\bar{x}_w(t)$ of system (24) that is defined and bounded on the whole time axis $(-\infty, +\infty)$. The existence of such $\bar{x}_w(t)$ will be shown using the following lemma.

Lemma 3 (Yakubovich 1964): *Consider system (24) with a given input $w(t)$ defined for all $t \in \mathbb{R}$. Let $\mathcal{D} \subset \mathbb{R}^n$ be a compact set that is positively invariant with respect to system (24). Then there is at least one solution $\bar{x}(t)$ satisfying $\bar{x}(t) \in \mathcal{D}$ for all $t \in (-\infty, +\infty)$.*

In order to apply this lemma, we need to prove the existence of a compact positively invariant set \mathcal{D} . Consider the function $W(x) := 1/2 x^T P x$. The upper derivative of this function along solutions of system (24) satisfies

$$\begin{aligned} \dot{W}^*(x, t) &= \sup_{\xi \in F(x, w(t))} x^T P \xi \\ &\leq \sup_{\xi \in F(x, w(t))} x^T P \xi - \inf_{\xi_1 \in F(0, w(t))} x^T P \xi_1 \\ &\quad + \sup_{\xi_2 \in F(0, w(t))} x^T P \xi_2. \end{aligned}$$

Notice that for the function $V(x_1, x_2)$ from the definition of quadratic stability it holds that

$$\begin{aligned} \dot{V}^*(x, 0, t) &= \sup_{\xi \in F(x, w(t))} x^T P \xi + \sup_{\xi_1 \in F(0, w(t))} (-x^T P \xi_1) \\ &= \sup_{\xi \in F(x, w(t))} x^T P \xi - \inf_{\xi_1 \in F(0, w(t))} x^T P \xi_1. \end{aligned}$$

Therefore,

$$\dot{W}^*(x, t) \leq \dot{V}^*(x, 0, t) + \sup_{\xi_2 \in F(0, w(t))} |x^T P \xi_2|. \quad (25)$$

By the quadratic convergence property it holds that

$$\dot{V}^*(x, 0, t) \leq -2\alpha V(x, 0) = -\alpha |x|_P^2, \quad (26)$$

where $|x|_P^2 = x^T P x$. At the same time, by the Cauchy inequality it holds that $|x^T P \xi_2| \leq |x|_P |\xi_2|_P$. Hence

$$\sup_{\xi_2 \in F(0, w(t))} |x^T P \xi_2| \leq |x|_P \sup_{\xi_2 \in F(0, w(t))} |\xi_2|_P. \quad (27)$$

Recall that the input $w(t)$ is bounded, i.e. $|w(t)| \leq R$ for all $t \in \mathbb{R}$, for some $R > 0$. By the assumption on the

right-hand side of system (4) (see §2), the function $f(x, w)$ takes bounded values on any compact set of (x, w) . Therefore the set $\{\xi \in \mathbb{R}^n: \xi \in F(0, w), |w| \leq R\}$ is bounded. Therefore, for some constant $\bar{c} > 0$ it holds that

$$\sup_{\xi_2 \in F(0, w(t))} |\xi_2|_P \leq \sup_{\substack{\xi_2 \in F(0, w) \\ |w| \leq R}} |\xi_2|_P \leq \bar{c}. \quad (28)$$

Combining inequalities (25)–(28) we obtain

$$\dot{W}^*(x, t) \leq |x|_P (-\alpha |x|_P + \bar{c}). \quad (29)$$

Hence, $\dot{W}^*(x, t) \leq 0$ for all $t \in \mathbb{R}$ and all x satisfying $|x|_P \geq \bar{c}/\alpha$. Taking into account the relation between the derivative and upper derivative of $W(x)$ along solutions $x(t)$ of system (24) (see (2)), we obtain

$$\dot{W}(x(t)) \leq 0$$

for almost all t such that $|x(t)|_P \geq \bar{c}/\alpha$. This implies that the set $\mathcal{D} := \{x: |x|_P \leq \bar{c}/\alpha\}$ is compact and positively invariant. By Lemma 3 there exists a solution $\bar{x}_w(t)$ that satisfies $\bar{x}_w(t) \in \mathcal{D}$ for all $t \in \mathbb{R}$.

Next, we need to show global exponential stability of $\bar{x}_w(t)$. By the quadratic convergence property it holds that

$$\dot{V}^*(x, \bar{x}_w(t), t) \leq -2\alpha V(x, \bar{x}_w(t)).$$

Consider some solution $x(t) := x(t, t_0, x_0)$ of system (24). Recall that the derivative of V along $x(t)$ and $\bar{x}_w(t)$ satisfies $\dot{V}(x(t), \bar{x}_w(t)) \leq \dot{V}^*(x(t), \bar{x}_w(t), t)$ for almost all t (see §2). Therefore,

$$\dot{V}(x(t), \bar{x}_w(t)) \leq -2\alpha V(x(t), \bar{x}_w(t))$$

for almost all $t \geq t_0$. Since $V(x_1, x_2)$ is a quadratic form with respect to the difference $(x_1 - x_2)$, the last inequality implies

$$|x(t) - \bar{x}_w(t)| \leq C e^{-\alpha(t-t_0)} |x(t_0) - \bar{x}_w(t_0)|,$$

where the number $C > 0$ depends only on the matrix P . Therefore, the global exponential and, thus, uniform asymptotic stability of the steady-state solution is proved. \square

7.2 Proof of Theorem 1

(ii) \Rightarrow (i) Denote the right-hand side of (7) by $f(x, w)$. Since P satisfies LMI (8), there exists a constant $\alpha > 0$ such that

$$P A_i + A_i^T P < -2\alpha P, \quad i = 1, \dots, l. \quad (30)$$

Let us show that for these α and P inequality (6) holds for all $x_1, x_2 \in \mathbb{R}^n$ and all $w \in \mathbb{R}^m$.

First, consider the case when both points x_1 and x_2 belong to the closure of the same cell Λ_i with the

dynamics $\dot{x} = A_i x + b_i + Dw$. Then, $f(x_1, w) = A_i x_1 + b_i + Dw$ and $f(x_2, w) = A_i x_2 + b_i + Dw$. Therefore,

$$\begin{aligned} & (x_1 - x_2)^T P(f(x_1, w) - f(x_2, w)) \\ &= (x_1 - x_2)^T P(A_i x_1 - A_i x_2) \\ &= \frac{1}{2}(x_1 - x_2)^T (PA_i + A_i^T P)(x_1 - x_2) \\ &\leq -\alpha(x_1 - x_2)^T P(x_1 - x_2). \end{aligned} \quad (31)$$

Thus, inequality (6) holds for such points x_1 and x_2 .

Next, we consider the case of x_1, x_2 from different cells. Denote $y_1 := x_1$, $y_p := x_2$ and y_i , $i = 2, \dots, p-1$, — the points of intersection of the line segment (x_1, x_2) with the switching surfaces such that any pair of points y_i, y_{i+1} lies in the closure of the same cell and the sequence y_1, y_2, \dots, y_p is ordered, see figure 6.

Denote $e := (x_1 - x_2)/|x_1 - x_2|_P$. Since all points y_i , $i = 1, \dots, p$, lie on the same line and they are ordered, then

$$e = \frac{y_i - y_{i+1}}{|y_i - y_{i+1}|_P}, \quad i = 1, \dots, p-1. \quad (32)$$

Taking this fact into account, we obtain

$$\begin{aligned} & (x_1 - x_2)^T P(f(x_1, w) - f(x_2, w)) \\ &= |x_1 - x_2|_P \sum_{i=1}^{p-1} e^T P(f(y_i, w) - f(y_{i+1}, w)) \\ &= |x_1 - x_2|_P \sum_{i=1}^{p-1} \frac{(y_i - y_{i+1})^T P(f(y_i, w) - f(y_{i+1}, w))}{|y_i - y_{i+1}|_P}. \end{aligned} \quad (33)$$

Since any pair of points y_i, y_{i+1} belongs to the closure of the same cell, from the first step of the proof we obtain

$$\begin{aligned} & (y_i - y_{i+1})^T P(f(y_i, w) - f(y_{i+1}, w)) \\ &\leq -\alpha(y_i - y_{i+1})^T P(y_i - y_{i+1}) \\ &= -\alpha|y_i - y_{i+1}|_P^2. \end{aligned}$$

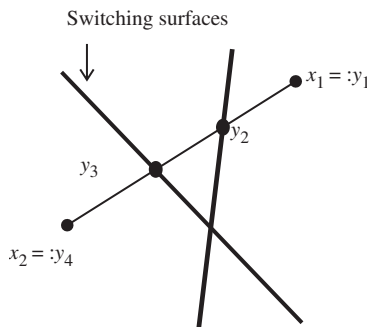


Figure 6. The points y_i on the line segment (x_1, x_2) : each pair y_i, y_{i+1} lies in the closure of the same cell and the sequence y_1, \dots, y_4 is ordered.

Substituting this inequality in (33) implies

$$\begin{aligned} & (x_1 - x_2)^T P(f(x_1, w) - f(x_2, w)) \\ &\leq -\alpha|x_1 - x_2|_P \sum_{i=1}^{p-1} |y_i - y_{i+1}|_P. \end{aligned} \quad (34)$$

Since all points y_i , $i = 1, \dots, p$, lie on the same line and they are ordered,

$$\sum_{i=1}^{p-1} |y_i - y_{i+1}|_P = |y_1 - y_p|_P = |x_1 - x_2|_P. \quad (35)$$

This fact together with (34) implies (6). This completes the proof of the implication (ii) \Rightarrow (i).

(i) \Rightarrow (ii). Consider the dynamics of system (7) in a cell Λ_i . The right-hand side of the system in Λ_i equals $f(x, w) = A_i x + b_i + Dw$. Therefore, quadratic convergence of system (7) implies

$$\begin{aligned} & (x_1 - x_2)^T PA_i(x_1 - x_2) \\ &\leq -\alpha(x_1 - x_2)^T P(x_1 - x_2) \end{aligned} \quad (36)$$

for all $x_1, x_2 \in \Lambda_i$. Consider a point x_* from the interior of Λ_i . Then there exists $\varepsilon > 0$ such that $(x_* + \varepsilon y) \in \Lambda_i$ for all $y \in \mathbb{R}^n$ satisfying $|y| \leq 1$. By substituting $x_1 := x_* + \varepsilon y$ and $x_2 := x_*$ in (36) and dividing both sides of the resulting inequality by ε^2 we obtain

$$y^T PA_i y = \frac{1}{2} y^T (PA_i + A_i^T P) y \leq -\alpha y^T P y$$

for all $y \in \mathbb{R}^n$ satisfying $|y| \leq 1$. This yields

$$PA_i + A_i^T P \leq -2\alpha P < 0. \quad (37)$$

Due to arbitrary choice of the cell Λ_i , we conclude that (37) holds for all $i = 1, \dots, l$. \square

7.3 Proof of Theorem 2

The theorem will be proved in the following order: (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).

(i) \Rightarrow (ii). According to Remark 3, quadratic convergence of system (12) implies that there exists a positive definite matrix $\bar{P} = \bar{P}^T > 0$ and a number $\alpha > 0$ such that for any x_1 and x_2 satisfying the inequalities $H^T x_1 > 0$ and $H^T x_2 < 0$ it holds that

$$\begin{aligned} & (x_1 - x_2)^T \bar{P}(A_1 x_1 + b_1 - A_2 x_2 - b_2) \\ &\leq -\alpha(x_1 - x_2)^T \bar{P}(x_1 - x_2). \end{aligned} \quad (38)$$

By denoting $e := x_1 - x_2$ and taking into account the fact that $-\alpha \bar{P} \leq -\bar{\beta} I$ for some $\bar{\beta} > 0$, we conclude that inequality (38) implies

$$e^T \bar{P}(A_1 e + \Delta A x_2 + \Delta b) \leq -\bar{\beta}|e|^2 \quad (39)$$

for all e and x_2 from the set $\Omega_1 := \{(e, x_2): H^T x_2 < 0, H^T e + H^T x_2 > 0\}$. Let us show that inequality (39) yields

$$e^T \bar{P}(A_1 e + \Delta A x_2) + \bar{\beta}|e|^2 \leq 0 \quad (40)$$

$$e^T \bar{P} \Delta b \leq 0 \quad (41)$$

for all $(e, x_2) \in \Omega_1$. Consider some point $(e, x_2) \in \Omega_1$. Then for all $\lambda > 0$ it holds that $(\lambda e, \lambda x_2) \in \Omega_1$. As follows from inequality (39), this yields

$$\lambda^2 (e^T \bar{P}(A_1 e + \Delta A x_2) + \bar{\beta}|e|^2) + \lambda e^T \bar{P} \Delta b \leq 0$$

for all $\lambda > 0$. One can easily check that this inequality is satisfied for all $\lambda > 0$ iff the inequalities (40) and (41) hold. Due to arbitrary choice of $(e, x_2) \in \Omega_1$, we conclude that inequalities (40) and (41) are satisfied for all $(e, x_2) \in \Omega_1$.

Repeating the same steps as in the first part of the proof, but this time for points x_1 and x_2 satisfying $H^T x_1 < 0$ and $H^T x_2 > 0$, we conclude that the inequality

$$e^T \bar{P}(A_1 e - \Delta A x_1) + \bar{\beta}|e|^2 \leq 0 \quad (42)$$

holds for all $(e, x_1) \in \Omega_2$, where $\Omega_2 := \{(e, x_1): H^T x_1 < 0, -H^T e + H^T x_1 > 0\}$. By denoting $\tilde{x}_1 := -x_1$, we see that

$$e^T \bar{P}(A_1 e + \Delta A \tilde{x}_1) + \bar{\beta}|e|^2 \leq 0 \quad (43)$$

holds for all $(e, \tilde{x}_1) \in \tilde{\Omega}_2$, where $\tilde{\Omega}_2 := \{(e, \tilde{x}_1): H^T \tilde{x}_1 > 0, H^T e + H^T \tilde{x}_1 < 0\}$.

No we can show that (13) is feasible. Combining inequalities (40) and (43) we obtain that the quadratic form $\mathcal{F}(e, \xi) := e^T \bar{P}(A_1 e + \Delta A \xi) + \bar{\beta}|e|^2$ satisfies

$$\mathcal{F}(e, \xi) \leq 0 \quad \text{for } (e, \xi): \mathcal{G}(e, \xi) < 0, \quad (44)$$

where $\mathcal{G}(e, \xi) := \xi^T H(H^T e + H^T \xi)$. Due to continuity of \mathcal{F} and non-strict inequality for \mathcal{F} in (44), the last inequality is equivalent to

$$\mathcal{F}(e, \xi) \leq 0 \quad \text{for } (e, \xi): \mathcal{G}(e, \xi) \leq 0. \quad (45)$$

Applying the S -procedure, see e.g., Boyd et al. (1994) and Yakubovich *et al.* (2004), we obtain that the conditional inequality (45) is equivalent to the unconditional inequality

$$\mathcal{F}(e, \xi) - \tau \mathcal{G}(e, \xi) \leq 0 \quad (46)$$

for some $\tau \geq 0$ and all $(e, \xi) \in \mathbb{R}^{2n}$. The equivalence holds because the S -procedure is lossless in case of one quadratic constraint, see e.g., Yakubovich *et al.* (2004). Notice that since the quadratic form $\mathcal{F}(e, \xi)$ is not negative semidefinite, $\tau \neq 0$ (otherwise the equivalence between (45) and (46) does not hold). Notice that inequality (46) is equivalent to the following LMI

$$\begin{pmatrix} \bar{P}A_1 + A_1^T \bar{P} + 2\bar{\beta}I & \bar{P}\Delta A - \tau H H^T \\ \Delta A^T \bar{P} - \tau H H^T & -2\tau H H^T \end{pmatrix} \leq 0. \quad (47)$$

Since $\tau > 0$, from this inequality we obtain (13) with $P := \bar{P}/(2\tau)$ and $\beta := \bar{\beta}/\tau$.

It remains to show that inequality (14) holds for the presented P and some $\gamma \geq 0$. To this end, consider inequality (41), which holds for all $(e, x_2) \in \Omega_1$. Notice that for all e satisfying $H^T e > 0$ there exists x_2 such that $(e, x_2) \in \Omega_1$. Therefore, $e^T \bar{P} \Delta b \leq 0$ for all e satisfying $H^T e > 0$. One can easily check that this is possible iff $\bar{P} \Delta b = -\bar{\gamma} H$ for some $\bar{\gamma} \geq 0$. After dividing both sides of the obtained equation by 2τ , we obtain (14) with $P = \bar{P}/(2\tau)$ and $\gamma := \bar{\gamma}/(2\tau)$. This finishes the proof of implication (i) \Rightarrow (ii).

(ii) \Rightarrow (iii) First, we will show that conditions (15)–(17) hold for some matrix $P = P^T > 0$, vector $G \in \mathbb{R}^n$ and some $\gamma \geq 0$. If $\gamma = 0$ this proves this implication. If $\gamma > 0$, then by dividing (15) and (17) by γ we obtain that relations (15) and (17) hold for $\tilde{P} := P/\gamma$ and $\tilde{\gamma} = 1$. This proves the remaining part of the implication.

Let us show that conditions (15)–(17) hold for some matrix $P = P^T > 0$, vector $G \in \mathbb{R}^n$ and some $\gamma \geq 0$. We only need to show (15) and (16), since (17) coincides with (14). One can easily see that inequality (13) implies $PA_1 + A_1^T P \leq -\beta I < 0$. Next we show that inequality $PA_2 + A_2^T P \leq -\beta I < 0$ holds. Denote the matrix in (13) by M . Then inequality (13) yields

$$\begin{pmatrix} x \\ -x \end{pmatrix}^T M \begin{pmatrix} x \\ -x \end{pmatrix} \leq 0 \quad (48)$$

for all $x \in \mathbb{R}^n$. After elaborating the left-hand side of (48) we obtain $x^T (PA_2 + A_2^T P + \beta I)x \leq 0$ for all $x \in \mathbb{R}^n$. Hence, we have shown (15). Let us show that (16) holds for some $G \in \mathbb{R}^n$. This is done in the same way as in Juloski *et al.* (2002). Suppose $\chi \in \ker(H^T)$. From the structure of the matrix M we obtain

$$\begin{pmatrix} 0 \\ \chi \end{pmatrix}^T M \begin{pmatrix} 0 \\ \chi \end{pmatrix} = 0.$$

Since $M = M^T < 0$, this equality implies $M(0, \chi^T)^T = 0$. Taking into account the structure of M , we obtain that $P\Delta A\chi = 0$. Since P is non-degenerate, we conclude that $\Delta A\chi = 0$. Thus we have shown that $\ker(H^T) \subset \ker(\Delta A)$. This relation, in turn, implies the existence of a vector $G \in \mathbb{R}^n$ such that $\Delta A = GH^T$. This concludes the proof of the implication (ii) \Rightarrow (iii).

(iii) \Rightarrow (i) Let us write the system (12) in the following form

$$\dot{x} = \tilde{f}(x, w) + b(x), \quad (49)$$

where

$$\tilde{f}(x, w) := \begin{cases} A_1 x + Dw, & \text{for } H^T x \geq 0 \\ A_2 x + Dw, & \text{for } H^T x < 0, \end{cases} \quad (50)$$

$$b(x) := \begin{cases} b_1, & \text{for } H^T x \geq 0 \\ b_2, & \text{for } H^T x < 0. \end{cases} \quad (51)$$

As follows from Remark 3, for quadratic convergence of system (49) it is sufficient that, for some matrix $P = P^T > 0$ and number $\alpha > 0$, the inequality

$$\begin{aligned} & (x_1 - x_2)^T P (\tilde{f}(x_1, w) + b(x_1) - \tilde{f}(x_2, w) - b(x_2)) \\ & \leq -\alpha(x_1 - x_2)^T P (x_1 - x_2) \end{aligned} \quad (52)$$

holds for all x_1 and x_2 such that $H^T x_1 \neq 0$ and $H^T x_2 \neq 0$, i.e. in the continuity points of the right-hand side of system (49). The vector-field $\tilde{f}(x, w)$ is piecewise affine. Moreover, as follows from condition (16) and Lemma 2, $\tilde{f}(x, w)$ is continuous. Since the matrices A_1 and A_2 satisfy (15) for some $P = P^T > 0$, then by Theorem 1 (see Remark 6) the inequality

$$\begin{aligned} & (x_1 - x_2)^T P (\tilde{f}(x_1, w) - \tilde{f}(x_2, w)) \\ & \leq -\alpha(x_1 - x_2)^T P (x_1 - x_2) \end{aligned} \quad (53)$$

holds for all x_1 and $x_2 \in \mathbb{R}^n$. Hence,

$$\begin{aligned} & (x_1 - x_2)^T P (\tilde{f}(x_1, w) + b(x_1) - \tilde{f}(x_2, w) - b(x_2)) \\ & \leq -\alpha(x_1 - x_2)^T P (x_1 - x_2) \\ & \quad + (x_1 - x_2)^T P (b(x_1) - b(x_2)). \end{aligned} \quad (54)$$

It remains to show that

$$(x_1 - x_2)^T P (b(x_1) - b(x_2)) \leq 0 \quad (55)$$

for all x_1 and x_2 such that $H^T x_i \neq 0$, $i = 1, 2$. If x_1 and x_2 belong to the same cell, i.e. either $H^T x_i > 0$, $i = 1, 2$ or $H^T x_i < 0$, $i = 1, 2$, then $b(x_1) = b(x_2)$ and, therefore, the left-hand side of (55) equals zero. If $H^T x_1 > 0$ and $H^T x_2 < 0$, then $b(x_1) - b(x_2) = b_1 - b_2 = \Delta b$. Taking into account equality (17), we see that the left-hand side of (55) satisfies

$$\begin{aligned} & (x_1 - x_2)^T P \Delta b = -\gamma(x_1 - x_2)^T H \\ & = -\gamma(H^T x_1 - H^T x_2) \leq 0. \end{aligned}$$

In the same way inequality (55) is proven for all x_1 and x_2 satisfying $H^T x_1 < 0$ and $H^T x_2 > 0$. Thus, we have shown that inequality (55) holds for all x_1 and x_2 such that $H^T x_i \neq 0$, $i = 1, 2$. Inequalities (55) and (54) jointly imply (52). This completes the proof of the implication (iii) \Rightarrow (i). \square

7.4 Proof of Theorem 3

Let $f(x, w)$ denote the right-hand side of (7), i.e.,

$$f(x, w) = A_i x + b_i + Dw, \quad \text{for } x \in \Lambda_i, \quad i = 1, \dots, l.$$

According to Remark 3, we only need to show that for the Lyapunov function $V(x_1, x_2) := 1/2(x_1 - x_2)^T P (x_1 - x_2)$ the inequality

$$\begin{aligned} \dot{V}(x_1, x_2, w) &= (x_1 - x_2)^T P (f(x_1, w) - f(x_2, w)) \\ &\leq -2\alpha V(x_1, x_2) \end{aligned} \quad (56)$$

holds for any $w \in \mathbb{R}^m$, x_1 and x_2 from the continuity domain of the function $f(x, w)$.

If x_1 and x_2 belong to the same cell, then (56) is obviously satisfied. Let us now consider the case of x_1 and x_2 belonging to the interior of two neighbouring cells Λ_i and Λ_j , respectively. In this case, we can consider our system as a bimodal system with the switching surface $H_{ij}^T x + h_{ij} = 0$. To apply Theorem 2 we need to transform the coordinates of the system such that the switching surface goes through the origin. This is achieved by shifting the coordinates as $\tilde{x} = x - x_*$ for some x_* lying on the switching surface, i.e. such that $H_{ij}^T x_* + h_{ij} = 0$. Then the system transforms into

$$\dot{\tilde{x}} = \begin{cases} A_i \tilde{x} + \tilde{b}_i + Dw, & H_{ij}^T \tilde{x} \geq 0 \\ A_j \tilde{x} + \tilde{b}_j + Dw, & H_{ij}^T \tilde{x} < 0, \end{cases} \quad (57)$$

with $\tilde{b}_i = b_i + A_i x_*$ and $\tilde{b}_j = b_j + A_j x_*$. For this system the conditions (18), (19) coincide with the corresponding conditions (15), (16) in Theorem 2. Condition (17) is satisfied due to condition (20) and the fact that

$$\begin{aligned} \tilde{b}_i - \tilde{b}_j &= b_i - b_j + (A_i - A_j)x_* = b_i - b_j + G_{ij}H_{ij}^T x_* \\ &= b_i - b_j - G_{ij}h_{ij}, \end{aligned}$$

where we have used (19). Hence all conditions of Theorem 2 are satisfied for system (57) and therefore the derivative $\dot{V}(\tilde{x}_1, \tilde{x}_2, w)$ along solutions of (57) satisfies $\dot{V}(\tilde{x}_1, \tilde{x}_2, w) \leq -2\alpha V(\tilde{x}_1, \tilde{x}_2)$ for any \tilde{x}_1 and \tilde{x}_2 not lying on the switching surface $H_{ij}^T \tilde{x} = 0$. Since $V(x_1, x_2) \equiv V(\tilde{x}_1, \tilde{x}_2)$, this implies that the derivative $\dot{V}(x_1, x_2, w)$ along solutions $x_1(t)$ and $x_2(t)$ of the original system also satisfies $\dot{V}(x_1, x_2, w) \leq -2\alpha V(x_1, x_2)$ for any x_1 and x_2 from the interior of Λ_i and Λ_j , respectively, i.e. (56) holds.

Next we consider the case of arbitrary x_1 and x_2 from the continuity domain of $f(x, w)$. Consider the straight line segment connecting the points x_1 and x_2 . Let $y_1 := x_1$, $y_p := x_2$ and y_i , $i = 2, \dots, p-1$, be the points on the line segment such that the points y_i , y_{i+1} for any $i = 1, \dots, p-1$, belong to the interior of two neighboring cells and the sequence y_1, \dots, y_p is ordered, see figure 7.

According to the first part of the proof we have

$$\begin{aligned} & (y_i - y_{i+1})^T P (f(y_i, w) - f(y_{i+1}, w)) \\ & \leq -\alpha(y_i - y_{i+1})^T P (y_i - y_{i+1}) \\ & = -\alpha|y_i - y_{i+1}|_P^2. \end{aligned}$$

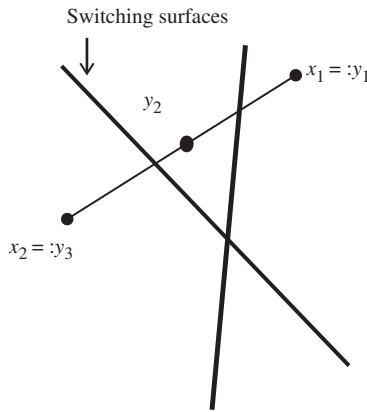


Figure 7. The points y_i on the line segment (x_1, x_2) : y_i and y_{i+1} belong to the interior of neighbouring cells and the points y_1, \dots, y_3 are ordered.

With this fact, by repeating the reasoning from the proof of Theorem 1, see formulas (32)–(35), we prove that inequality (56) holds for any x_1 and x_2 from the continuity domain of $f(x, w)$. This completes the proof of the theorem. \square

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