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The local approximate output regulation problem: convergence region estimates

A. Pavlov*,†, N. van de Wouw and H. Nijmeijer

Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

SUMMARY

In this paper, the problem of local approximate output regulation is considered. The presented results answer the question: given a controller solving the local approximate output regulation problem, how to estimate the set of initial conditions for which approximate output regulation occurs. The results are illustrated by a disturbance rejection problem for the TORA system. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: non-linear systems; approximate output regulation; disturbance rejection; convergence region

1. INTRODUCTION

In this paper, we consider the problem of asymptotic regulation of the output of a dynamical system, which is subject to disturbances generated by an external system. This problem is known as the output regulation problem. For non-linear systems, solutions to the *local* output regulation problem were given in References [1, 2] (see also References [3–5]). Even though the local output regulation problem may be solvable, it can be extremely difficult to find a controller solving it. This is due to the fact that finding such a controller requires solving the so-called *regulator equations*, which are mixed partial differential and non-linear algebraic equations. To overcome this difficulty, it was proposed in Reference [6] to design controllers based on approximate solutions to the regulator equations. For such a controller, if the initial conditions of the closed-loop system and the exosystem are sufficiently small, the regulation error declines to small values, which are of the same order of magnitude as the error in the regulator equations. This is called local approximate output regulation. Methods for finding such approximate solutions to the local output regulation problem were presented in References [3, 7, 8]. Controllers proposed in these papers solve the approximate output regulation problem in *some* neighbourhood of the origin. In practice, it is important to estimate this neighbourhood

*Correspondence to: A. Pavlov, Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
†E-mail: A.Pavlov@tue.nl

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of admissible initial conditions. Without finding such estimates, approximate solutions to the local output regulation problem may not be satisfactory from an engineering point of view.

For the case of *exact* local output regulation, such an estimation problem was addressed in References [9–11]. In this paper, we extend these results to the case of *approximate* output regulation. As in References [9–11], the analysis in this paper is based on the results of Demidovich [12, 13], which give sufficient conditions for every trajectory in a certain set to be exponentially stable. More information related to such properties of dynamical systems can be found in References [9, 14–16].

The paper is organized as follows. In Section 2, we recall the local (approximate) output regulation problem and formulate the problem of estimating the set of admissible initial conditions. In Section 3, some auxiliary technical results are presented. Section 4 contains the main estimation results. In Section 5, the obtained results are applied to a disturbance rejection problem in the TORA system (see References [17, 18] for details about the TORA system). Conclusions are presented in Section 6. The proofs of all results are given in the appendix.

The notations used in the paper are the following: $\mathscr{A}^{\mathrm{T}}$ is the transpose of matrix $\mathscr{A}$; norm of a vector is denoted as $|z| = (z^{\mathrm{T}}z)^{1/2}$; for a positive definite matrix $P = P^{\mathrm{T}} > 0$, we define the vector norm $|\cdot|_P$ as $|z|_P := \sqrt{z^{\mathrm{T}}Pz}$; $\|P\|$ is the operator norm of the matrix P induced by the vector norm $|\cdot|$; I is the identity matrix; the largest eigenvalue of a symmetric matrix $J = J^{\mathrm{T}}$ is denoted $\Lambda(J)$ and $\mathscr{D}F_z(z)$ is the Jacobian matrix of $F(z)$.

2. ESTIMATION PROBLEM STATEMENT

First, we recall the problem of local output regulation. Following Reference [1], consider systems modelled by equations of the form

$$\dot{x} = f(x, u, w) \tag{1}$$

$$e = h(x, w), \quad y = h_m(x, w) \tag{2}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, regulated output $e \in \mathbb{R}^{l_r}$, measured output $y \in \mathbb{R}^{l_m}$ and exogenous input $w \in R^m$ generated by the linear exosystem

$$\dot{w} = Sw \tag{3}$$

The exogenous signal $w(t)$ can be viewed as a disturbance in Equation (1) or as a reference signal in (2). The functions f, h and h_m are at least continuously differentiable and $f(0,0,0) = 0$, $h(0,0) = 0$, $h_m(0,0) = 0$. It is assumed that exosystem (3) is *neutrally stable*, i.e. the equilibrium $w = 0$ is Lyapunov stable in forward and backward time (see Reference [4]). The assumption of linearity of the exosystem is introduced in order to avoid unnecessary technical complications. All results presented below can be extended to the case of a general neutrally stable exosystem. Due to the neutral stability assumption, the spectrum of S consists of simple eigenvalues on the imaginary axis and, possibly, multiple eigenvalues at zero. Without loss of generality, we assume that S is skew-symmetric and thus any solution of system (3) has the property $|w(t)| \equiv Const$. Notice, that if the right-hand side of (1) depends on a vector ψ of unknown constant parameters, w and ψ can be united and treated together as an external signal (w, ψ) generated by an extended exosystem given by Equations (3) and $\dot{\psi} = 0$. This extended exosystem also satisfies the assumptions given above. Here we assume that such extension has already been performed and that (3) corresponds to an extended exosystem.

The local output regulation problem is to find, if possible, an output feedback of the form

$$\dot{\xi} = \eta(\xi, y) \tag{4}$$

$$u = \theta(\xi, y) \tag{5}$$

with $\eta(0,0) = 0$ and $\theta(0,0) = 0$ such that (a) $e(t) = h(x(t), w(t)) \to 0$ as $t \to \infty$ along every solution of the system

$$\dot{x} = f(x, \theta(\xi, h_m(x, w)), w) \tag{6}$$

$$\dot{\xi} = \eta(\xi, h_m(x, w)) \tag{7}$$

$$\dot{w} = Sw \tag{8}$$

starting close enough to the origin; (b) for $w(t) = 0$, the equilibrium point $(x, \xi) = (0, 0)$ of the closed-loop system (6), (7) is locally exponentially stable.

Denote $z := (x^{\mathrm{T}}, \xi^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{n+k}$ (k is the dimension of ξ). Then, the closed-loop system (6),(7) can be written as

$$\dot{z} = F(z, w) \tag{9}$$

$$e = \bar{h}(z, w) := h(x, w)$$

where $F(z, w)$ is the right-hand side of (6), (7). It is well-known (see References [1, 3]) that a controller solves the local output regulation problem if and only if the corresponding closed-loop system (9) satisfies the following conditions:

($\mathscr{A}$) The Jacobian matrix $\mathscr{D}F_z(0, 0)$ is Hurwitz,
($\mathscr{B}$) There exists a mapping $z = \pi(w)$ defined in a neighbourhood $\mathscr{W}$ of the origin, with $\pi(0) = 0$, such that

$$\begin{aligned} \frac{\partial \pi}{\partial w}(w)Sw &= F(\pi(w), w) \\ 0 &= \bar{h}(\pi(w), w) \quad \text{for all } w \in \mathscr{W} \end{aligned} \tag{10}$$

Even though the local output regulation can be solvable, it can be extremely difficult to find a controller that solves it. For such controller, the closed-loop system would satisfy conditions ($\mathscr{A}$) and ($\mathscr{B}$). Condition ($\mathscr{B}$) is the one that is most difficult to satisfy. At the same time, in many cases it is easier to find a controller that satisfies Equations (10) in condition ($\mathscr{B}$) approximately (see References [3, 6, 7]), i.e.

($\mathscr{B}^*$) There exists a mapping $z = \tilde{\pi}(w)$ defined in a neighbourhood $\mathscr{W}$ of the origin, with $\tilde{\pi}(0) = 0$, such that

$$\begin{aligned} \frac{\partial \tilde{\pi}}{\partial w}(w)Sw &= F(\tilde{\pi}(w), w) + \varepsilon_1(w) \\ 0 &= \bar{h}(\tilde{\pi}(w), w) + \varepsilon_2(w) \end{aligned} \tag{11}$$

for all $w \in \mathcal{W}$, where $\varepsilon_1(w)$ and $\varepsilon_2(w)$ are small (in some sense) continuous functions satisfying $\varepsilon_1(0) = 0$ and $\varepsilon_2(0) = 0$.

It is known (see Reference [6]), that if the closed-loop system satisfies conditions ($\mathcal{A}$) and ($\mathcal{B}^*$), then for all sufficiently small initial conditions $z(0)$ and $w(0)$ the regulated output $e(t)$ converges to $\tilde{e}(w(t))$, where $\tilde{e}(w)$ is of the same order of magnitude as $\varepsilon_1(w)$ and $\varepsilon_2(w)$. Since it is required that the initial conditions must be sufficiently small, we come to the following estimation problem: given the closed-loop system (9) and the neutrally stable exosystem (3) satisfying conditions ($\mathcal{A}$) and ($\mathcal{B}^*$), estimate the region of admissible initial conditions for which approximate output regulation is attained.

We will give a solution to this estimation problem based on the functions $F(z, w)$, $\tilde{\pi}(w)$ and $\varepsilon_1(w)$ which are found at the stage of controller design [3, 6]. To simplify the subsequent analysis, it is assumed that the closed-loop system (9) and the mappings $\tilde{\pi}(w)$, $\varepsilon_1(w)$ and $\varepsilon_2(w)$ are globally defined for all $z \in \mathbb{R}^{n+k}$ and $w \in \mathbb{R}^m$ (i.e. $\mathcal{W} = \mathbb{R}^m$). If this assumption does not hold, one should restrict the subsequent results to sets $\mathcal{Z} \subset \mathbb{R}^{n+k}$ and $\mathcal{W} \subset \mathbb{R}^m$ for which $F(z, w)$, $\tilde{\pi}(w)$, $\varepsilon_1(w)$ and $\varepsilon_2(w)$ are well-defined.

Before proceeding with solving the estimation problem, we discuss the main idea of the solution. First, we find two sets $\mathcal{C} \subseteq \mathbb{R}^{n+k}$ and $\mathcal{W}_c \subseteq \mathbb{R}^m$ having the following property: if $w(t) \in \mathcal{W}_c$ for $t \geqslant 0$, then any two solutions $z_1(t)$ and $z_2(t)$ of system (9) lying in $\mathcal{C}$ for all $t \geqslant 0$ converge to each other exponentially. We call such set $\mathcal{C}$ a *convergence set* and the set $\mathcal{W}_c$ a *companion* to the set $\mathcal{C}$. Such sets exist, due to condition ($\mathcal{A}$). This condition implies that near the origin, for small $w(t)$, the closed-loop system (9) behaves like a linear asymptotically stable system and, in particular, all its solutions are uniformly exponentially stable. Second, we find a set $\mathcal{Y} \subset \mathcal{C} \times \mathcal{W}_c$ of initial conditions $(z(0), w(0))$ such that any trajectory $(z(t), w(t))$ starting in this set satisfies the following conditions: $w(t) \in \mathcal{W}_c$, $\tilde{\pi}(w(t)) \in \mathcal{C}$ and $z(t) \in \mathcal{C}$ for all $t \geqslant 0$. As follows from (11), $\tilde{z}(t) := \tilde{\pi}(w(t))$ can be considered as a solution of the perturbed system

$$\dot{z} = F(z, w) + \varepsilon_1(w) \tag{12}$$

and along this solution the regulated output equals $-\varepsilon_2(w(t))$. Since $z(t)$ is uniformly exponentially stable (due to the choice of $\mathcal{C}$ and $\mathcal{W}_c$), a small perturbation $\varepsilon_1(w(t))$ implies, in the limit, a small difference between $z(t)$ and $\tilde{\pi}(w(t))$ with the upper bound proportional to the upper bound of $|\varepsilon_1(w(t))|$ (see Reference [19, Chapter 5]). Consequently, the regulated output $e(t)$, in the limit for $t \to +\infty$, will be close to $\varepsilon_2(w(t))$. Thus, if both $\varepsilon_1(w(t))$ and $\varepsilon_2(w(t))$ are small, so is the regulated output $e(t)$. Thus, $\mathcal{Y}$ is an estimate of the set of initial conditions for which approximate output regulation is attained. This reasoning will be made precise in Section 4.

3. CONVERGENCE SETS AND THE DEMIDOVICH CONDITION

In this section, we present and discuss a technical result about convergence sets for a system with input w given by

$$\dot{z} = F(z, w(t)) \quad \text{where} \quad z \in \mathbb{R}^{n+k}, \quad w \in \mathbb{R}^m, \quad F(\cdot, \cdot) \in C^1 \tag{13}$$

The next lemma gives sufficient conditions for sets $\mathcal{C} \subseteq \mathbb{R}^{n+k}$ and $\mathcal{W}_c \subseteq \mathbb{R}^m$ to be a convergence set and its companion, respectively.

Lemma 1 (Demidovich [12])
Suppose, a convex set $\mathscr{C} \subseteq \mathbb{R}^{n+k}$ and a set $\mathscr{W}_c \subseteq \mathbb{R}^m$ satisfy the Demidovich condition

$$\sup_{z\in\mathscr{C},\ w\in\mathscr{W}_c} \Lambda(P\mathscr{D}F_z(z,w) + \mathscr{D}F_z^{\mathrm{T}}(z,w)P) =: -\alpha < 0 \tag{14}$$

for some positive definite matrix $P = P^{\mathrm{T}} > 0$. Then, for any continuous input $w(t)$ such that $w(t) \in \mathscr{W}_c$ for $t \geqslant 0$, any two solutions $z(t)$ and $\bar{z}(t)$ of system (13) lying in $\mathscr{C}$ for all $t \geqslant 0$ satisfy

$$|z(t) - \bar{z}(t)| \leqslant C\mathrm{e}^{-\beta t}|z(0) - \bar{z}(0)| \tag{15}$$

for some $\beta > 0$ and $C > 0$ that are independent of the particular $z(t)$, $\bar{z}(t)$ and $w(t)$.

The proof of this result is based on the Lyapunov-like function $V(z,\bar{z}) = (z-\bar{z})^{\mathrm{T}}P(z-\bar{z})$. Condition (14) guarantees that for any $z \in \mathscr{C}$, $\bar{z} \in \mathscr{C}$ and $w \in \mathscr{W}_c$, the following relation holds:

$$2(z-\bar{z})^{\mathrm{T}}P(F(z,w) - F(\bar{z},w)) \leqslant -\frac{\alpha}{\|P\|}(z-\bar{z})^{\mathrm{T}}P(z-\bar{z}) \tag{16}$$

Due to exponential stability of the solution $\bar{z}(t)$, small perturbations of the right-hand side of (13) lead, in the limit, to small differences between $\bar{z}(t)$ and solutions of the perturbed system (see e.g. Reference [19]). We formulate this statement in a way that is convenient for our purposes.

Lemma 2
Consider system (13) and the perturbed system

$$\dot{z} = F(z, w(t)) + \varepsilon(t) \tag{17}$$

where $\varepsilon(t)$ is a continuous perturbation term. Suppose $\mathscr{C}$ and $\mathscr{W}_c$ satisfy the conditions of Lemma 1. Let $w(t) \in \mathscr{W}_c$ for all $t \geqslant 0$ and $\bar{z}(t)$ be a solution of (13) such that the ellipsoid $E_P(\bar{z}(t), r) := \{z : V(z, \bar{z}(t)) \leqslant r^2\}$ is contained in $\mathscr{C}$ for all $t \geqslant 0$. If the perturbation term satisfies $|\varepsilon(t)|_P \leqslant \alpha r/(2\|P\|)$ for $t \geqslant 0$, then any solution of the perturbed system (17) starting in $z(0) \in E_P(\bar{z}(0), r)$ satisfies

$$\limsup_{t\to+\infty} |z(t) - \bar{z}(t)|_P \leqslant \frac{2\|P\|}{\alpha} \limsup_{t\to+\infty} |\varepsilon(t)|_P \tag{18}$$

In order to solve the estimation problem stated in Section 2, we need to find sets $\mathscr{C}$ and $\mathscr{W}_c$ satisfying the Demidovich condition. If $\mathscr{D}F_z(0,0)$ is Hurwitz (this is the case in the output regulation problem), one can choose a matrix $P = P^{\mathrm{T}} > 0$ satisfying the matrix inequality $P\mathscr{D}F_z(0,0) + \mathscr{D}F_z^{\mathrm{T}}(0,0)P < 0$. By continuity, $P\mathscr{D}F_z(z,w) + \mathscr{D}F_z^{\mathrm{T}}(z,w)P$ is negative definite at least for small z and w. Hence, the Demidovich condition (14) is satisfied for $\mathscr{C}(\mathscr{R}) := \{z : |z| < \mathscr{R}\}$ and $\mathscr{W}(\rho) := \{w : |w| < \rho\}$ for some small $\mathscr{R}$ and ρ. If $P\mathscr{D}F_z(z,w) + \mathscr{D}F_z^{\mathrm{T}}(z,w)P$ depends only on part of the coordinates z, then the Demidovich condition is satisfied for $\mathscr{C}_N(\mathscr{R}) := \{z : |Nz| < \mathscr{R}\}$ and $\mathscr{W}(\rho) := \{w : |w| < \rho\}$, where the matrix N is such that Nz consists of the coordinates that are present in $P\mathscr{D}F_z(z,w) + \mathscr{D}F_z^{\mathrm{T}}(z,w)P$. Having chosen the matrix N, the numbers ρ and $\mathscr{R}$ can be found numerically (in some simple cases this can be done analytically).

4. MAIN RESULTS

Having found the sets $\mathscr{C}_N(\mathscr{R})$ and $\mathscr{W}_c(\rho)$ for which the closed-loop system (9) satisfies the Demidovich condition, we can solve the estimation problem stated in Section 2. Prior to formulating the solution, let us introduce the following functions:

$$m_N(w_0) := \sup_{t\geqslant 0} |N\tilde{\pi}(w(t,w_0))|, \quad q(w_0) := \sup_{t\geqslant 0} |\varepsilon_1(w(t,w_0))|_P \tag{19}$$

where $w(t,w_0)$ is a solution of exosystem (3) satisfying $w(0,w_0)=w_0$. Denote d to be the smallest number such that the inequality $|Nz|\leqslant d|z|_P$ is satisfied for all $z\in\mathbb{R}^{n+k}$. The number d can be found from the formula $d=\|NP^{-1/2}\|$. Indeed,

$$d = \sup_{|z|_P=1} |Nz| = \sup_{|P^{1/2}z|=1} |Nz| = \sup_{|\tilde{z}|=1} |NP^{-1/2}\tilde{z}| = \|NP^{-1/2}\|$$

The following theorem gives an estimate of the set of admissible initial conditions in the form of a neighbourhood of the approximate output-zeroing manifold $z=\tilde{\pi}(w)$.

Theorem 1

Consider the closed-loop system (9) and the exosystem (3) satisfying conditions ($\mathscr{A}$) and ($\mathscr{B}^*$). Suppose, the closed-loop system (9) satisfies the Demidovich condition (14) with $\mathscr{C}_N(\mathscr{R}) := \{z : |Nz|<\mathscr{R}\}$ and $\mathscr{W}_c(\rho) := \{w : |w|<\rho\}$ for some $\mathscr{R}>0$, $\rho>0$ and some matrix N. Then, any trajectory $(z(t),w(t))$ of the closed-loop system (9) and the exosystem (3) starting in the set

$$\mathscr{Y} := \left\{(z_0,w_0) : |w_0|<\rho,\ m_N(w_0) + \frac{2d\|P\|}{\alpha} q(w_0) < \mathscr{R}, |z_0-\tilde{\pi}(w_0)|_P < \frac{1}{d}(\mathscr{R}-m_N(w_0))\right\} \tag{20}$$

satisfies

$$\limsup_{t\to+\infty} |z(t)-\tilde{\pi}(w(t))|_P \leqslant \frac{2\|P\|}{\alpha} \limsup_{t\to+\infty} |\varepsilon_1(w(t))|_P \tag{21}$$

and thus

$$\limsup_{t\to+\infty} |e(t)| \leqslant \bar{C} \limsup_{t\to+\infty} |\varepsilon_1(w(t))|_P + \limsup_{t\to+\infty} |\varepsilon_2(w(t))| \tag{22}$$

Herein, for all trajectories $(z(t),w(t))$ such that $|w(t)|\leqslant K$ for some $K>0$, the constant $\bar{C}>0$ can be chosen independent of the particular $z(0)$ and $w(0)$. □

Remark

Notice, that if the closed-loop system (9) satisfies the Demidovich condition (14) globally, i.e. for $\mathscr{C}=\mathbb{R}^{n+k}$ and $\mathscr{W}_c=\mathbb{R}^m$, then (21) and (22) hold globally.

The relation between the sets $\mathscr{Y}$, $\mathscr{C}_N(\mathscr{R})$ and $\mathscr{W}_c(\rho)$ is shown schematically in Figure 1. If we want the closed-loop system (9) and exosystem (3) to start in the set $\mathscr{Y}$, we need to guarantee that, first, the exosystem starts in a point w_0 in the set $\mathscr{M} := \{w_0 : |w_0|<\rho, m_N(w_0) + 2d\|P\|/\alpha\ q(w_0)<\mathscr{R}\}$ and, second, that the closed-loop system (9) starts in the set $\mathscr{A}(w_0) := \{z_0 : (z_0,w_0)\in\mathscr{Y}\}$. As can be seen in Figure 2, the sets $\mathscr{A}(w_0)$ may be different for different values of w_0. Thus, knowledge on w_0 is important. In practice, however, we may not know the exact value of w_0. For example, if the exosystem generates disturbances, then, knowing the level of disturbances, we can establish that $w_0\in\mathscr{M}$, but still the exact value of w_0 is unknown. In order to cope with this difficulty, in the next result we find sets Z_0 and W_0 such that in whatever

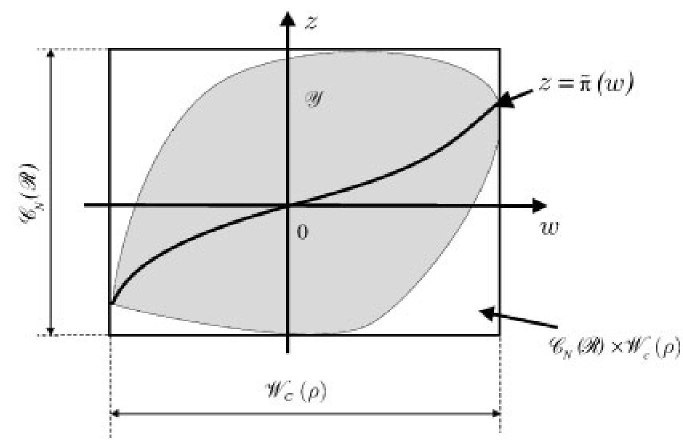

Figure 1. Relation between the sets $\mathscr{Y}$, $\mathscr{C}_N(\mathscr{R})$ and $\mathscr{W}_c(\rho)$: $\mathscr{Y}$ is an invariant set inside $\mathscr{C}_N(\mathscr{R}) \times \mathscr{W}_c(\rho)$.

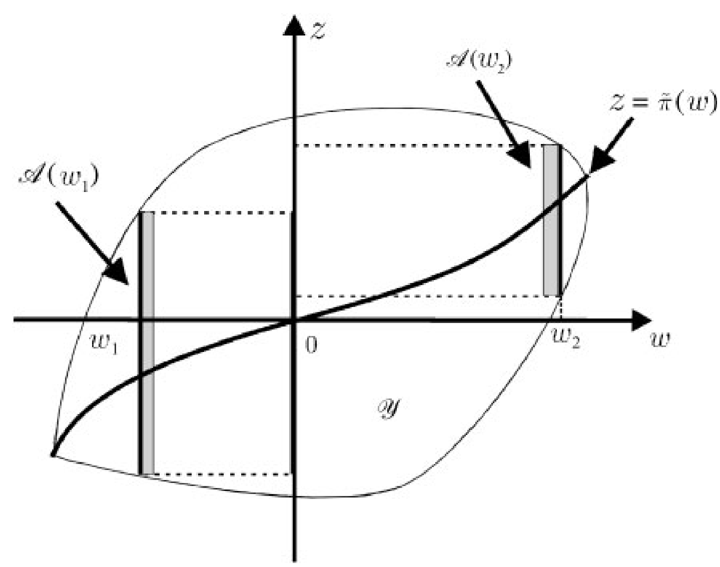

Figure 2. The sets $\mathscr{Y}$ and $\mathscr{A}(w)$: for different w_1 and w_2, the sets $\mathscr{A}(w_1)$ and $\mathscr{A}(w_2)$ may differ.

point $w_0 \in W_0$ the exosystem is initialized, output regulation will occur if the closed-loop system starts in $z_0 \in Z_0$. Prior to formulating the result, we define the functions

$$\begin{aligned} \delta(r) &:= \sup_{|w_0|<r} (|N\tilde{\pi}(w_0)| + d|\tilde{\pi}(w_0)|_P), \quad R(r) := (\mathscr{R} - \delta(r))/d \\ \eta(r) &:= \sup_{|w_0|<r} \left(|N\tilde{\pi}(w_0)| + \frac{2d\|P\|}{\alpha} |\varepsilon_1(w_0)|_P \right) \end{aligned} \tag{23}$$

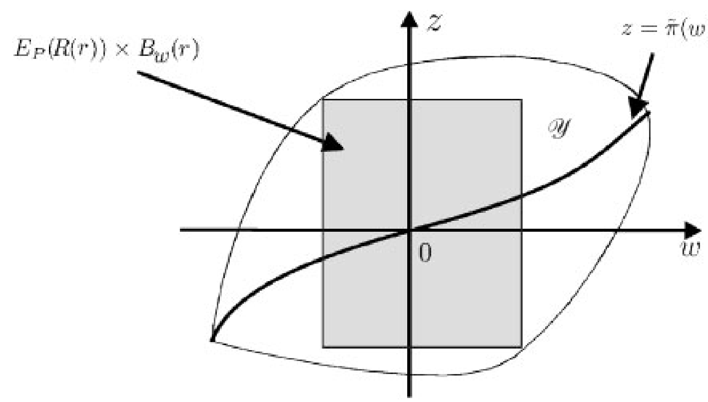

Figure 3. Relation between the sets $\mathscr{Y}$ and $E_P(R(r))$, $B_w(r)$.

Let $r_* > 0$ be the largest number such that $r_* \leqslant \rho$, $\delta(r) < \mathscr{R}$ and $\eta(r) < \mathscr{R}$ for all $r \in [0, r_*)$. The estimates for the sets of admissible $z(0)$ and $w(0)$ are given by the next theorem.

Theorem 2
The conclusion of Theorem 1 holds for any trajectory $(z(t), w(t))$ starting in

$$z(0) \in E_P(R(r)) := \{z : |z|_P < R(r)\}, \quad w(0) \in B_w(r) := \{w : |w| < r\}$$

for any $r \in [0, r_*)$.

The proof of this theorem is based on the fact that for every $r \in [0, r_*)$ the set $E_P(R(r)) \times B_w(r)$ is a subset of $\mathscr{Y}$, as shown in Figure 3.

5. EXAMPLE

We illustrate the application of Theorem 2. Consider the so-called TORA-system (Translational Oscillator with a Rotational Actuator) described by equations of the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon \sin x_3 + \mu D$$

$$\dot{x}_3 = x_4, \quad \dot{x}_4 = v \tag{24}$$

$$e = (x_1 - \varepsilon \sin x_3)$$

where μ and $\varepsilon < 1$ are some positive parameters, v is a control input and D is a disturbance force. This system is a non-linear benchmark mechanical system that was introduced in Reference [17] (see also Reference [18]). The control problem is to find a controller such that e tends to zero in the presence of a disturbance D generated by the system

$$\dot{w}_1 = \Omega w_2, \quad \dot{w}_2 = -\Omega w_1, \quad D = \lambda \arctan(w_1/\lambda) \tag{25}$$

For simplicity, we assume that both x and w are measured, i.e. $y = (x, w)$. This is a particular case of the output regulation problem. It is very difficult to find an exact solution to this problem. At the same time, we can relatively easily find an approximate solution: $v = c(w) + K(x - \tilde{\pi}(w))$, where the mappings $\tilde{\pi}(w) \in \mathbb{R}^4$ and $c(w) \in \mathbb{R}$ are defined by the formulae

$$\begin{aligned} &\tilde{\pi}_1(w) := -\frac{\mu w_1}{\Omega^2}, \quad \tilde{\pi}_2(w) := -\frac{\mu w_2}{\Omega}, \quad \tilde{\pi}_3(w) := -\arcsin\left(\frac{\mu w_1}{\Omega^2 \varepsilon}\right) \\ &\tilde{\pi}_4(w) := -\frac{\mu \Omega w_2}{\sqrt{\Omega^4 \varepsilon^2 - \mu^2 w_1^2}} \end{aligned} \tag{26}$$

$$c(w) := \frac{\mu \Omega^2 w_1 (\Omega^4 \varepsilon^2 - \mu^2 (w_1^2 + w_2^2))}{\left(\sqrt{\Omega^4 \varepsilon^2 - \mu^2 w_1^2}\right)^3} \tag{27}$$

and the matrix K is such that the closed-loop system has an asymptotically stable linearization at the origin. Indeed, one can easily check that for such controller the closed-loop system satisfies conditions ($\mathscr{A}$) and ($\mathscr{B}^*$) with the specified $\tilde{\pi}(w)$, $\varepsilon_1(w) := (0, \mu(\lambda \arctan(w_1/\lambda) - w_1), 0, 0)^{\mathrm{T}}$ and $\varepsilon_2(w) \equiv 0$ (see References [3, 6, 7] for details on finding approximate solutions to the local output regulation problem). Let us apply Theorem 2 to estimate the set of admissible $(x(0), w(0))$ (since the controller is static, then $z = x$) for the following values of the parameters: $\varepsilon = 0.5$, $\mu = 0.04$, $\lambda = 3$, $\Omega = 1$, $K = (12, -4, -8, -5)$.

First, we must choose a matrix $P = P^{\mathrm{T}} > 0$ such that $P\mathscr{D}F_x(0,0) + (\mathscr{D}F_x(0,0))^{\mathrm{T}} P < 0$. We find such P from the Lyapunov equation $P\mathscr{D}F_x(0,0) + (\mathscr{D}F_x(0,0))^{\mathrm{T}} P = -Q$, where Q is the diagonal matrix $\mathrm{diag}(2, 8, 1, 1)$. For convenience, P is normalized such that $\|P\| = 1$. Since $\mathscr{D}F_x(x, w)$ only depends on x_3, the matrix N for the set $\mathscr{C}_N(\mathscr{R})$ is chosen equal to $N = (0, 0, 1, 0)$, i.e. such that $Nx = x_3$. So, the convergence set $\mathscr{C}$ is sought in the form $\mathscr{C}_N(\mathscr{R}) := \{x : |x_3| < \mathscr{R}\}$ (see Section 3 for details). Since $\mathscr{D}F_x(x, w)$ does not depend on w, the companion set $\mathscr{W}_c$ can be taken equal to $\mathbb{R}^2$. For the convergence set $\mathscr{C}_N(\mathscr{R})$ we choose $\mathscr{R} = 0.88$. The corresponding α equals to $\alpha = 0.083$. We have chosen arbitrary $\mathscr{R}$ from the range of $\mathscr{R}$'s for which the corresponding α is positive. Such range is determined numerically. Finally, after computing $R(r)$, $\eta(r)$ and r_*, we obtain estimates of the admissible initial conditions set $E_P(R(r)) \times B_w(r)$, where $R(r)$ is given in Figure 4. Theorem 2 provides the estimates for $r \in [0, r_*)$. In our case, $r_* \approx 2.3$ (for $r = r_*$, the function $\eta(r)$ reaches $\mathscr{R}$). For $r > r_*$, Theorem 2 does not guarantee that *both* $x(t)$ starting in $E_P(R(r))$ and $\tilde{\pi}(w(t))$ with $w(t)$ starting in $B_w(r)$ will lie in the convergence set $\mathscr{C}_N(\mathscr{R})$. Thus, Lemma 2 cannot be applied and inequalities (21) and (22) may not hold.

Note, that the mappings $\tilde{\pi}(w)$ and $c(w)$ and, thus, the closed-loop system are defined only for $|w_1| < \Omega^2 \varepsilon/\mu$. For the given values of the system parameters this constraint is given by $|w_1| < 12.5$. The obtained estimates satisfy this condition. The estimates are fairly conservative. According to simulations, for a fixed level of disturbance r, approximate output regulation still occurs for $x(0) \in E_P(\bar{R}(r))$ with $\bar{R}(r)$ about 4 times larger than the obtained $R(r)$. One possible reason for such conservativeness is the specific choice of the matrix P and the number $\mathscr{R}$. A different choice for P and $\mathscr{R}$ may result in better estimates. At the moment, it is an open question how to choose these parameters in order to obtain the best (in some sense) estimates.

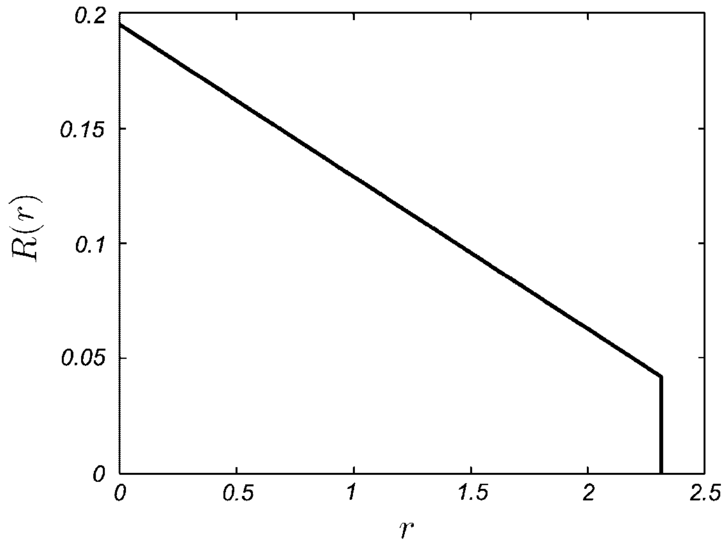

Figure 4. Estimates for the TORA system: $R(r)$ and r for the estimates $E_P(R(r)) \times B_w(r)$.

6. CONCLUSIONS

In this paper, we have considered the problem of estimating the set of admissible initial conditions for a solution to the local approximate output regulation problem. The presented solutions to this estimation problem are based on the so-called Demidovich condition. The obtained estimates consist of initial conditions for which the trajectories of the forced closed-loop system converge to a certain neighbourhood of the approximate output-zeroing manifold. The results are illustrated by application to a disturbance rejection problem in the TORA system. Since the exosystem is allowed to generate constant signals, the obtained results are also applicable for systems with parametric uncertainties. Although the analysis in the paper was performed under the assumption of linearity of the exosystem, the results can be extended to the case of general neutrally stable exosystems.

The obtained estimates are, in general, fairly conservative, since they are based on quadratic stability analysis and strongly depend on the choice of the parameters N, P and $\mathcal{R}$. Despite this conservatism, the results can be rather useful in the following situations. First, one can directly use the estimates in practice (for certain simple systems they may be quite satisfactory). Second, if the estimates are too conservative, one can use them as a starting point for obtaining less conservative estimates by means of, for example, backward integration. The third way is to use the estimates as a criterion for choosing/tuning certain controller parameters. Since controller design admits some freedom in choosing certain controller parameters (such as the matrix K in the TORA example), one can choose parameters which guarantee larger estimates. For example, one can aim at finding controller parameters that guarantee satisfaction of the Demidovich condition globally. For such controller, the upper bound of the limit values of the regulation error would be proportional to the upper bound on the error in the regulator equations, regardless of the initial conditions, i.e. approximate output regulation is attained globally.

APPENDIX A

Proof of Lemma 2

Denote $\zeta := z - \bar{z}$. It satisfies the equation

$$\dot{\zeta} := F(z, w(t)) + \varepsilon(t) - F(\bar{z}, w(t)) \tag{A1}$$

Consider the Lyapunov function $V(\zeta) := \zeta^{\mathrm{T}} P \zeta$. Its derivative satisfies

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2\zeta^{\mathrm{T}} P(F(z, w(t)) - F(\bar{z}, w(t)) + \varepsilon(t))$$

Notice, that in the region $|\zeta|_P \leqslant r$ both $\bar{z}(t)$ and $z = \zeta + \bar{z}(t)$ belong to $\mathscr{C}$. Since $w(t)$ belongs to $\mathscr{W}_c$ for all $t \geqslant 0$, we can apply Lemma 1. By formula (16) and by the Cauchy inequality, we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant -\frac{\alpha}{\|P\|}|\zeta|_P^2 + 2|\zeta|_P|\varepsilon(t)|_P \leqslant -\frac{\alpha}{\|P\|}|\zeta|_P^2 + 2|\zeta|_P \lambda(t_0) \quad \text{for } t \geqslant t_0 \geqslant 0 \tag{A2}$$

where $\lambda(t_0) := \sup_{t \geqslant t_0} |\varepsilon(t)|_P$. By the conditions of the lemma, $\lambda(t_0) < \alpha r/(2\|P\|)$ for any $t_0 \geqslant 0$. Thus, from (A2) we can conclude that the ellipsoid $\bar{E}_P(r) := \{\zeta : V(\zeta) < r^2\}$ is invariant with respect to (A1). Application of Theorem 5.1 from [19] implies that for any solution starting in $\bar{E}_P(r)$ and any η satisfying $2\|P\|/\alpha\lambda(t_0) < \eta < r$ there exists $T > 0$ such that $|\zeta(t)|_P \leqslant \eta$ for all $t \geqslant t_0 + T$. Due to the arbitrary choice of $\eta > 2\|P\|/\alpha\lambda(t_0)$, any solution of (A1) starting in $\bar{E}_P(r)$ satisfies

$$\limsup_{t \to +\infty} |\zeta(t)|_P \leqslant \frac{2\|P\|}{\alpha} \lambda(t_0)$$

Since the left-hand side does not depend on t_0, we can conclude that

$$\limsup_{t \to +\infty} |\zeta(t)|_P \leqslant \frac{2\|P\|}{\alpha} \limsup_{t_0 \to +\infty} \lambda(t_0) = \frac{2\|P\|}{\alpha} \limsup_{t \to +\infty} |\varepsilon(t)|_P \qquad \square$$

Proof of Theorem 1

We need to show that (21) holds for any solution $(z(t), w(t))$ that starts in $(z(0), w(0))$ satisfying the relations: $|w(0)| < \rho$, $m_N(w(0)) + 2d\|P\|/\alpha\, q(w(0)) < \mathscr{R}$ and $z(0) \in E_P(\tilde{\pi}(w(0)), r)$, where $E_P(\bar{z}, r) := \{z : |z - \bar{z}|_P < r\}$ and $r := (\mathscr{R} - m_N(w(0)))/d$. Due to the conditions on the initial conditions and the properties of the exosystem, $|w(t)| \equiv |w(0)| < \rho$ and the solution $\bar{z}(t) := \tilde{\pi}(w(t))$ of the system

$$\dot{z} = F(z, w(t)) + \varepsilon_1(w(t)) \tag{A3}$$

satisfies

$$|N\bar{z}(t)| \leqslant \sup_{t \geqslant 0} |N\tilde{\pi}(w(t))| = m_N(w(0)) < \mathscr{R}$$

Hence, $\bar{z}(t) \in C_N(\mathscr{R})$ and $w(t) \in \mathscr{W}_c(\rho)$ for all $t \geqslant 0$. Let us show that $E_P(\bar{z}(t), r) \subset \mathscr{C}_N(\mathscr{R})$ for all $t \geqslant 0$. Suppose $z \in E_P(\bar{z}(t), r)$ for some $t \geqslant 0$. Then,

$$|Nz| \leqslant |N\bar{z}(t)| + |N(z - \bar{z}(t))| \leqslant m_N(w(0)) + \mathrm{d}|z - \bar{z}(t)|_P < m_N(w(0)) + \mathrm{d}r = \mathscr{R}$$

Consequently, $E_P(\bar{z}(t), r) \subset \mathscr{C}_N(\mathscr{R})$ for all $t \geqslant 0$. The sets $\mathscr{C}_N(\mathscr{R})$ and $\mathscr{W}_c(\rho)$ satisfy the conditions of Lemma 1. As follows from the second inequality in the definition of $\mathscr{Y}$, the term $\varepsilon_1(w(t))$ satisfies

$$|\varepsilon_1(w(t))|_P \leqslant \sup_{t \geqslant 0} |\varepsilon_1(w(t))|_P = q(w(0)) < \frac{\alpha}{2\|P\|} r$$

Thus, by Lemma 2, we obtain that any solution of system (9) starting in $z(0) \in E_P(\bar{z}(0), r)$ satisfies (21). It remains to show that (22) holds with the number $\bar{C} > 0$ being independent from particular $z(0)$, $w(0)$ as long as $|w(0)| < K$ for some $K > 0$. As follows from (21), for any $K > 0$ any solution of the closed-loop system (9) and the exosystem (3) starting in $(z(0), w(0)) \in \mathscr{Y}$, where $|w(0)| < K$, converges to the compact set

$$\mathscr{Y}_K := \left\{ (z, w) : |w| \leqslant K,\ \ |z - \tilde{\pi}(w)|_P \leqslant \frac{2\|P\|}{\alpha} |\varepsilon_1(w)|_P \right\}$$

Since $\bar{h}(z, w)$ is continuously differentiable, it is Lipschitz over any compact set. Thus, there exists $L > 0$ such that $|\bar{h}(z_1, w) - \bar{h}(z_2, w)| \leqslant L|z_1 - z_2|_P$ for all (z_1, w), $(z_2, w) \in \mathscr{Y}_K$. Hence, in the limit we obtain:

$$\begin{aligned}
\limsup_{t \to +\infty} |\bar{h}(z(t), w(t))| &\leqslant \limsup_{t \to +\infty} (|\bar{h}(\tilde{\pi}(w(t)), w(t))| + |\bar{h}(z(t), w(t)) - \bar{h}(\tilde{\pi}(w(t)), w(t))|) \\
&\leqslant \limsup_{t \to +\infty} |\varepsilon_2(w(t))| + \limsup_{t \to +\infty} L|z(t) - \tilde{\pi}(w(t))|_P \\
&\leqslant \limsup_{t \to +\infty} |\varepsilon_2(w(t))| + \limsup_{t \to +\infty} \frac{2\|P\|L}{\alpha} |\varepsilon_1(w(t))|_P
\end{aligned}$$

Thus, the constant $\bar{C}$ in (22) can be chosen equal to $\bar{C} := 2\|P\|L/\alpha$. □

Proof of Theorem 2

It is sufficient to show that $E_P(R(r)) \times B_w(r) \subset \mathscr{Y}$ for any $r \in [0, r_*)$. Then, the statement of Theorem 2 follows from Theorem 1. Suppose $z_0 \in E_P(R(r))$ and $w_0 \in B_w(r)$ for some fixed $r \in [0, r_*)$. According to the definition of $\mathscr{Y}$, we first need to show that $|w_0| < \rho$. This is true due to the fact that $|w_0| < r < r_* \leqslant \rho$. Next, we show that $m_N(w_0) + 2d\|P\|/\alpha\, |q(w_0)|_P < \mathscr{R}$. By the definition of $\eta(r)$, it holds that $|N\tilde{\pi}(w)| + 2d\|P\|/\alpha\, |\varepsilon_1(w)|_P \leqslant \eta(r)$ for all $|w| < r$. The choice of $|w_0| < r$ implies $|w(t, w_0)| \equiv |w_0| < r$. Hence, by the definition of $m_N(w_0)$ and $q(w_0)$ we obtain

$$m_N(w_0) = \sup_{t \geqslant 0} |N\tilde{\pi}(w(t, w_0))| \leqslant \sup_{|w| < r} |N\tilde{\pi}(w)|$$

$$q(w_0) = \sup_{t \geqslant 0} |\varepsilon_1(w(t, w_0))|_P \leqslant \sup_{|w| < r} |\varepsilon_1(w)|_P$$

Thus, we obtain

$$m_N(w_0) + \frac{2d\|P\|}{\alpha}|q(w_0)|_P \leqslant \sup_{|w|<r}\left(|N\tilde{\pi}(w)| + \frac{2d\|P\|}{\alpha}|\varepsilon_1(w)|_P\right) = \eta(r)$$

The choice of $r < r_*$ implies that $\eta(r) < \mathscr{R}$ and consequently $m_N(w_0) + 2d\|P\|/\alpha\,|q(w_0)|_P < \mathscr{R}$.

Next, we need to show that $|z_0 - \tilde{\pi}(w_0)|_P < (\mathscr{R} - m_N(w_0))/d$. The triangle inequality implies

$$|z_0 - \tilde{\pi}(w_0)|_P \leqslant |z_0|_P + |\tilde{\pi}(w_0)|_P \tag{A4}$$

By the choice of z_0 and by the definition of $R(r)$,

$$|z_0|_P < R(r) = (\mathscr{R} - \delta(r))/d = \left(\mathscr{R} - \sup_{|w|<r}(|N\tilde{\pi}(w)| + d|\tilde{\pi}(w)|_P)\right)\bigg/ d$$

$$\leqslant (\mathscr{R} - m_N(w_0))/d - |\tilde{\pi}(w_0)|_P$$

Substituting this inequality in (A4), we obtain $|z_0 - \tilde{\pi}(w_0)|_P < (\mathscr{R} - m_N(w_0))/d$. This completes the proof. □

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