# Tracking control for sampled-data systems with uncertain time-varying sampling intervals and delays

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#### SUMMARY

In this paper, a solution to the approximate tracking problem of sampled-data systems with uncertain, time-varying sampling intervals and delays is presented. Such time-varying sampling intervals and delays can typically occur in the field of networked control systems. The uncertain, time-varying sampling and network delays cause inexact feedforward, which induces a perturbation on the tracking error dynamics, for which a model is presented in this paper. Sufficient conditions for the input-to-state stability (ISS) of the tracking error dynamics with respect to this perturbation are given. Hereto, two analysis approaches are developed: a discrete-time approach and an approach in terms of delay impulsive differential equations. These ISS results provide bounds on the steady-state tracking error as a function of the plant properties, the control design and the network properties. Moreover, it is shown that feedforward preview can significantly improve the tracking performance and an online extremum seeking (nonlinear programming) algorithm is proposed to online estimate the optimal preview time. The results are illustrated on a mechanical motion control example showing the effectiveness of the proposed strategy and providing insight into the differences and commonalities between the two analysis approaches. Copyright © 2009 John Wiley & Sons, Ltd.

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#### 1. INTRODUCTION

In this paper, we study the tracking control problem for sampled-data systems with uncertain, time-varying sampling intervals and delays. The cause for such time-varying and uncertain sampling intervals and delays can be twofold. First, consider the case in which the hardware/software architecture of the control system is such that multiple processes are running on the processor used to compute the control law. As a consequence, delays may be introduced by the fact that the computation of the next control action is delayed due to the fact that the processor is busy running other

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processes (e.g. related to other control-loops or other processes). Moreover, even in the case when requests for measurements are scheduled in a time-sliced manner again the fact that the processor may be busy running other processes first (causing the request for a new measurement to be sent out later) may induce time-varying sampling intervals. Another motivation for studying the problem at hand can be recognized in the field of networked control systems (NCS), in which the communication between the actuators, sensors and the controllers takes place over a communication network. The benefits of using such a communication network, as opposed to dedicated point-to-point wiring, is increased architectural flexibility, decreased maintenance costs and system wiring [1, 2]. Typical application areas to which these benefits appeal are cooperative (mobile) robotics, haptics and mobile sensor networks [3]. The presence of the communication network, however, induces non-ideal behaviour in the form of uncertain, and time-varying, sampling intervals, network delays and packet loss [1].

Different types of models for NCSs with uncertain, and time-varying, sampling intervals and network delays have been proposed in the literature. In many works, see, e.g. [4-8], an approach using exact discretizations of the continuous-time plant dynamics is employed in the face of network delays. In [9, 10], NCSs, with uncertain sampling intervals and network delays, are analysed using results for impulsive delay differential equations thereby avoiding such discretizations. For both cases, stability criteria have been proposed. We refer to [7, 8, 11] for stability conditions for the discrete-time approach and to [9] for stability results for the approach in terms of impulsive delay differential equations.

To this date, the work on NCSs has largely focussed on modelling, stability and stabilization problems. Tracking control, however, poses additional challenges, some of which are specifically due to the communication network. In tracking control, typical high-performance designs include feedforward control thereby inducing the desired solution in the controlled system, whereas feedback assures convergence to the desired solution and favourable robustness and disturbance attenuation properties. Owing to the delays and variation in sampling intervals, the feedforward control signal generally arrives at the actuator later than intended, leading to a (network-induced) feedforward error and reduced tracking performance. Consequently, only approximate tracking can be achieved. In the NCS literature, the tracking problem has received very little attention. Recent works related to the tracking control of systems over networks are [12, 13]. In [12], an  $H_{\infty}$ -approach towards the tracking control problem of NCSs with network delays (and constant sampling intervals) is presented; however, the fact that the feedforward generally experiences delays is not taken into account. In [13], the optimal tracking control problem is studied with a focus on the effects of quantization in the feedforward. In the current paper, we propose a novel NCS model for tracking control.

Here, we propose control designs rendering the NCS input-to-state stable (ISS) with respect to the feedforward error. Based on the ISS property we provide an asymptotic upper bound for the tracking error depending on the properties of the plant, the controller and the network. Such knowledge is, e.g. instrumental in providing upper bounds on the sampling interval guaranteeing a minimum level of steady-state tracking performance. Input-to-state (and input-output) stability properties of (nonlinear) NCSs have been studied in [14, 15]. In these works, specific attention is given to the role of the network protocol in guaranteeing such stability properties. Herein, NCSs with time-varying sampling intervals and multiple-packet communication are considered; however, no network delays are taken into account.

Moreover, we propose to use to concept of feedforward preview in order to reduce the feedforward error. Preview control is regularly applied to improve tracking performance of both continuous-time and discrete-time systems when some preview on the desired trajectory is available, see e.g. [16–20]. Here, we propose to exploit preview to improve the tracking performance of NCSs that is limited by the presence of network delays and variable sampling intervals. In order to do so, knowledge on the present sampling interval time and the communication delay is needed, which is clearly not available due to the uncertain and time-varying nature of these quantities. Here, we propose an extremum seeking control approach [21, 22] towards minimizing the feedforward error. Using this approach the preview time is adapted online based on a tracking error performance measure that is measured online. It is shown that this approach can attain a significant performance increase, even in the face of changing network conditions (i.e. changing stochastic properties of the sampling interval lengths and network delays).

The outline of the paper is as follows. In Section 2, an NCS model for tracking is proposed and the approximate tracking problem is formalized. In Section 3, two approaches for analysing the ISS properties of this NCS model are addressed; one approach based on a discretization of the continuous-time NCS dynamics and the other based on a formulation in terms of impulsive delay systems. Moreover, asymptotic bounds for the tracking error are provided in Section 4. The dependency of the feedforward error on the network properties and the feedforward preview time is illuminated in Section 4. Next, the extremum seeking control algorithm is introduced. In Section 5, an example is presented illustrating both the benefit of the ISS results and the effectiveness of the extremum seeking preview control strategy. Finally, conclusions and an outlook on future work are given in Section 6.

*Notation.* A function  $\gamma:[0,\infty) \to [0,\infty)$  is said to be of class- $\mathscr{G}$  if it is continuous, zero at zero and nondecreasing. It is of class- $\mathscr{K}$  if it is of class- $\mathscr{G}$  and strictly increasing. It is of class- $\mathscr{K}_{\infty}$  if it is of class- $\mathscr{K}$  and unbounded. A continuous function  $\beta:[0,\infty) \times$  $[0,\infty) \to [0,\infty)$  is said to be of class- $\mathscr{K}\mathscr{L}$  if  $\beta(.,t)$  is of class- $\mathscr{K}$  for each  $t \ge 0$  and  $\beta(s,.)$  is monotonically decreasing to zero for each s > 0. We denote the transpose of a matrix **A** by **A**<sup>T</sup> and we write **P**>0 (or **P**<0) when *P* is a symmetric positive (or negative) definite matrix. We write a symmetric matrix  $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}$  as  $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{*} & \mathbf{C} \end{bmatrix}$ . When there is no confusion we write  $\mathbf{x}(t)$  as  $\mathbf{x}$ .

#### 2. AN NCS MODEL FOR TRACKING CONTROL

The two-channel NCS is schematically depicted in Figure 1. It consists of a continuous-time plant and a discrete-time controller, which receives information from the plant only at the sampling instants  $s_k$ . Owing to the fact that we allow for a variable sampling



Figure 1. Schematic overview of the two-channel NCS with variable sampling and network delays.

interval  $h_k$ , the samplings instants  $s_k = \sum_{i=0}^{k-1} h_i$ ,  $\forall k \ge 1, s_0 = 0$ , are non-equidistantly spaced. Moreover, the computation times and the network delays result in a sensor-to-controller delay  $\tau_k^{sc}$  and in a controllerto-actuator delay  $\tau_k^{ca}$ , which have to be taken into account. Similar to [23], the sensor acts in a timedriven (though variable) fashion and the controller and actuator (including the zero-order-hold (ZOH) in Figure 1) act in an event-driven fashion in the sense that the controller and the actuator update their outputs as soon as they receive a new sample. Hereto, we assume that all sensors are sampled synchronously and there is a single sensor-sending source.

The two-channel NCS in Figure 1 is equivalent to a one-channel NCS with the total delay  $\tau_k := \tau_k^{sc} + \tau_k^{ca}$ given that the controller is static and time-invariant [1]. In remainder of the paper we consider such a onechannel NCS. We call  $\tau_k$  the (*k*th total) delay in the loop and  $t_k := s_k + \tau_k$  the (kth) control input update time. The sampling times  $\{s_0, s_1, s_2, s_3, \ldots\}$  and the input update times  $\{t_0, t_1, t_2, t_3, ...\}$  form strictly increasing sequences in  $[s_0, \infty)$  for some initial time  $s_0$ . In practice the sampling times sequence is a strictly increasing sequence and with some assumptions we guarantee that the control input update times are also a strictly increasing sequence. In Section 3.1, we assume that  $\tau_k \leq h_k := s_{k+1} - s_k, \forall k \in \mathbb{N}$ , and necessarily the control input update times form a strictly increasing sequence (note that  $s_k \leq t_k \leq s_{k+1}$ ). When the total delay can be potentially larger than the sampling intervals then the samples may arrive at the destination out of order  $(t_k \ge t_{k+1} \text{ for some } k \in \mathbb{N})$ . In this case, we drop the out-dated sample and we only index the samples that arrive at the destination and by doing that both the sampling times and the input update times form strictly increasing sequences. By indexing the samples that arrive at the destination, we can capture the effect of

packet dropout in the network as well. For example even when the sampling times are uniform but there are packet dropouts in the network, because we do not index the dropped samples, it appears that the sampling times are non-uniform. Although we would not emphasis on the packet dropout in the network, the NCSs with variable sampling intervals and delays are general enough to capture the packet dropout effect.

The continuous-time model of the plant can then be given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}^*(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$
  
$$\mathbf{u}^*(t) = \mathbf{u}_k \quad \text{for } t \in [t_k, t_{k+1}), \ t_0 = \tau_0$$
(1)

with **A** and **B** the continuous-time system and input matrices,  $\mathbf{x}(t) \in \mathbb{R}^n$  the state,  $t \in \mathbb{R}$  the time,  $t_k = s_k + \tau_k$ the instants at which a control update is effectuated,  $\tau_k$ the delay experienced by the sample sent at sampling instants  $s_k$ , and  $\mathbf{u}_k = \mathbf{u}(s_k) \in \mathbb{R}^m$  the delayed discretetime input, and  $\mathbf{x}_k = \mathbf{x}(s_k) \in \mathbb{R}^n$  the state at sampling times. For the sake of simplicity, we assume that we measure the entire state, i.e.  $\mathbf{y}_k = \mathbf{x}_k$ , at the sampling instants  $s_k$ , with  $\mathbf{y}_k = \mathbf{y}(s_k)$ . Note that this NCS model accounts for both uncertain time-varying sampling intervals  $h_k = s_{k+1} - s_k$ , with  $h_k \in [h_{\min}, h_{\max}]$  for all k and  $h_{\min} > 0$ , and uncertain time-varying delays  $\tau_k$ , with  $\tau_k \in [\tau_{\min}, \tau_{\max}]$  for all k.

In what follows we introduce the tracking problem, control signal construction, the tracking error dynamics, and we argue that the ISS property of the error dynamics is the relevant notion to study the effect of the network on the tracking problem.

*Control signal construction*: We desire the system to asymptotically track a desired trajectory  $\mathbf{x}^{d}(t)$  which is only known to the controller. The proposed control law consists of a feedforward part and a feedback part. Herein, the exact feedforward  $\mathbf{u}_{e}^{\text{ff}}(t)$  should be selected such that the desired solution  $\mathbf{x}^{d}(t)$  is a solution of the continuous-time system

$$\dot{\mathbf{x}}^{d}(t) = \mathbf{A}\mathbf{x}^{d}(t) + \mathbf{B}\mathbf{u}_{e}^{\mathrm{ff}}(t)$$
(2)

Here, we assume that  $\mathbf{x}_d(t)$  is at least  $\mathscr{C}^2$ , guaranteeing that  $\mathbf{u}_e^{\text{ff}}(t)$  is at least  $\mathscr{C}^1$ . We propose the following

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tracking control law for (1):

$$\mathbf{u}_{k}(\mathbf{x}_{k}, \mathbf{x}^{d}(s_{k}), \mathbf{u}_{e}^{\mathrm{ff}}(s_{k})) = \mathbf{u}_{e}^{\mathrm{ff}}(s_{k}) - \mathbf{K}(\mathbf{x}_{k} - \mathbf{x}^{d}(s_{k})) \quad (3)$$

that consists of the superposition of a sampled feedforward component  $\mathbf{u}_{e}^{\text{ff}}(s_{k})$  with a linear tracking error feedback component with feedback gain matrix  $\mathbf{K} \in \mathbb{R}^{m \times n}$ . We employ time-stamping for measurements; hence, the sampling time  $s_{k}$  is known to the controller which enables the computation of the control command (3) at time  $s_{k} + \tau_{k}^{\text{sc}}$ . Although in this paper we employ a control command constructed according to (3), alternatively we could use  $\mathbf{u}_{k} =$  $\mathbf{u}_{e}^{\text{ff}}(s_{k} + \tau_{k}^{\text{sc}}) - \mathbf{K}(\mathbf{x}_{k} - \mathbf{x}^{d}(s_{k}))$ , since the delay  $\tau_{k}^{\text{sc}}$  is known to the controller due to time stamping. Moreover, if the estimate of the delays and the sampling intervals were available we could use

$$\mathbf{u}_{k} = \mathbf{u}_{e}^{\mathrm{ff}}(s_{k} + T_{p}) - \mathbf{K}(\mathbf{x}_{k} - \mathbf{x}^{d}(s_{k}))$$
(4)

where  $T_p$  is the so-called preview time which is a function of the delays and sampling intervals in the control loop. In Section 4, we exploit such feedforward preview to improve the tracking performance.

Clearly, the implemented continuous-time feedforward  $\mathbf{u}^{\text{ff}}(t)$  in (1), (3) is given by

$$\mathbf{u}^{\mathrm{ff}}(t) = \mathbf{u}_{e}^{\mathrm{ff}}(s_{k}) \quad \text{for } t \in [t_{k}, t_{k+1})$$
(5)

and differs from the exact feedforward  $\mathbf{u}_{e}^{\text{ff}}(t)$  due to the ZOH and the network delays. Therefore, we decompose the implemented feedforward in an exact feedforward part  $\mathbf{u}_{e}^{\text{ff}}(t)$  and a feedforward error  $\Delta \mathbf{u}^{\text{ff}}(t)$ :  $\mathbf{u}^{\text{ff}}(t) = \mathbf{u}_{e}^{\text{ff}}(t) + \Delta \mathbf{u}^{\text{ff}}(t)$ , with the feedforward error simply defined by  $\Delta \mathbf{u}^{\text{ff}}(t) = \mathbf{u}_{e}^{\text{ff}}(s_{k}) - \mathbf{u}_{e}^{\text{ff}}(t)$  for  $t \in [t_{k}, t_{k+1})$ .

*Closed-loop system*: Applying the control law (3) to system (1) yields the following closed-loop NCS dynamics:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_1(\mathbf{x}_k - \mathbf{x}^d(s_k)) + \mathbf{B}_2\mathbf{u}_e^{\text{ff}}(t) + \mathbf{B}_2\Delta\mathbf{u}^{\text{ff}}(t)$$
(6)

for  $t \in [t_k, t_{k+1})$  and with  $\mathbf{B}_1 := -\mathbf{B}\mathbf{K}$  and  $\mathbf{B}_2 := \mathbf{B}$ . The initial condition  $\bar{\mathbf{x}}(0) := [\mathbf{x}^T(0) \mathbf{x}^T(s_{-1})]^T$  for this system consists of both the initial state at time  $s_0 = 0$ , i.e.  $\mathbf{x}(0) = \mathbf{x}_0$ , and the *hold* state  $\mathbf{x}(s_{-1})$  at time  $s_{-1} < 0$  due to the fact that in the time interval  $t \in [0, \tau_0]$ , the feedback part of the control action is given by  $\mathbf{u}_{-1} = -\mathbf{K}$ 

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 $(\mathbf{x}(s_{-1}) - \mathbf{x}^d(s_{-1}))$ . Hence, the network delays cause the initial state to involve a past state.

*Tracking error dynamics*: We define the tracking error **e** by  $\mathbf{e} = \mathbf{x} - \mathbf{x}^d$ . By combining (2) and (6) we can formulate the continuous-time tracking error dynamics as follows:

$$\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t) + \mathbf{B}_1 \mathbf{e}(s_k) + \mathbf{B}_2 \Delta \mathbf{u}^{\text{ff}}(t)$$
(7)

for  $t \in [t_k, t_{k+1})$  and with initial condition  $\bar{\mathbf{e}}(0) := [\mathbf{e}^{\mathrm{T}}(0) \mathbf{e}^{\mathrm{T}}(s_{-1})]^{\mathrm{T}}$ .

In this work, we consider the approximate tracking problem. Herein, we aim to ensure ultimate boundedness of the tracking error, i.e.  $\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}^d(t) \leq \varepsilon$  for  $t \rightarrow \infty$  for some small  $\varepsilon > 0$ . Some tracking error is to be expected in the NCS setting, as the implemented feedforward signal  $\mathbf{u}^{\text{ff}}(t)$  in (5) will never equal the exact feedforward  $\mathbf{u}_{e}^{\mathrm{ff}}(t)$ . The reasons for non-exact feedforward are, first, the fact that the control signal (and therefore also the feedforward signal) will be passed through a ZOH and, second, the fact that the network delays (in particular, the controller-to-actuator delay  $\tau_k^{ca}$ ) in general cause the feedforward to be implemented too late. Thereby, a feedforward error  $\Delta \mathbf{u}^{\text{ff}}(t)$  is introduced. In the following section, we propose sufficient conditions for the ISS of the continuous-time tracking error dynamics (7) with respect to the input  $\Delta \mathbf{u}^{\text{ff}}(t)$ . An ISS property of the tracking error dynamics guarantees that the controller solves an approximate tracking problem.

Moreover, since such ISS properties of linear sampled-data systems with time-varying delays and sampling intervals are of interest in a wider context we consider systems of the form

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{B}_1 \mathbf{z}(s_k) + \mathbf{B}_2 \mathbf{w}(t)$$
  
for  $t \in [t_k, t_{k+1})$  (8)

with initial condition  $\bar{\mathbf{z}}(0) := [\mathbf{z}^{T}(0) \ \mathbf{z}^{T}(s_{-1})]^{T}$ . Herein, the time-varying input  $\mathbf{w}(t)$  may be the feedforward error (as above). Alternatively, in the scope of the disturbance rejection problem one may consider it to represent external perturbations or, in the scope of the design of observer-based output-feedback schemes, it may represent the observer error perturbing the closed-loop system (where the ISS-property is shown to be instrumental in providing a separation principle). In the remainder of this note, we consider the case that the

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sampling intervals are uncertain and taken from the set  $h_k \in [h_{\min}, h_{\max}], \forall k$ . Moreover, the delay is also uncertain and taken from the set  $\tau_k \in [\tau_{\min}, \tau_{\max}]$ .

#### 3. INPUT-TO-STATE STABILITY OF SAMPLED-DATA SYSTEMS WITH TIME-VARYING DELAYS AND SAMPLING INTERVALS

In this section, we propose sufficient conditions for the ISS of the continuous-time dynamics (8) with respect to the input  $\mathbf{w}(t)$  (i.e. also guaranteeing the ISS property for the continuous-time tracking error dynamics (7)). In the subsequent subsections, we propose two approaches towards proving such ISS properties: in the first approach, the dynamics are largely analysed in a discrete-time setting, whereas in the second approach the dynamics are analysed using delay impulsive differential equations. We will see later that, depending on the problem, either approach can be favourable over the other when considering the stability bounds and the ISS gains provided.

Suppose that a sequence of sampling instants  $s_k$  and delays  $\tau_k$  is denoted by  $\{s_k, \tau_k\}$ . We say that system (8) is *uniformly* input-to-state stable (ISS) over a given class  $\mathscr{S}$  of admissible sequences  $\{s_k, \tau_k\}$  if there exist a  $\mathscr{KL}$ -function  $\beta(r, s)$  and a  $\mathscr{K}$ -function  $\gamma(r)$  such that, for any initial condition  $\bar{\mathbf{z}}(0)$  and any bounded input  $\mathbf{w}(t)$ , the solution  $\mathbf{z}(t)$ , for  $t \ge 0$ , of system (8) satisfies

$$|\mathbf{z}(t)| \leq \beta(|\bar{\mathbf{z}}(0)|, t) + \gamma \left( \sup_{0 \leq s \leq t} |\mathbf{w}(s)| \right)$$
(9)

with functions  $\beta$  and  $\gamma$  that are independent of the choice of the particular sequence  $\{s_k, \tau_k\}$ . We would like to have the ISS property for any sequence of delays such that  $\tau_{\min} \leq \tau_k \leq \tau_{\max}, \forall k \in \mathbb{N}$  and any sequence of sampling intervals such that  $h_{\min} \leq s_{k+1} - s_k \leq h_{\max}$  for given  $h_{\min}, h_{\max}, \tau_{\min}, \tau_{\max}$  where  $0 \leq h_{\min} \leq h_{\max}$  and  $0 \leq \tau_{\min} \leq \tau_{\max}$ . Consequently, the class of admissible sequences is characterized by:

$$\mathcal{S} := \{\{s_k, \tau_k\} : h_{\min} \leqslant s_{k+1} - s_k \leqslant h_{\max}, \\ \tau_{\min} \leqslant \tau_k \leqslant \tau_{\max}\}$$
(10)

#### 3.1. Discrete-time approach

Under the assumption that  $\tau_k < h_k$ ,  $\forall k$ , the discretization of (8) at the sampling instants  $s_k$  gives the discrete-time system, which will form the basis of our analysis:

$$\mathbf{z}_{k+1} = \mathbf{e}^{\mathbf{A}h_k} \mathbf{z}_k + \int_0^{h_k - \tau_k} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_1 \mathbf{z}_k$$
$$+ \int_{h_k - \tau_k}^{h_k} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_1 \mathbf{z}_{k-1} + \bar{\mathbf{w}}_k \qquad (11)$$

where  $\mathbf{\bar{w}}_k := \int_0^{h_k} e^{\mathbf{A}s} \mathbf{B}_2 \mathbf{w}(h_k + s_k - s) \, ds$  and  $\mathbf{z}_k = \mathbf{z}(s_k)$ . Since  $\tau_k < h_k, \forall k$ , we can define an extended state for the system (11) by  $\boldsymbol{\xi}_k := (\mathbf{z}_k^T \mathbf{z}_{k-1}^T)^T$  and we obtain the following discrete-time state-space system:

$$\chi_{k+1} = \tilde{\mathbf{A}}(h_k, \tau_k) \boldsymbol{\xi}_k + \tilde{\mathbf{B}} \bar{\mathbf{w}}_k \quad \text{with } \tau_k < h_k$$
$$h_k \in [h_{\min}, h_{\max}], \ \tau_k \in [\tau_{\min}, \tau_{\max}] \quad (12)$$

with  $\xi_k \in \mathbb{R}^{2n}$ ,

$$\tilde{\mathbf{A}}(h_k, \tau_k) = \begin{bmatrix} e^{\mathbf{A}h_k} + \int_0^{h_k - \tau_k} e^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_1 & \int_{h_k - \tau_k}^{h_k} e^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_1 \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$$
$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}$$
(13)

Later, we will use that  $\mathbf{z}_k = \mathbf{C}_z \boldsymbol{\xi}_k$ , with  $\mathbf{C}_z = [\mathbf{I} \ \mathbf{0}]$  being an  $n \times 2n$ -matrix.

Before we formulate conditions for the ISS of system (8), we recall results on the global asymptotic stability of the equilibrium point  $\boldsymbol{\xi} = \boldsymbol{0}$  of the discrete-time system (12) for the case that  $\bar{\mathbf{w}}_k = \mathbf{0}$  for all k (i.e. the case of stabilization). Clearly, the discrete-time system (12), with  $\bar{\mathbf{w}}_k = \mathbf{0}$ , represents a switching discrete-time system for which stability can be guaranteed using a common quadratic Lyapunov function approach. More specifically, stability is guaranteed if the following (infinite) set of matrix inequalities is feasible:

$$\mathbf{P} = \mathbf{P}^{\mathrm{T}} > 0$$
  

$$\tilde{\mathbf{A}}^{\mathrm{T}}(h_k, \tau_k) \mathbf{P} \tilde{\mathbf{A}}(h_k, \tau_k) - (1 - \alpha) \mathbf{P} < 0 \qquad (14)$$
  

$$\forall \{s_k, \tau_k\} \in \mathscr{S}$$

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for some  $0 < \alpha < 1$ . Based on these stability results, we will show (see Theorem 1) that the ISS of (8) is guaranteed if the following (infinite) set of matrix inequalities is feasible:

$$\mathbf{P} = \mathbf{P}^{T} > 0$$

$$\begin{bmatrix} \tilde{\mathbf{A}}^{T}(h_{k}, \tau_{k}) \mathbf{P} \tilde{\mathbf{A}}(h_{k}, \tau_{k}) - (1 - \alpha) \mathbf{P} & \tilde{\mathbf{A}}^{T}(h_{k}, \tau_{k}) \mathbf{P} \tilde{\mathbf{B}} \\ \tilde{\mathbf{B}}^{T} \mathbf{P} \tilde{\mathbf{A}}(h_{k}, \tau_{k}) & \tilde{\mathbf{B}}^{T} \mathbf{P} \tilde{\mathbf{B}} - c_{4} \mathbf{I} \end{bmatrix} < 0$$

$$\forall \{s_{k}, \tau_{k}\} \in \mathscr{S}$$
(15)

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for some  $0 < \alpha < 1$  and  $c_4 > 0$ . Sufficient conditions for the feasibility of (14) in terms of (finite sets of) LMIs are proposed in [11] based on a (real) Jordan form representation of the NCS.<sup>‡</sup> A similar approach can be followed to formulate (a finite set of) LMI conditions for the feasibility of (15). Based on the sufficiency of (15) for ISS (see the proof of Theorem 1), the necessary derivations of the finite set of LMIs are similar to those in [11]; we refer to [24] for details. For the sake of brevity, we omit such technicalities here.

Let us now present the result on the ISS of the continuous-time dynamics (8).

#### Theorem 1

Consider the sampled-data system (8), with uncertain time-varying sample instants  $s_k$  and uncertain time-varying delays  $\tau_k$  taken from the class  $\mathscr{S}$  as in (10), with  $\tau_k < s_{k+1} - s_k \forall k$ . Suppose there exist a matrix **P** and scalars  $0 < \alpha < 1$  and  $c_4 > 0$  for which (15) is satisfied. Then, the system (8) is uniformly ISS over the class  $\mathscr{S}$ , with  $\tau_k < s_{k+1} - s_k$ ,  $\forall k$ , with respect to the time-varying input **w**(*t*) and with

$$\beta(|\bar{\mathbf{z}}(0)|, t) = |\bar{\mathbf{z}}(0)| \begin{cases} \max(g_{1,0}, g_{1,1}, g_{1,2}) & \text{for } t \in [0, s_2) \\ g_{1,k} & \text{for } k \ge 2, \\ t \in [s_k, s_{k+1}) \end{cases}$$
$$\gamma\left(\sup_{0 \le s \le t} |\mathbf{w}(s)|\right) = g_2 \sup_{0 \le s \le t} |\mathbf{w}(s)| \tag{16}$$

<sup>&</sup>lt;sup>‡</sup>In [11], it shown that a similar stability analysis approach can be taken for the large delay case.



Figure 2. Construction of the function  $\overline{\beta}(s, t)$  from  $\beta(s, t)$ .

where

$$g_{1,0} = (c_1 + c_2)$$

$$g_{1,1} = \left(c_1 \| \mathbf{C}_z \mathbf{P}^{-1/2} \| \sqrt{\bar{\alpha} \lambda_{\max}(\mathbf{P})} + c_2\right)$$

$$g_{1,k} = \| \mathbf{C}_z \mathbf{P}^{-1/2} \| \left(c_1 \sqrt{\bar{\alpha}^k \lambda_{\max}(\mathbf{P})} + c_2 \sqrt{\bar{\alpha}^{k-1} \lambda_{\max}(\mathbf{P})}\right), \quad k \ge 2$$

$$g_2 = c_3 \left(1 + (c_1 + c_2) \| \mathbf{C}_z \mathbf{P}^{-1/2} \| \sqrt{\frac{c_4}{\alpha}}\right)$$
(17)

 $c_1$ ,  $c_2$  and  $c_3$  defined in (A9), (A11), and  $\bar{\alpha} = 1 - \alpha$ .

#### Proof

For the proof, see Appendix A.1.

Note that  $\beta(s, t)$  satisfies all the conditions of a class- $\mathscr{KL}$  function except that for fixed *s* it is only nonincreasing and not continuous everywhere because for  $s_0 \leq t < s_2$  and  $s_k \leq t < s_{k+1}, \forall k \geq 2$  the function  $\beta(s, t)$  is flat and it reduces at  $t = s_k, \forall k \geq 3$ . However, it is easy to construct a  $\overline{\beta}(s, t) \in \mathscr{KL}$  from  $\beta(s, t)$ , see Figure 2.

Clearly, this result implies that the state  $\mathbf{z}$  of the sampled-data system is globally uniformly ultimately bounded and the asymptotic bound is given by  $\limsup_{t\to\infty} |\mathbf{z}(t)| \leq g_2 \sup_{t\geq 0} |\mathbf{w}(t)|$ , with  $g_2$  as in (17). Note that all parameters in (17), (A9) and (A11) are known and depend on the system dynamics (matrices  $\mathbf{A}$ ,  $\mathbf{B}_1$  and  $\mathbf{B}_2$ ), the network (maximum and minimum sampling intervals  $h_{\text{max}}$  and  $h_{\text{min}}$ , respectively, and maximum and minimum delays  $\tau_{\text{max}}$  and  $\tau_{\text{min}}$ , respectively) and the parameters  $\alpha$ ,  $c_4$  and matrix  $\mathbf{P}$  satisfying (15).

#### Remark 1

An asymptotic bound for the state at the sampling instants can directly be derived from (A5) and the fact that  $\lim_{k\to\infty} D_k = \lim_{k\to\infty} \sum_{i=1}^k \overline{\alpha}^{i-1} = 1/\alpha$ , yielding

$$\limsup_{k \to \infty} |\mathbf{z}_k| \leq \|\mathbf{C}_z \mathbf{P}^{-1/2}\| \sqrt{\frac{c_4}{\alpha}} \sup_{k \geq 1} |\bar{\mathbf{w}}_k|$$
$$\leq c_3 \|\mathbf{C}_z \mathbf{P}^{-1/2}\| \sqrt{\frac{c_4}{\alpha}} \sup_{t \geq 0} |\mathbf{w}(t)|$$
$$=: \bar{g}_2 \sup_{t \geq 0} |\mathbf{w}(t)|$$
(18)

This bound can in many practical cases (e.g. for sufficiently small sampling intervals) be sufficient and it is typically much less conservative since  $\bar{g}_2 < g_2$ . The difference between  $g_2$  and  $\bar{g}_2$  originates from the need to upperbound the intersample behavior of  $\mathbf{z}$ , thereby introducing additional conservatism.

#### 3.2. Delay impulsive approach

In the previous section, we established sufficient conditions for the ISS property of the system (8) by adopting the discrete-time NCS analysis approach. As used in this paper, this approach required the delays to be smaller than the sampling interval, i.e.  $\tau_k \leq h_k, \forall k$ . Without this assumption, i.e. for large delays, the discrete-time approach yields increasingly complex models [11, 25]; however, the approach remains applicable, see [24]. We now formulate the system (8) as a delay impulsive system, which avoids an increase in complexity when dealing with large delays.

Impulsive dynamical systems exhibit continuous evolutions described by ordinary differential equations and instantaneous state jumps or impulses. We refer to impulsive dynamical systems with delay in the jump equation as delay impulsive systems. First we consider a more general system of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}_k(\mathbf{x}(t), t, \mathbf{w}(t)), \quad t \in [t_k, t_{k+1})$$
(19a)

$$\mathbf{x}(t_{k+1}) = \mathbf{g}_k(\mathbf{x}(t_{k+1}^-), \mathbf{x}(s_{k+1}^-), t_{k+1}^-), \quad k \in \mathbb{N} \quad (19b)$$

where  $\mathbf{f}_k$ ,  $\mathbf{g}_k$  are locally Lipschitz functions such that  $\mathbf{f}_k(0, t, 0) = 0$ ,  $\mathbf{g}_k(0, 0, t) = 0$ ,  $\forall t \in \mathbb{R}_{\geq 0}$ . For the system (19), we assess the ISS property over the set  $\mathscr{S}$  of

impulse-delay sequences defined in (10) using the tools developed for delay differential equations in [26]. Given a Lyapunov-like function

$$V: \mathbb{R}^n \times [0, \rho_{\max}] \times [-h_{\max} - \tau_{\max}, \infty) \quad \to [0, \infty) \quad (20)$$

we use the shorthand notation  $V(t) := V(\mathbf{x}(t), \rho(t), t)$ , where  $\rho(t) := t - t_k$ ,  $t \in [t_k, t_{k+1})$  characterizes the time between impulses.  $\rho(t)$  is a continuous function of time with derivative equal to one almost everywhere except at the update times  $t_k$ . We denote its upper bound by  $\rho_{\max} := \sup_{t \ge 0} \rho(t)$ , which is a function of  $h_{\min}$ ,  $h_{\max}$ ,  $\tau_{\min}$ ,  $\tau_{\max}$ . We define  $t_d := h_{\max} + \tau_{\max}$  and  $|\mathbf{x}_m(t)| :=$  $\max_{-t_d \le \theta \le 0} |\mathbf{x}(t + \theta)|$ , for  $t \ge 0$ .

In the proof of Theorem 2, we rely on the following technical lemma.

#### Lemma 1

Let  $\mu$ ,  $\alpha_3 > 0$ . If  $V(t) \ge \mu$  implies  $dV(t)/dt \le -\alpha_3 V(t)$ , then we have  $V(t) \le \max\{V(t_0) e^{-\alpha_3(t-t_0)}, \mu\}$ .

The proof of the lemma is analogous to the proof of Theorem 4.18 in [27].

#### Theorem 2

Assume that there exist  $\alpha_1, \alpha_2 \in \mathscr{K}_{\infty}, \gamma_v, \gamma_w \in \mathscr{G}$ , a scalar  $\alpha_3 > 0$ , and a *function* V as in (20), such that for any impulse-delay sequence  $\{s_k, \tau_k\} \in \mathscr{S}$  the corresponding solution **x** to (19) satisfies:

$$\alpha_1(|\mathbf{x}(t)|) \leqslant V(t) \leqslant \alpha_2(|\mathbf{x}(t)|),$$
  
$$\forall \rho \in [0, \rho_{\max}], \quad \forall t \ge 0$$
(21)

$$V(t) \ge \max\{\gamma_v(V_m(t)), \gamma_w(\|\mathbf{w}\|_{t_0})\}$$

$$\Rightarrow \frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant -\alpha_3 V(t) \quad \forall t \ge 0 \tag{22}$$

$$\gamma_v(s) < s \quad \forall s > 0 \tag{23}$$

and that

$$V(t_{k+1}) \leqslant \lim_{t \uparrow t_{k+1}} V(t) \quad \forall k \in \mathbb{N}$$
(24)

Then, the system (19) is uniformly ISS over the class  $\mathscr{S}$  of impulse-delay sequences with  $\gamma(s) := \alpha_1^{-1}(\gamma_w(s))$ ,  $\beta(s,t) := \alpha_1^{-1}(e^{-\lfloor (t+t_d)/(T+t_d) \rfloor \alpha_3 T} \alpha_2(s))$ , where T > 0 is small enough such that  $\gamma_v(s) \leq s e^{-\alpha_3 T} \forall s \leq V_m(t_0)$ .

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Proof

For the proof, see Appendix A.2.

Similar to Theorem 1, we note that  $\beta(s, t)$  satisfies all the conditions of a class- $\mathscr{K}\mathscr{L}$  function except that for fixed *s* it is only non-increasing and not continuous everywhere because for  $n(T+t_d) \leq t < (n+1)(T+t_d)$ ,  $\forall n \in \mathbb{N}$  the function  $\beta(s, t)$  is flat and it reduces at t =n(t+T),  $\forall n \in \mathbb{N}$ . However, it is easy to construct a  $\overline{\beta}(s, t) \in \mathscr{K}\mathscr{L}$  from  $\beta(s, t)$ .

The system (8) can be written as a delay impulsive system of the form

$$\dot{\boldsymbol{\zeta}}(t) = \mathbf{F}\boldsymbol{\zeta}(t) + \bar{\mathbf{B}}_2 \mathbf{w}(t), \quad t \in [t_k, t_{k+1}) \quad (25a)$$

$$\boldsymbol{\zeta}(t_{k+1}) = \begin{bmatrix} \mathbf{z}(t_{k+1}) \\ \mathbf{z}(s_{k+1}) \end{bmatrix}, \quad k \in \mathbb{N}$$
(25b)

with the initial condition  $\zeta(0) := [\mathbf{z}^{T}(0) \ \mathbf{z}^{T}(s_{-1})]^{T}$ ,  $\zeta(t) := [\mathbf{z}^{T}(t) \ \mathbf{v}_{1}^{T}(t)]^{T}$ ,  $\mathbf{v}_{1}(t) := \mathbf{z}(s_{k})$ , for  $t \in [t_{k}, t_{k+1})$ , and

$$\mathbf{F} := \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \bar{\mathbf{B}}_2 := \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{0} \end{bmatrix}$$

We employ a Lyapunov candidate function of the form

$$V(t) := \mathbf{z}^{\mathrm{T}} \mathbf{P} \mathbf{z} + (\rho_{\mathrm{max}} - \rho) (\mathbf{z} - \mathbf{v}_2)^{\mathrm{T}} \mathbf{X} (\mathbf{z} - \mathbf{v}_2)$$
(26)

where  $V(t):=V(\tilde{\zeta}, \rho)$ ,  $\rho(t):=t-t_k$ ,  $\mathbf{v}_2:=\mathbf{z}(t_k)$ ,  $t \in [t_k, t_{k+1})$ ,  $\tilde{\zeta}:=[\mathbf{z}^T \mathbf{v}_2^T]^T$  and **P**, **X** are symmetric positive definite matrices. Note that V(t) is positive (for any **z** and **v**<sub>2</sub> not both equal to zero) and satisfies (21). Along jumps this Lyapunov function does not increase since the first term remains unchanged and the second term is non-negative before the jumps and it becomes zero right after the jumps and consequently (24) holds. We choose  $\gamma_w(s):=ps$ ,  $0 ; hence, (23) holds and we choose <math>\gamma_w(s):=g_w s^2$ ,  $g_w > 0$ . If the matrix inequalities that appear below in Theorem 3 are feasible then (22) is satisfied and consequently Theorem 2 guarantees that system (25) is uniformly ISS over the class  $\mathscr{S}$  of sampling-delay sequences.

#### Theorem 3

Assume that there exist positive scalars  $\alpha$ ,  $\lambda_i$ ,  $1 \le i \le 4$ ,  $g_w$ , p < 1 and symmetric positive definite matrices **P**, **X** and (not necessarily symmetric) matrices **N**<sub>1</sub>, **N**<sub>2</sub> that

satisfy the following matrix inequalities:

$$\begin{bmatrix} \mathbf{M}_{1} + \rho_{\max}\mathbf{M}_{2} & \mathbf{N}_{1}\mathbf{A} & \mathbf{N}_{1}\mathbf{B}_{1} & \mathbf{N}_{1}\mathbf{B}_{2} \\ * & -\tau_{\max}^{-1}\lambda_{1}\mathbf{P} & \mathbf{0} & \mathbf{0} \\ * & * & -\tau_{\max}^{-1}\lambda_{3}\mathbf{P} & \mathbf{0} \\ * & * & * & -\tau_{\max}^{-1}\lambda_{2}\mathbf{I} \end{bmatrix} < 0$$
(27a)  
$$\begin{bmatrix} \mathbf{M}_{1} + \rho_{\max}\mathbf{M}_{3} & \mathbf{N}_{1}\mathbf{A} & \mathbf{N}_{1}\mathbf{B}_{1} & \mathbf{N}_{1}\mathbf{B}_{2} & (\mathbf{N}_{1} + \mathbf{N}_{2})A & (\mathbf{N}_{1} + \mathbf{N}_{2})\mathbf{B}_{2} \\ * & -\tau_{\max}^{-1}\lambda_{1}\mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & -\tau_{\max}^{-1}\lambda_{3}\mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\tau_{\max}^{-1}\lambda_{2}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & -\rho_{\max}^{-1}\lambda_{1}\mathbf{P} & \mathbf{0} \\ * & * & * & * & * & -\rho_{\max}^{-1}\lambda_{2}\mathbf{I} \end{bmatrix} < 0$$
(27b)

where  $\bar{\mathbf{F}}:=[\mathbf{A} \ \mathbf{B}_1 \ \mathbf{0} \ \mathbf{B}_2], \beta_1:=(\lambda_1+\lambda_2 g_w^{-1})\tau_{\max}p+\lambda_4+\alpha,$  $\beta_2:=\lambda_1 p+\lambda_2 g_w^{-1}p,$ 

$$\mathbf{M}_{1} := \begin{bmatrix} \mathbf{P} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{\bar{F}} + \mathbf{\bar{F}}^{\mathrm{T}} \begin{bmatrix} \mathbf{P} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}^{\mathrm{T}} - \mathbf{N}_{1} [\mathbf{I} - \mathbf{I} \mathbf{0} \mathbf{0}]$$
$$- [\mathbf{I} - \mathbf{I} \mathbf{0} \mathbf{0}]^{\mathrm{T}} \mathbf{N}_{1}^{\mathrm{T}} - \mathbf{N}_{2} [\mathbf{I} \mathbf{0} - \mathbf{I} \mathbf{0}]$$
$$- [\mathbf{I} \mathbf{0} - \mathbf{I} \mathbf{0}]^{\mathrm{T}} \mathbf{N}_{2}^{\mathrm{T}} - \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{X} [\mathbf{I} \mathbf{0} - \mathbf{I} \mathbf{0}]$$
$$- \lambda_{4} g_{w} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} [\mathbf{0} \mathbf{0} \mathbf{0} \mathbf{I}] + \beta_{1} \begin{bmatrix} \mathbf{P} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} [\mathbf{I} \mathbf{0} \mathbf{0} \mathbf{0}]$$
$$+ \lambda_{3} p \tau_{\max} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{P} \\ \mathbf{0} \end{bmatrix} [\mathbf{0} \mathbf{0} \mathbf{I} \mathbf{0}]$$

$$\mathbf{M}_{2} := \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{X} \mathbf{\bar{F}} + \mathbf{\bar{F}}^{\mathrm{T}} \mathbf{X} \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix}^{\mathrm{T}}$$
$$+ (\beta_{1} + \beta_{2} \rho_{\max}) \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{X} \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix}^{\mathrm{T}}$$
$$\mathbf{M}_{3} := \beta_{2} \begin{bmatrix} \mathbf{P} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} [\mathbf{I} \ \mathbf{0} \ \mathbf{0} \ \mathbf{0}] + (\mathbf{N}_{1} + \mathbf{N}_{2}) \mathbf{B}_{1} [\mathbf{0} \ \mathbf{I} \ \mathbf{0} \ \mathbf{0}]$$
$$+ \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{B}_{1}^{\mathrm{T}} (\mathbf{N}_{1} + \mathbf{N}_{2})^{\mathrm{T}}$$
(28)

Then, the system (8) is uniformly ISS over the class  ${\mathscr S}$  of sampling-delay sequences with respect to the

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time-varying input  $\mathbf{w}(t)$ , i.e. inequality (9) is satisfied with the functions  $\beta$ ,  $\gamma$  defined in (16) with

$$g_{2} := \sqrt{\frac{g_{w}}{\lambda_{\min}(\mathbf{P})}}$$

$$g_{1}(t) := \sqrt{\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}} p^{\lfloor (\alpha(t+t_{d}))/(\alpha t_{d} - \log p) \rfloor}$$
(29)

Proof

For the proof, see Appendix A.3.

#### Remark 4

The conditions in Theorem 3 depend on  $\tau_{\text{max}}$  and  $\rho_{\text{max}}$ which is the maximum of the input update interval. With regard to the fact that  $\rho_{\text{max}} \leq \tau_{\text{max}} + h_{\text{max}}$ , we can replace  $\rho_{\text{max}}$  by  $\tau_{\text{max}} + h_{\text{max}}$  and formulate the conditions in Theorem 3 in terms of  $\tau_{\text{max}}$  and  $h_{\text{max}}$ . Generally, in NCSs there will be no relation between the size of the delays and the sampling intervals and in the application of Theorem 3 we will use  $\rho_{\text{max}} = \tau_{\text{max}} + h_{\text{max}}$ .

We note that the conditions in Theorem 3 do not explicitly depend on the values of  $h_{\min}$  and  $\tau_{\min}$ . Consequently, this approach towards modelling NCSs may result in more conservative conditions in comparison with those obtained using the discrete-time approach when  $0 \ll h_{\min} \simeq h_{\max}$  or  $0 \ll \tau_{\min} \simeq \tau_{\max}$ . These topics will be studied in the examples presented in Section 5.

#### 4. TRACKING CONTROL PERFORMANCE

The results in Theorems 1 and 3 on the ISS property of sampled-data systems can directly be used in the scope of the tracking problem of NCSs with variable sampling intervals and delays as stated in Section 2. Namely, when applied to the tracking error dynamics (7) the satisfaction of the conditions of either of the results guarantees that the approximate tracking problem is solved and an ultimate bound on the tracking error can be provided. In Section 4.1, we provide a bound on  $|\Delta \mathbf{u}^{\text{ff}}(t)|$ , i.e.  $\sup_{t \in \mathbb{R}} |\Delta \mathbf{u}^{\text{ff}}(t)|$ , based on the properties of the exact feedforward  $\Delta \mathbf{u}_{e}^{\text{ff}}(t)$ ,  $\tau_{\min}$ ,  $\tau_{\max}$ ,  $h_{\min}$  and  $h_{\max}$ . Using this knowledge, we can use the results of

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the previous section to solve the approximate tracking problem and to explicitly construct the bound on the tracking error in Section 4.2.

Moreover, if the feedforward error is zero (i.e.  $\Delta \mathbf{u}^{\text{ff}}(t) = \mathbf{0}$ ,  $\forall t$ ), then the equilibrium point  $\mathbf{e} = \mathbf{0}$  of the tracking error dynamics (7) is globally uniformly asymptotically stable. Such case could be encountered when the feedforward would not be implemented digitally, but would be implemented in an analogue fashion directly at the actuator (hence, avoiding the feedforward errors due to the ZOH and the network delays). Clearly, the latter case is very uncommon and therefore we will propose ways to improve the steady-state performance by decreasing the feedforward error in Sections 4.1 and 4.3.

#### 4.1. Preview feedforward

In Section 3, we have shown that the ultimate bound on the tracking error depends linearly on the bound on the feedforward error. We propose a strategy to decrease the feedforward error caused by the network delays and the ZOH. This strategy involves the exploitation of the preview of the desired trajectory (and therefore of the exact feedforward). In other words, instead of implementing the control law (3) we propose to implement the control law (4), where  $T_p$  is the so-called preview time. Note that in many practical situations it is reasonable to assume that we can preview the desired trajectory for some small  $T_p$  and can also compute the exact feedforward using (2) with some preview.

Let us therefore study how the feedforward error depends on the properties of the exact feedforward, the network delays, the sampling intervals and the preview time. We denote each scalar component of the exact feedforward  $\mathbf{u}_{e}^{\text{ff}}(t)$  by  $u_{e,i}^{\text{ff}}(t)$ , i = 1, ..., m. In the time interval  $t \in [t_k, t_{k+1})$ , the delayed ZOH feedforward signal with preview time  $T_p$  is given by  $\mathbf{u}^{\text{ff}}(t) = \mathbf{u}_{e}^{\text{ff}}(s_k + T_p)$ . Consequently, each component of the feedforward error  $\Delta u_i^{\text{ff}}(t)$  in this time interval satisfies:

$$\Delta u_i^{\rm ff}(t) = u_i^{\rm ff}(t) - u_{e,i}^{\rm ff}(t)$$
  
=  $u_{e,i}^{\rm ff}(s_k + T_p) - u_{e,i}^{\rm ff}(t) \quad \forall t \in [t_k, t_{k+1})$  (30)

i = 1, ..., m. Using the mean value theorem, we can write

$$\begin{aligned} |\Delta u_i^{\rm ff}(t)| &= |u_{e,i}^{\rm ff}(s_k + T_p) - u_{e,i}^{\rm ff}(t)| \\ &\leq \gamma_{1,i} |T_p + s_k - t| \quad \forall t \in [t_k, t_{k+1}) \quad (31) \end{aligned}$$

with  $\gamma_{1,i} = \sup_{t \in \mathbb{R}} |\partial u_{e,i}^{\text{ff}}(t) / \partial t|$ , i = 1, ..., m. Note that such  $\gamma_{1,i}$ , i = 1, ..., m, are well-defined due to the assumption that  $\dot{\mathbf{x}}_d(t)$  is at least  $\mathscr{C}^2$ , guaranteeing that  $\mathbf{u}_e^{\text{ff}}(t)$  is at least  $\mathscr{C}^1$ .

Let us now, for the sake of the transparency of the notation, consider the case of a constant delay  $\tau$  and a constant sampling interval *h* and formulate the optimal preview time (in terms of minimizing the feedforward error) for that case. We will revert to the case of time-varying delays and sampling intervals later. Based on (31), we can then provide the following bound for each component of the feedforward error on  $\mathbb{R}$ :

$$|\Delta u_i^{\text{ff}}(t)| \leq \gamma_{1,i} \max(|T_p - \tau|, |T_p - \tau - h|) \quad \forall t \in \mathbb{R} \quad (32)$$

for i = 1, ..., m. Independent of the particular value of  $i \in \{1, ..., m\}$ , the optimal choice for the preview time  $T_p$  is such that it is the solution of the following minimization problem:  $\min_{T_p} [\max(|T_p - \tau|, |T_p - \tau - h|)]$ . By a straightforward analysis it can be shown that this minimization problem exhibits a global minimum at  $T_p = T_{p_{\min}}$ , with

$$T_{p_{\min}} = \frac{h}{2} + \tau \tag{33}$$

Clearly, it appeals to our intuition that this optimal preview exactly compensates for the 'effective delay'  $(h/2) + \tau$  (of half a sampling interval due to the sample and hold and the network delay) by previewing the feedforward by this amount of time. Resuming, the optimal preview time is given by  $T_{p_{\min}} = (h/2) + \tau$  and each component of the feedforward error  $u_{e,i}^{\text{ff}}(t), i \in \{1, ..., m\}$ , is minimized by using the same preview time.

In the problem setting of this paper, the delays are randomly taken from the set  $\tau \in [\tau_{\min}, \tau_{\max}]$  and the sampling intervals are randomly taken from the set  $h \in [h_{\min}, h_{\max}]$ . For the sake of simplicity, we adopt the assumption that both  $\tau$  and h are uniformly distributed in their respective admissible bounded

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intervals. Clearly, since in every sampling interval we are facing a new unknown delay and sampling interval, we can only hope to choose the preview time optimally in an average sense (with the average sampling interval given by  $(h_{\min}+h_{\max})/2$  and the average delay given by  $(\tau_{\min}+\tau_{\max})/2$  due to the assumption on the uniform distributions). Based on (33), the optimal preview in an average sense (not necessarily in a worst-case sense) would then be given by

$$T_{p_{\min}} = \frac{h_{\min} + h_{\max}}{4} + \frac{\tau_{\min} + \tau_{\max}}{2} \tag{34}$$

Let us now asses the possible impact of the preview time  $T_p$  on the bound on the feedforward error using (32):

$$\begin{aligned} |\Delta u_i^{\Pi}(t)| &\leq \max_{\substack{\tau \in \{\tau_{\min}, \tau_{\max}\}, \\ h \in \{h_{\min}, h_{\max}\}}} [\gamma_{1,i} \max(|T_p - \tau|, \\ |T_p - \tau - h|)] \\ &=: R_i, \quad i \in \{1, \dots, m\} \end{aligned}$$
(35)  
$$\Rightarrow |\Delta \mathbf{u}^{\text{ff}}(t)| \leq \sqrt{\sum_{i=1}^m R_i^2} =: R$$

 $\forall t \in \mathbb{R}$ . Equation (35) can be used to derive the bound on the feedforward error for the case of no preview  $(T_p = 0)$ , which we will denote by  $R_0$ , and the case of the optimal preview time  $T_{p_{\min}}$ , which we will denote by  $R_{\min}$ :

$$R_{0} = \sqrt{\sum_{i=1}^{m} \gamma_{1,i}^{2}} (\tau_{\max} + h_{\max})$$

$$R_{\min} = \sqrt{\sum_{i=1}^{m} \gamma_{1,i}^{2}} \max_{\tau \in \{\tau_{\min}, \tau_{\max}\}} [\max(|T_{p_{\min}} - \tau|, || T_{p_{\min}} - \tau - h_{\max}|)]$$
(36)

with  $T_{p_{min}}$  as in (34). Note that in traditional control systems (as opposed to NCSs), the delay is absent (or constant) and the sampling interval is constant and small (i.e. the sample frequency is typically small with respect to the frequencies to be tracked), which guarantees a small feedforward error and high tracking performance. In NCSs, however, the delays and the sampling intervals are typically randomly varying

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and typically not small; therefore, the preview of the feedforward may significantly improve tracking performance as illustrated by the examples presented in Section 5.

#### 4.2. Solution to the approximate tracking problem

Let us now state the following corollary, based on Theorems 1 and 3 and the bound on the feedforward error defined in the previous section, on the steady-state tracking performance achieved by applying the tracking controller (4), the optimal preview time  $T_{p_{\min}}$  as in (34) and  $\mathbf{u}_{e}^{f}(t)$  satisfying (2), to the NCS (1).

Corollary 1 (Tracking for uncertain time-varying network delays and sampling intervals). Consider the NCS (1), with sampling-delay sequences  $\{s_k, \tau_k\} \in \mathcal{S}$  and  $\mathcal{S}$  defined by (10). Moreover, consider the controller (4), a fixed preview time  $T_p$  and  $\mathbf{u}_e^{\mathrm{ff}}(t)$  satisfying (2). If either the LMIs (15) or the matrix inequalities (27)–(28) are feasible, with  $\mathbf{B}_1 = -\mathbf{B}\mathbf{K}$  and  $\mathbf{B}_2 = \mathbf{B}$ , then the tracking error dynamics (7) is uniformly input-to-state stable (ISS) with respect to the feedforward error  $\Delta \mathbf{u}^{\mathrm{ff}}(t)$  over the class  $\mathcal{S}$  of sampling-delay sequences (10). Moreover, the tracking error is globally uniformly ultimately bounded with the asymptotic bound computed from the following methods:

- *Method 1*: if the LMIs (15) are feasible, then the asymptotic bound on the tracking error is given by  $\limsup_{t\geq 0} |\mathbf{e}(t)| \leq g_2 R$ , with  $g_2$  given in (17) and R given in (35);
- *Method 2*: if the matrix inequalities (27)–(28) are feasible, then the asymptotic bound on the tracking error is given by  $\limsup_{t\geq 0} |\mathbf{e}(t)| \leq g_2 R$ , with  $g_2$  given in (29) and *R* given in (35).

For the case of the optimal preview time  $T_{p_{\min}}$  as in (34), the asymptotic bound for the tracking error is given by  $\limsup_{t \ge 0} |\mathbf{e}(t)| \le g_2 R_{\min}$ , with  $R_{\min}$  as in (36).

## 4.3. Extremum seeking control strategy for online performance improvement

In order to achieve the performance as proposed in Corollary 1, we would need to know the optimal

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preview time as in (34); in other words we would need to know the bounds of  $\tau_{\min}$ ,  $\tau_{\max}$ ,  $h_{\min}$  and  $h_{\max}$  of the distributions of the network delay and the sampling interval (and the form of the distributions themselves). Note that in practice  $\tau_{\min}$ ,  $\tau_{\max}$ ,  $h_{\min}$  and  $h_{\max}$  are not known exactly. These quantities may even undergo sudden changes due to, for example, a change in network load.

Consequently, the implementation of the optimal preview is not possible *a priori*. Therefore, in this section we propose an extremum seeking control technique to online estimate the optimal preview, which we now know to exist from the analysis in Section 4.1. Herewith, we show to be able to achieve online tracking performance improvement.

A schematic representation of the tracking control strategy including the extremum seeking control part is depicted in Figure 3. Herein, the feedforward controller generates feedforward signals with a preview  $T_p$ , i.e.  $\mathbf{u}_k^{\text{ff}} = \mathbf{u}_e^{\text{ff}}(s_k + T_p)$ . The preview time  $T_p$  is provided by an extremum seeking algorithm that aims to minimize a performance criterion, based on the tracking error. It is important that the calculation of the performance measure P and the update of  $T_p$ , by the extremum seeking algorithm, operates at a larger time scale than the rest of the control system. More specifically, the performance measure assessment and the extremum seeking algorithm produce an update for the preview time  $T_p$  every M samples. Consequently, the performance measure update  $P_l$ , generated at sampling instant  $s_{lM}$ , is based on the sampled tracking error  $\mathbf{e}_k$  for the samples  $k \in [(l-1)M+1, lM]$ . Here, we adopt the following performance measure:

$$P_{l}(T_{p,l}) = \sum_{i=1}^{n} \frac{\sum_{k=(l-1)M+1}^{lM} |e_{k}[i]|}{\max_{k \in [(l-1)M+1, lM]} |x_{k}^{d}[i]|}, \ l \ge 1 \quad (37)$$

Herein,  $e_k[i]$  and  $x_k^d[i]$  are the *i*th component of the tracking-error vector and the vector of desired states, respectively. In  $P_l$ , we account for the integral absolute value of each component of the tracking error vector over the samples  $k \in [(l-1)M+1, lM]$ . Moreover, each component of the tracking error vector is scaled by the maximum absolute value of the corresponding desired state in the same interval. The latter scaling is incorporated to ensure equal importance of the tracking errors

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Figure 3. Schematic overview of the control tracking control strategy including the extremum seeking control to find the optimal preview.

in all the states. Furthermore, note that we explicate the dependency of the performance measure  $P_l$  on the implemented preview time  $T_{p,l}$  that has been used in the time interval  $[s_{(l-1)M}, s_{lM}]$ . Of course, this dependency is implicit in the sense that  $P_l$  depends on  $T_{p,l}$  due to the fact that  $T_{p,l}$  affects the feedforward error and thereby the tracking error.

Note that in order for (37) to define a mapping between  $P_l$  and  $T_{p,l}$ , the time interval  $[s_{(l-1)M}, s_{lM}]$ should be infinitely long to ensure that all transients have vanished and therefore  $P_l$  indeed reflects the steady-state tracking error for the preview  $T_{p,l}$ . However, in practice we clearly have to consider finite update intervals, where these are chosen such that transient effects have sufficiently vanished. However, in that case the dynamics of the extremum seeking algorithm may interact with the dynamics of the closed-loop (in fact the extremum seeking algorithm is part of the closed-loop, see Figure 3). For an indepth analysis of the conditions under which it can be shown that such strategy still converges to the minimum while using finite time-intervals to evaluate the objective function (in our case  $P_l$ ), see [22]. Here, we refrain from such rigorous analysis and choose the time-intervals large compared with the settling time of the closed-loop system to ensure a separation of the time-scales of the closed-loop system and the extremum seeking algorithm thereby ensuring a decoupling of the two. However, we can guarantee stability of the entire closed-loop system by the following reasoning. First, the extremum seeking algorithm will only take preview times from a bounded set, typically  $T_p \in [0, h_{\text{max}}/2 + \tau_{\text{max}}]$ . Second, for every bounded

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preview time  $T_p$  we can guarantee the boundedness of the feedforward error, see (32). Finally, the ISS property of the closed-loop system with respect to the feedforward error, as guaranteed by the design of the feedback gain **K**, implies that the tracking error will be bounded for any bounded feedforward error.

In the extremum seeking control algorithm, the preview time  $T_{p,l}$  is updated at time  $s_{lM}$ . Hence, every M samples an update for  $T_{p,l}$  is produced, based on an evaluation of the performance objective function  $P_l(T_{p,l} + \Delta T_{p,l})$ . Herein,  $\Delta T_{p,l}$  is a random perturbation on  $T_{p,l}$ . This perturbation is effectuated to seek for 'better' preview times or, ultimately, to seek for the optimum preview time minimizing the performance objective function  $P_l(T_{p,l})$ . Here, we propose the following nonlinear programming algorithm that aims to minimize the performance object function  $P_l$  by changing  $T_{p,l}$ :

$$T_{p,l+1} = \begin{cases} T_{p,l} & \text{if } P_l(T_{p,l} + \Delta T_{p,l}) \\ \geqslant P_{\text{best},l} \\ T_{p,l} + \Delta T_{p,l} & \text{if } P_l(T_{p,l} + \Delta T_{p,l}) \\ < P_{\text{best},l} \\ \end{cases}$$
(38)  
$$P_{\text{best},l+1} = \begin{cases} \kappa P_{\text{best},l} & \text{if } P_l(T_{p,l} + \Delta T_{p,l}) \\ \geqslant P_{\text{best},l} \\ P_l(T_{p,l} + \Delta T_{p,l}) & \text{if } P_l(T_{p,l} + \Delta T_{p,l}) \\ < P_{\text{best},l} \end{cases}$$

with a constant  $\kappa \ge 1$ . Furthermore, we adopt a simple two-valued distribution for  $\Delta T_{p,l}$ , in other words

 $\Delta T_{p,l} = \pm \Delta T_p$ , with probability 0.5, where  $\Delta T_p$  determines the coarseness of the incremental search-grid in the preview time. For  $\kappa = 1$ ,  $P_{\text{best},l+1}$  is the best performance (i.e. lowest  $P_l$ ) obtained so far:  $P_{\text{best},l+1} =$  $\min_{m \leq l+1} P_m$ . Note that in this case, by construction, this algorithm guarantees that  $P_{\text{best},l}$  is a non-increasing sequence. In other words, the extremum seeking algorithm only updates the preview time if it leads to improved performance. This does not imply that the actual performance is always guaranteed to decrease. Namely, if one or more of the parameters  $h_{\min}$ ,  $h_{\max}$ ,  $\tau_{min}$  or  $\tau_{max}$  change due to changing network conditions (e.g. network load), the performance indicator  $P_l$ can take a different value for the same preview time  $T_{p,l}$ . In order to guarantee high performance in the face of such changing network conditions, we propose the adapted extremum seeking algorithm with  $\kappa > 1$ . Typically,  $\kappa$  is chosen slightly higher than 1. Such choice for  $\kappa$  implements the following idea. Suppose that an adapted preview time  $T_{p,l} + \Delta T_{p,l}$  does not lead to improved performance due to the fact that the network conditions have changed and a sudden decrease in performance occurs as a consequence (i.e. the optimal preview time to be estimated by the extremum seeking technique has changed). Then,  $\kappa > 1$  gives  $P_{\text{best},l+1} > P_{\text{best},l}$ , which ensures that the algorithm gradually gains more freedom in changing the preview time until adaptation of  $T_p$  leads to performance improvement again.

#### 5. ILLUSTRATIVE EXAMPLE

We consider an example of a motion control system from the document printing domain. In general, a paper path, consisting of rollers, driven by motors, is used to move a paper through the printer. Here, the motor controllers share the CPU-time of one processor, which is connected to the motors and sensors via a network resulting in unpredictable time-varying sampling intervals and delays in the control loop.

We limit ourselves to one single motor driving one roller-pair, as depicted in Figure 4. Still, the controller is connected to the motor via the network. In the motorroller model, the motor is assumed to behave ideally



Figure 4. Schematic overview of the motor-roller example.

and slip between the paper and the rollers is neglected, which gives:

$$\ddot{x}_s = \frac{nr_{\rm R}}{J_{\rm M} + n^2 J_{\rm R}} u \tag{39}$$

with  $J_{\rm M} = 1.95 \times 10^{-5} \,\rm kg \,m^2$  the inertia of the motor,  $J_{\rm R} = 6.5 \times 10^{-5} \,\rm kg \,m^2$  the inertia of the roller-pair,  $r_{\rm R} = 14 \times 10^{-3} \,\rm m$  the radius of the roller, n = 0.2 the transmission ratio between motor and upper roller,  $x_s$  the sheet position and u the motor torque.

The continuous-time state-space representation of (39), where the delays are accounted for in the discretetime input  $u_k$  is given by (1), with  $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ , with  $b := nr_{\mathrm{R}}/(J_{\mathrm{M}} + n^2 J_{\mathrm{R}})$ , and  $\mathbf{x}(t) = (x_s(t) \ \dot{x}_s(t))^{\mathrm{T}}$ . We adopt a feedback controller of the form  $u_k = -\mathbf{K}\mathbf{x}_k$ , with  $K = (K_1 \ K_2)$ .

#### 5.1. Tracking error bounds

In this section, we apply the results on ISS of sampleddata systems in Section 3 to upperbound the steadystate tracking error. This information can be used to derive requirements on, e.g. the maximum sampling interval or the maximum delay allowed to guarantee a certain steady-state tracking performance.

Consider the system parameters as introduced before and the following controller parameters:  $K_1 =$ 50 and  $K_2 = 1.18$ . These control gains have been designed to achieve nominal performance of the closed-loop systems in the absence of the network. Moreover, consider an harmonic desired trajectory:  $\mathbf{x}_d(t) = (A_d \sin(\omega t) \ A_d \omega \cos(\omega t))^T$ , with  $A_d = 0.01$ and  $\omega = 2\pi$ . The exact feedforward is given by  $u_e^{\text{ff}}(t) =$  $-(A_d \omega^2/b) \sin(\omega t)$ , with  $b = nr_{\text{R}}/(J_{\text{M}} + n^2 J_{\text{R}})$ .

Let us first consider the case of a constant sampling interval, but with time-varying and uncertain delays



Figure 5. Tracking error bounds for a constant sampling interval  $h=5\times10^{-3}$  s and time-varying and uncertain delays in the set  $[0, \tau_{\text{max}}]$ .

in the set  $[0, \tau_{max}]$ . Figure 5 depicts the error bounds as provided in Corollary 1 for  $\tau_{\text{max}} < h$ , where no feedforward preview is used (the effect of feedforward preview will be studied in Section 5.2). The results for the discrete-time approach are obtained using a finite set of LMIs guaranteeing the satisfaction of (15), based on a (real) Jordan form approach as in [11]. Note that, for the discrete-time approach, also the bound for the tracking error at the sampling times  $s_k$  ( $\bar{g}_2 R_0$ ) is included by means of the dotted line. Figure 5 shows that by using the discrete time approach, ISS can be guaranteed up to  $\tau_{\text{max}} = 0.94$  h, but using the delay impulsive approach ISS can only be guaranteed up to  $\tau_{\text{max}} = 0.33$  h. Hence, the discrete-time approach allows to prove ISS for a larger range of delays. However, the delay impulsive modelling/analysis approach provides much tighter (ISS) bounds on the tracking error (note that the scale of the vertical axis is logarithmic). Note that the overestimation of the bound on the tracking error for the discrete-time modelling approach is significantly worsened due to upperbounding the intersample behaviour (compare the solid and dotted lines in Figure 5).

Next, we consider the case in which the sampling interval is variable, i.e.  $h \in [h_{\min}, h_{\max}]$ , and the delay is zero. Figure 6 depicts the error bounds as provided



Figure 6. Tracking error bounds for variable sampling intervals  $h \in [h_{\min}, h_{\max}]$  and no delays, where  $h_{\max} = 1.5h_{\min}$ .

in Corollary 1. Note that, in this example, we take  $h_{\min} = h_{\max}/1.5$ , hence,  $h_{\min} \neq 0$ . Using the discretetime approach, we can assure ISS almost up to  $h_{\text{max}} =$  $1.34 \times 10^{-2}$  s, which is the sampling interval for which the system with a constant sampling interval (and no delay) becomes unstable (see the dashed vertical line in Figure 6). This fact shows that the proposed ISS conditions are not conservative from a stability perspective. Using the delay impulsive approach, ISS can only be guaranteed up to  $h_{\text{max}} = 9 \times 10^{-3}$  s. However, the delay impulsive approach clearly provides significantly less conservative bounds on the tracking error. Moreover, Figure 6 shows that the bounds on the tracking error increase progressively for increasing  $h_{\text{max}}$  (and  $h_{\text{min}}$ ). This increase is due to, first, the fact that the ISS gain  $g_2$  increases for increasing  $h_{\text{max}}$  and, second, the fact that the bound on the feedforward error  $R_0$  in (36) increases for increasing  $h_{\text{max}}$ . This type of plot is instrumental in determining an upperbound on the maximum sampling interval needed to guarantee a minimum level of steady-state tracking performance.

Comparing the two analysis approaches and related ISS results in this example, we can conclude that the discrete-time modelling approach seems to allow to prove ISS for larger ranges of sampling intervals and delays, whereas the delay impulsive approach clearly provides much tighter (ISS) bounds on the tracking error. The latter observations clearly show that both analysis approaches can be favourable, depending on the purpose of the analysis, and provide complementary analysis tools for the tracking problem for sampleddata systems with time-varying delays and sampling intervals.

#### 5.2. Online performance improvement

In this section, the combined method of feedforward preview and the online extremum seeking technique to estimate the optimal preview time is applied to the motion control example to illustrate its effectiveness in attaining online performance improvement in the face of uncertain network conditions (sampling intervals and delays). Moreover, we will show that if the network conditions change, the extremum seeking technique is able to adapt to such a new situation.

Consider the system and control parameters as introduced before. Again, we consider an harmonic desired trajectory:  $\mathbf{x}_d(t) = (A_d \sin(\omega t) \ A_d \omega \cos(\omega t))^{\mathrm{T}}$ , with  $A_d = 0.01$  and  $\omega = 20\pi$ . The uncertain sampling intervals are varying randomly (with a uniform distribution) in the set  $h_k \in [h_{\min}, h_{\max}]$ ,  $\forall k$ . Moreover, the delays are varying randomly (with a uniform distribution) in the set  $\tau_k \in [\tau_{\min}, \tau_{\max}]$ ,  $\forall k$ . In practice, no exact knowledge on  $h_{\min}, h_{\max}, \tau_{\min}$  and  $\tau_{\max}$  is available, especially if network conditions can suddenly change.

First consider a simulation of the closed-loop system, in which  $h_{\min}=3.50 \times 10^{-3}$  s,  $h_{\max}=4.2 \times 10^{-3}$  s,  $\tau_{\min}=0$  s and  $\tau_{\max}=h_{\min}/2$ . We present the results of a simulation with initial conditions  $\mathbf{x}(0)=[0.02\ 0]^{\mathrm{T}}$  and an initial estimate of the preview time  $T_p=3h_{\max}/2$ (which is the optimal preview time for  $h_k=h_{\max}$ ,  $\tau_k=h_{\max}$ ,  $\forall k$ , see (33) with  $h_{\max}/2=2.1 \times 10^{-3}$  s. Moreover, the parameters of the extremum seeking algorithm are chosen as follows:  $\kappa=1$ , M=20, where the choice for  $\kappa=1$  is made because we consider the case of stationary network conditions. In other words, once the 'optimal' preview time is estimated accurately, there is no need for adaptation of the preview time anymore. In Figure 7, the simulated sequence of sampling intervals  $h_k$  is depicted and, in Figure 8, the

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Figure 7. Sequence of sampling intervals.



Figure 8. Sequence of delays.

simulated sequence of delays  $\tau_k$  is shown. In Figure 9, the position tracking error  $x_s - A_d \sin(\omega t)$  is depicted both for the case without feedforward preview and the case of preview adapted online using the extremum seeking algorithm. In the latter case, after a fast transient the tracking error steadily decreases. This decrease of the tracking is due to a decrease of the



Figure 9. Position tracking error.



Figure 10. Feedforward error.

feedforward error, see Figure 10. The decreasing feedforward error is, in turn, effectuated by an improved estimation of the (average) effective delay  $h/2+\tau$ , i.e. a better estimate of the 'optimal' preview time  $T_p$  is implemented, see Figure 11. Figure 9 clearly shows the performance improvement achieved by the adaptive preview strategy. Finally, Figure 12 depicts the performance indicator evolution  $P_l$ , as defined in (37).

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Figure 11. Sequence of 'effective delays'  $h_k/2 + \tau_k$  and the implemented preview time  $T_p$ .



Figure 12. Evolution of the tracking performance indicator  $P_l$  as in (37).

Indeed, since  $\kappa = 1$ ,  $P_l$  is a non-increasing sequence and the figure reflects that a significant performance improvement is attained by the online adaptation of the preview time.



Figure 13. Position tracking error.



Figure 14. Sequence of 'effective delays'  $h_k/2 + \tau_k$  and the implemented preview time  $T_p$ .

Second, consider a simulation of the closedloop system, in which  $h_{\min} = 3.50 \times 10^{-3}$  s,  $h_{\max} = 5.25 \times 10^{-3}$  s. Moreover, the minimum and maximum delay satisfy  $\tau_{\min} = h_{\min}/2$ ,  $\tau_{\max} = 0.6h_{\min}$  for the first 1000 samples and  $\tau_{\min} = 0.2h_{\min}$ ,  $\tau_{\max} = 0.3h_{\min}$  for the second 1000 samples. The change of the delay distribution reflects a sudden change in network conditions, such as the network load. Moreover, the parameter  $\kappa$  of the extremum seeking algorithm is now chosen as  $\kappa = 1.1$ , reflecting a certain level of freedom for the extremum seeking algorithm to respond to changing network conditions. In this case the initial preview time is taken to be zero. In Figure 13, the position tracking error  $x_s - A_d \sin(\omega t)$  is depicted, while in Figure 14 the 'effective delay' (used to set the preview time  $T_p$ ) is shown. Clearly, in the first part of the simulation the 'effective delay'  $h/2 + \tau$  is estimated well, yielding a good estimate of the optimal preview time and an improved tracking performance. At time t = 4.4 s, the network conditions change resulting in a changed 'effective delay'. Figure 14 shows that, due to the fact that  $\kappa > 1$ , the algorithm adapts and estimates the new 'effective delay'. Again, this results in an improved tracking performance, see Figure 13.

#### 6. CONCLUSIONS

In this paper, a solution to the approximate tracking problem of sampled-data systems with uncertain, timevarying sampling intervals and delays is presented (typically encountered in the domain of networked control systems (NCS)). The uncertain, time-varying sampling intervals and network delays cause inexact feedforward, which induces a perturbation on the tracking error dynamics. A model for the tracking error dynamics in the face of these uncertainties is presented. For this model, sufficient conditions in terms of LMIs for the ISS of the tracking error dynamics with respect to this perturbation are given. Hereto, two NCS analysis approaches are used; first, an approach based on an exact discretization of the plant dynamics and, second, an approach in terms of delay impulsive differential equations. These ISS results provide bounds on the steady-state tracking error as a function of the plant properties, the control design and the network properties. Such error bounds can readily be used to formulate design rules regarding the maximum sampling interval or the maximum delay allowed to guarantee a certain steady-state tracking performance.

Moreover, it is shown that feedforward preview can significantly improve the tracking performance by reducing the feedforward error. An online extremum seeking algorithm from the nonlinear programmingdomain is proposed to online estimate the optimal preview time. The proposed strategy is also shown to be suitable to effectively deal with changing network conditions (such as, e.g. network load), leading to changing stochastic properties of the uncertain sampling intervals and network delays.

The results are illustrated on a mechanical motion control example showing the effectiveness of the proposed strategy and providing insight into the differences and commonalities between the two analysis approaches. More specifically, in the presented motion control example, the discrete-time modelling approach allows to prove ISS (and thus a bounded tracking error) for larger ranges of sampling intervals and delays. On the other hand, the delay impulsive approach provides much tighter (ISS) bounds (and therewith tighter bounds on the tracking error). It can be concluded that the presented approaches provide complementary analysis tools for the tracking problem for sampleddata systems with time-varying delays and sampling intervals.

We foresee that the results in this work may be useful in the context of synchronization of NCSs and the cooperative control of networks of systems communicating over (wireless) networks.

#### APPENDIX A

#### A.1. Proof of Theorem 1

Consider the discrete-time system (12), (13) describing the tracking error dynamics and the candidate Lyapunov function  $V = \boldsymbol{\xi}^{\mathrm{T}} \mathbf{P} \boldsymbol{\xi}$ , with  $\mathbf{P} = \mathbf{P}^{\mathrm{T}} > 0$  satisfying (14). The increment of V along solutions of (12), (13) satisfies

$$\Delta V_{k} = V_{k+1} - V_{k} = \boldsymbol{\xi}_{k+1}^{\mathrm{T}} \mathbf{P} \boldsymbol{\xi}_{k+1} - \boldsymbol{\xi}_{k}^{\mathrm{T}} \mathbf{P} \boldsymbol{\xi}_{k}$$
$$= \boldsymbol{\xi}_{k}^{\mathrm{T}} (\tilde{\mathbf{A}}^{\mathrm{T}}(h_{k}, \tau_{k}) \mathbf{P} \tilde{\mathbf{A}}(h_{k}, \tau_{k}) - \mathbf{P}) \boldsymbol{\xi}_{k}$$
$$+ 2 \boldsymbol{\xi}_{k}^{\mathrm{T}} \tilde{\mathbf{A}}^{\mathrm{T}}(h_{k}, \tau_{k}) \mathbf{P} \tilde{\mathbf{B}} \bar{\mathbf{w}}_{k} + \bar{\mathbf{w}}_{k}^{\mathrm{T}} \tilde{\mathbf{B}}^{\mathrm{T}} \mathbf{P} \tilde{\mathbf{B}} \bar{\mathbf{w}}_{k} \quad (A1)$$

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As the LMIs (15) are feasible, we have that

$$\begin{aligned} \boldsymbol{\xi}_{k}^{\mathrm{T}}(\tilde{\mathbf{A}}^{\mathrm{T}}(h_{k},\tau_{k})\mathbf{P}\tilde{\mathbf{A}}(h_{k},\tau_{k})-\mathbf{P})\boldsymbol{\xi}_{k} \\ +2\boldsymbol{\xi}_{k}^{\mathrm{T}}\tilde{\mathbf{A}}^{\mathrm{T}}(h_{k},\tau_{k})\mathbf{P}\tilde{\mathbf{B}}\bar{\mathbf{w}}_{k}+\bar{\mathbf{w}}_{k}^{\mathrm{T}}\tilde{\mathbf{B}}^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{B}}\bar{\mathbf{w}}_{k} \\ <&-\alpha\boldsymbol{\xi}_{k}^{\mathrm{T}}\mathbf{P}\boldsymbol{\xi}_{k}+c_{4}\bar{\mathbf{w}}_{k}^{\mathrm{T}}\bar{\mathbf{w}}_{k} \end{aligned}$$
(A2)

Combining (A1) and (A2) yields that  $\Delta V_k$  satisfies the following inequality:  $\Delta V_k < -\alpha \boldsymbol{\xi}_k^T \mathbf{P} \boldsymbol{\xi}_k + c_4 \bar{\mathbf{w}}_k^T \bar{\mathbf{w}}_k$ . By denoting  $|\boldsymbol{\xi}_k|_P^2 = \boldsymbol{\xi}_k^T \mathbf{P} \boldsymbol{\xi}_k$ , we can rewrite this inequality as  $|\boldsymbol{\xi}_{k+1}|_P^2 - |\boldsymbol{\xi}_k|_P^2 < -\alpha |\boldsymbol{\xi}_k|_P^2 + c_4 \sup_{1 \le l \le k} |\bar{\mathbf{w}}_l|^2$ . This implies that

$$\begin{aligned} |\boldsymbol{\xi}_{k+1}|_{P}^{2} &< \bar{\alpha} |\boldsymbol{\xi}_{k}|_{P}^{2} + c_{4} \sup_{1 \leq l \leq k} |\bar{\mathbf{w}}_{l}|^{2} \\ &\Rightarrow |\boldsymbol{\xi}_{k}|_{P}^{2} \\ &< \bar{\alpha}^{k} |\boldsymbol{\xi}_{0}|_{P}^{2} + c_{4} D_{k} \sup_{1 \leq l \leq k} |\bar{\mathbf{w}}_{l}|^{2} \\ &\qquad k \geq 1 \end{aligned}$$
(A3)

with  $0 < \bar{\alpha} = 1 - \alpha < 1$  and  $D_k = \sum_{i=1}^k \bar{\alpha}^{i-1}$ . Using (A3), the fact that  $\lambda_{\min}(\mathbf{P}) |\boldsymbol{\xi}_k|^2 \leq |\boldsymbol{\xi}_k|_P^2 \leq \lambda_{\max}(\mathbf{P}) |\boldsymbol{\xi}_k|^2$  and  $|\mathbf{z}_k|^2 \leq ||\mathbf{C}_z \mathbf{P}^{-1/2}||^2 |\boldsymbol{\xi}_k|_P^2$ , we can establish the following inequality on regarding the norms of the tracking error  $\mathbf{z}_k$  at the sampling instants:

$$\frac{|\mathbf{z}_{k}|^{2}}{\|\mathbf{C}_{z}\mathbf{P}^{-1/2}\|^{2}} \leq |\mathbf{\xi}_{k}|_{P}^{2} < \bar{\alpha}^{k} |\mathbf{\xi}_{0}|_{P}^{2} + c_{4}D_{k} \sup_{1 \leq l \leq k} |\bar{\mathbf{w}}_{l}|^{2}$$
$$\leq \bar{\alpha}^{k} \lambda_{\max}(\mathbf{P}) |\mathbf{\xi}_{0}|^{2}$$
$$+ c_{4}D_{k} \sup_{1 \leq l \leq k} |\bar{\mathbf{w}}_{l}|^{2}, \quad k \geq 1$$
(A4)

Now, we use the fact that  $|\boldsymbol{\xi}_0|^2 \!=\! |\boldsymbol{z}_0|^2 \!+\! |\boldsymbol{z}_{-1}|^2$  in (A4) to obtain

$$\frac{|\mathbf{z}_{k}|^{2}}{\|\mathbf{C}_{z}\mathbf{P}^{-1/2}\|^{2}} < \bar{\alpha}^{k}\lambda_{\max}(\mathbf{P})(|\mathbf{z}_{0}|^{2}+|\mathbf{z}_{-1}|^{2})$$

$$+c_{4}D_{k}\sup_{1\leqslant l\leqslant k}|\bar{\mathbf{w}}_{l}|^{2}$$

$$\Rightarrow |\mathbf{z}_{k}| < \|\mathbf{C}_{z}\mathbf{P}^{-1/2}\|$$

$$\times \left(\sqrt{\bar{\alpha}^{k}\lambda_{\max}(\mathbf{P})}\sqrt{|\mathbf{z}_{0}|^{2}+|\mathbf{z}_{-1}|^{2}}\right)$$

$$+\sqrt{c_{4}D_{k}}\sup_{1\leqslant l\leqslant k}|\bar{\mathbf{w}}_{l}|\right), \quad k \ge 1 \quad (A5)$$

Let us now revert to the continuous-time sampled-data system (8) and study its evolution for  $t \in [s_k, s_{k+1}]$ :

$$\mathbf{z}(s_k + \tilde{t}) = \mathbf{e}^{\mathbf{A}\tilde{t}}\mathbf{z}_k + \int_0^{\tilde{t}} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_1 \mathbf{z}_{k-1}$$
$$+ \int_0^{\tilde{t}} \mathbf{e}^{\mathbf{A}s} \mathbf{B}_2 \mathbf{w}(s_k + \tilde{t} - s) \, \mathrm{d}s$$
for  $0 \leq \tilde{t} < \tau_k$ 

$$\mathbf{z}(s_k + \tilde{t}) = \mathbf{e}^{\mathbf{A}\tilde{t}} \mathbf{z}_k + \int_{\tilde{t} - \tau_k}^{\tilde{t}} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_1 \mathbf{z}_{k-1} \qquad (A6)$$
$$+ \int_0^{\tilde{t} - \tau_k} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_1 \mathbf{z}_k$$
$$+ \int_0^{\tilde{t}} \mathbf{e}^{\mathbf{A}s} \mathbf{B}_2 \mathbf{w}(s_k + \tilde{t} - s) \, \mathrm{d}s$$
for  $\tau_k \leqslant \tilde{t} < h_k$ 

Consequently, we establish the following bounds on the tracking error in this time interval:

$$|\mathbf{z}(s_{k}+\tilde{t})| \leq |\mathbf{e}^{\mathbf{A}\tilde{t}}\mathbf{z}_{k}| + \left| \int_{0}^{\tilde{t}} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_{1} \mathbf{z}_{k-1} \right|$$

$$+ \left| \int_{0}^{\tilde{t}} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_{2} \right| \sup_{s_{k} \leq s \leq t_{k}} |\mathbf{w}(s)|$$
for  $0 \leq \tilde{t} < \tau_{k}$ 

$$|\mathbf{z}(s_{k}+\tilde{t})| \leq |\mathbf{e}^{\mathbf{A}\tilde{t}}\mathbf{z}_{k}| + \left| \int_{\tilde{t}-\tau_{k}}^{\tilde{t}} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_{1} \mathbf{z}_{k-1} \right| \quad (A7)$$

$$+ \left| \int_{0}^{\tilde{t}-\tau_{k}} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_{1} \mathbf{z}_{k} \right|$$

$$+ \left| \int_{0}^{\tilde{t}} \mathbf{e}^{\mathbf{A}s} \, \mathrm{d}s \mathbf{B}_{2} \right| \sup_{s_{k} \leq s \leq s_{k+1}} |\mathbf{w}(s)|$$
for  $\tau_{k} \leq \tilde{t} < h_{k}$ 

Using Wazewski's inequalities, see [28, 29]:  $|e^{A\tilde{t}}\mathbf{z}| \leq |\mathbf{z}|e^{\lambda_{\max}\tilde{t}}$ , with  $\lambda_{\max} = \frac{1}{2}\max(\operatorname{eig}(\mathbf{A} + \mathbf{A}^{T}))$ , the terms in

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the above inequality can be upperbounded to obtain:

$$|\mathbf{z}(s_{k}+\tilde{t})| \leq \bar{c}_{1}|\mathbf{z}_{k}|+\bar{c}_{2}|\mathbf{z}_{k-1}|$$

$$+\bar{c}_{3}\sup_{s_{k}\leq s\leq t_{k}}|\mathbf{w}(s)| \quad \text{for } 0\leq\tilde{t}<\tau_{k}$$

$$|\mathbf{z}(s_{k}+\tilde{t})| \leq (\hat{c}_{1}+\hat{c}_{2})|\mathbf{z}_{k}|+\tilde{c}_{2}|\mathbf{z}_{k-1}|$$

$$+\hat{c}_{3}\sup_{s_{k}\leq s\leq s_{k+1}}|\mathbf{w}(s)| \quad \text{for } \tau_{k}\leq\tilde{t}< h_{k}$$
(A8)

with

$$\bar{c}_{1} = \max(1, e^{\lambda_{\max}\tau_{\max}})$$

$$\bar{c}_{2} = \|\mathbf{B}_{1}\| \begin{cases} \frac{e^{\lambda_{\max}\tau_{\max}} - 1}{\lambda_{\max}} & \text{if } \lambda_{\max} \neq 0\\ \tau_{\max} & \text{if } \lambda_{\max} = 0 \end{cases}$$

$$\bar{c}_{3} = \|\mathbf{B}_{2}\| \begin{cases} \frac{e^{\lambda_{\max}\tau_{\max}} - 1}{\lambda_{\max}} & \text{if } \lambda_{\max} \neq 0\\ \tau_{\max} & \text{if } \lambda_{\max} = 0 \end{cases}$$

$$\left\{ \max(e^{\lambda_{\max}h_{\max}}, e^{\lambda_{\max}\tau_{\max}}) & \text{if } \lambda_{\max} = 0 \end{cases}$$

$$\hat{c}_1 = \begin{cases} \max\left(e^{\lambda_{\max}n_{\max}}, e^{\lambda_{\max}\tau_{\min}}\right) & \text{if } \lambda_{\max} \neq 0\\ 1 & \text{if } \lambda_{\max} = 0 \end{cases}$$
(A9)

$$\hat{c}_{2} = \|\mathbf{B}_{1}\| \begin{cases} \frac{e^{\lambda_{\max}(h_{\max} - \tau_{\min})} - 1}{\lambda_{\max}} & \text{if } \lambda_{\max} \neq 0\\ h_{\max} - \tau_{\min} & \text{if } \lambda_{\max} = 0 \end{cases}$$

$$\tilde{c}_{2} = \|\mathbf{B}_{1}\| \begin{cases} \frac{e^{\lambda_{\max}h_{\max}} - 1}{\lambda_{\max}} & \text{if } \lambda_{\max} \neq 0\\ \tau_{\max} & \text{if } \lambda_{\max} = 0 \end{cases}$$
$$\hat{c}_{3} = \|\mathbf{B}_{2}\| \begin{cases} \frac{e^{\lambda_{\max}h_{\max}} - 1}{\lambda_{\max}} & \text{if } \lambda_{\max} \neq 0\\ h_{\max} & \text{if } \lambda_{\max} = 0 \end{cases}$$

Consequently, the inequality in (A7) can be replaced by

$$|\mathbf{z}(s_k + \tilde{t})| \leq c_1 |\mathbf{z}_k| + c_2 |\mathbf{z}_{k-1}|$$
  
+ $c_3 \sup_{s_k \leq s \leq s_{k+1}} |\mathbf{w}(s)| \text{ for } 0 \leq \tilde{t} < h_k \quad (A10)$ 

with  $c_1$ ,  $c_2$  and  $c_3$  defined by

$$c_{1} = \max(\bar{c}_{1}, \hat{c}_{1} + \hat{c}_{2})$$

$$c_{2} = \max(\bar{c}_{2}, \tilde{c}_{2})$$

$$c_{3} = \max(\bar{c}_{3}, \hat{c}_{3}) = \hat{c}_{3}$$
(A11)

We will exploit (A5) in (A10); however in order to do so, we first formulate the following upperbound on  $\sup_{1 \le l \le k} |\bar{\mathbf{w}}_l|$ :

$$\sup_{1 \leq l \leq k} |\bar{\mathbf{w}}_{l}| = \sup_{1 \leq l \leq k} \left| \int_{0}^{h_{l}} e^{\mathbf{A}s} \mathbf{B}_{2} \mathbf{w}(h_{l} + s_{l} - s) \, ds \right|$$

$$\leq \sup_{1 \leq l \leq k} \int_{0}^{h_{l}} \|e^{\mathbf{A}s}\| \|\mathbf{B}_{2}\| \|\mathbf{w}(h_{l} + s_{l} - s)\| \, ds$$

$$\leq \|\mathbf{B}_{2}\| \sup_{0 \leq s \leq t} |\mathbf{w}(s)|$$

$$\times \begin{cases} \frac{e^{\lambda_{\max}h_{\max}} - 1}{\lambda_{\max}} & \text{if } \lambda_{\max} \neq 0\\ h_{\max} & \text{if } \lambda_{\max} = 0 \end{cases}$$

$$= c_{3} \sup_{0 \leq s \leq s_{k+1}} |\mathbf{w}(s)| & \text{for } k \geq 1 \quad (A12) \end{cases}$$

with  $c_3$  defined in (A9), (A11). Now, using (A5), (A12) and the fact that  $\sqrt{|\mathbf{z}_0|^2 + |\mathbf{z}_{-1}|^2} = |\bar{\mathbf{z}}(0)|$  in (A10) yields

• for k = 0:

$$|\mathbf{z}(s_k+\tilde{t})| \leq (c_1+c_2)|\bar{\mathbf{z}}(0)| + c_3 \sup_{0 \leq s \leq s_{k+1}} |\mathbf{w}(s)|$$

• for k = 1:

$$|\mathbf{z}(s_k + \tilde{t})| \leq (c_1 \|\mathbf{C}_z \mathbf{P}^{-\frac{1}{2}} \| \sqrt{\bar{\alpha} \lambda_{\max}(\mathbf{P})} + c_2) |\bar{\mathbf{z}}(0)|$$
$$+ c_3 (1 + c_1 \sqrt{c_4} \|\mathbf{C}_z \mathbf{P}^{-\frac{1}{2}}\|)$$
$$\times \sup_{0 \leq s \leq s_{k+1}} |\mathbf{w}(s)|$$

• for  $k \ge 2$ :

$$|\mathbf{z}(s_k + \tilde{t})| \leq \|\mathbf{C}_z \mathbf{P}^{-1/2}\| \left( c_1 \sqrt{\bar{\alpha}^k \lambda_{\max}(\mathbf{P})} + c_2 \sqrt{\bar{\alpha}^{k-1} \lambda_{\max}(\mathbf{P})} \right) |\bar{\mathbf{z}}(0)| + c_3 \left( 1 + (c_1 \sqrt{c_4 D_k}) \right) |\mathbf{z}(0)|$$

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$$+c_2\sqrt{c_4D_{k-1}}\|\mathbf{C}_{z}\mathbf{P}^{-1/2}\|\right)$$
$$\times \sup_{0\leqslant s\leqslant s_{k+1}}|\mathbf{w}(s)|$$

for  $0 \le \tilde{t} < h_k$ . Now, note that  $D_k$  is a strictly increasing (since  $0 < \bar{\alpha} < 1$ ) geometric series which exhibits a limit for  $k \to \infty$ :  $\lim_{k\to\infty} D_k = \lim_{k\to\infty} \sum_{i=1}^{\infty} \bar{\alpha}^{i-1} = 1/(1 - \bar{\alpha}) = 1/\alpha$ . Concluding, we can show that the continuous-time tracking error dynamics is ISS with respect to the time-varying input  $\mathbf{w}(t)$ , since

$$|\mathbf{z}(t)| \leq g_1(t) |\bar{\mathbf{z}}(0)| + g_2 \sup_{0 \leq s \leq t} |\mathbf{w}(s)| \quad \text{for } t \geq 0$$
 (A13)

with  $g_1(t)$  a decreasing function of t according to

$$g_{1}(t) = \begin{cases} \max(g_{1,0}, g_{1,1}, g_{1,2}) \\ \text{for } t \in [0, s_{2}) \\ g_{1,k} \\ \text{for } k \ge 2, \ t \in [s_{k}, s_{k+1}) \end{cases}$$
(A14)

with

$$g_{1,0} = (c_1 + c_2)$$

$$g_{1,1} = (c_1 \| \mathbf{C}_z \mathbf{P}^{-1/2} \| \sqrt{\bar{\alpha} \lambda_{\max}(\mathbf{P})} + c_2)$$

$$g_{1,k} = \| \mathbf{C}_z \mathbf{P}^{-1/2} \| \left( c_1 \sqrt{\bar{\alpha}^k \lambda_{\max}(\mathbf{P})} + c_2 \sqrt{\bar{\alpha}^{k-1} \lambda_{\max}(\mathbf{P})} \right), \quad k \ge 2$$
(A15)

Note that  $g_1(t)$  is a decreasing function, with  $\lim_{t\to\infty} g_1(t) = 0$ , because  $g_{1,k}$ ,  $k \ge 2$ , is a strictly decreasing sequence, with  $\lim_{k\to\infty} g_{1,k} = 0$ . Moreover,  $g_2$  in (A13) is given by

$$g_2 = c_3 \left( 1 + (c_1 + c_2) \| \mathbf{C}_z \mathbf{P}^{-1/2} \| \sqrt{\frac{c_4}{\alpha}} \right)$$
(A16)

#### A.2. Proof of Theorem 2

The proof of this theorem closely follows the proof of Theorem 1 of [26]. Moreover, we define  $\|\mathbf{x}_m\|_{t_0} := \sup_{s \ge t_0} |\mathbf{x}_m(s)| = \sup_{s \ge t_0 - t_d} |\mathbf{x}(s)|$ . From the definitions

we have

$$V_m(t) \leq \max\{|V_m(t_0)|\phi(t-t_0), \|V\|_{t_0}\}$$

for 
$$t \ge t_0$$
 (A17)

where  $\phi(s) := 0.5(1 - \text{sgn}(s - t_d))$  and from Lemma 1 and Equation (22), we conclude

$$V(t) \leq \max\{V(t_0)e^{-\alpha_3(t-t_0)}, \\ \gamma_v(\|V_m\|_{t_0}), \gamma_w(\|\mathbf{w}\|_{t_0})\}$$
(A18)

for  $t \ge t_0$ . From (A17), we have

$$\|V_m\|_{t_0} \leq \max\{|V_m(t_0)|\phi(t-t_0), \|V\|_{t_0}\}$$
(A19)

and from (A18) that

$$\|V\|_{t_0} \leq \max\{V(t_0), \gamma_v(\|V_m\|_{t_0}), \gamma_w(\|\mathbf{w}\|_{t_0})\}$$
(A20)

By combining (A19) and (A20) we have

$$\|V_{m}\|_{t_{0}} \leq \max\{\|V_{m}(t_{0})\|\phi(t-t_{0}), V(t_{0}),$$
  

$$\gamma_{v}(\|V_{m}\|_{t_{0}}), \gamma_{w}(\|\mathbf{w}\|_{t_{0}})\}$$
  

$$\leq \max\{\|V_{m}(t_{0})\|\phi(t-t_{0}), V(t_{0}), \gamma_{w}(\|\mathbf{w}\|_{t_{0}})\}$$
  

$$\leq \max\{\|V_{m}(t_{0})\|, \gamma_{w}(\|\mathbf{w}\|_{t_{0}})\}$$

in which we used the fact that for all  $a, b \ge 0$ , if  $a \le \max\{b, \gamma_v(a)\}$  then  $a \le b$  given that  $\gamma_v(a) < a$ . Then, we conclude boundedness of the solution,  $\|\mathbf{x}_m\|_{t_0} \le \max\{\alpha_1^{-1}(\alpha_2(|\mathbf{x}_m(t_0)|)), \alpha_1^{-1}(\gamma_w(\|\mathbf{w}\|_{t_0}))\}$ . For the proof of convergence, we choose T such that  $\gamma_v(s) \le s e^{-\alpha_3 T}$  for  $\forall s \le V_m(t_0)$ , and from (A18) we have

$$\|V_m\|_{t_0+t_d+T} \leq \|V\|_{t_0+T}$$
  
$$\leq \max\{V(t_0)e^{-\alpha_3 T},$$
  
$$\gamma_{\upsilon}(\|V_m\|_{t_0}), \gamma_{\upsilon}(\|\mathbf{w}\|_{t_0})\}$$
  
$$\leq \max\{\|V_m\|_{t_0}e^{-\alpha_3 T}, \gamma_{\upsilon}(\|\mathbf{w}\|_{t_0})\}$$

 $||V_m||_{t_0+2(t_d+T)} \leq ||V||_{t_0+2T+t_d}$ 

$$\leq \max\{\|V_m\|_{t_0+T+t_d} e^{-\alpha_3 T}, \gamma_w(\|\mathbf{w}\|_{t_0})\}$$
$$\leq \max\{\|V_m\|_{t_0} e^{-\alpha_3 2T}, \gamma_w(\|\mathbf{w}\|_{t_0})\}$$

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and following the same steps, we conclude that, for any  $n \in \mathbb{N}$ ,  $||V_m||_{t_0+n(t_d+T)} \leq \max\{||V_m||_{t_0}e^{-\alpha_3 nT}, \gamma_w(||\mathbf{w}||_{t_0})\}$ , and from (A19),

$$\|V_m\|_{t_0+n(t_d+T)} \leq \max\{V_m(t_0)e^{-\alpha_3 nT}, \gamma_w(\|\mathbf{w}\|_{t_0})\}$$
(A21)

Let us now use the following facts:  $\alpha_1(\|\mathbf{x}_m\|_{t_0+n(T+t_d)}) \leq \|V_m\|_{t_0+n(T+t_d)}, V_m(t_0) \leq \alpha_2(|\mathbf{x}_m(t_0)|)$ , and, for  $t_0+nT+(n-1)t_d \leq t \leq t_0 + (n+1)T + nt_d$ ,  $|\mathbf{x}(t)| \leq \|\mathbf{x}_m\|_{t_0+n(T+t_d)}$ . Using these facts and (A21) we can show that  $|\mathbf{x}(t)| \leq \|\mathbf{x}_m\|_{t_0+n(T+t_d)} \leq \max\{\alpha_1^{-1}(e^{-\alpha_3 n T} \alpha_2(|\mathbf{x}_m(t_0)|)), \alpha_1^{-1}(\gamma_w(|\mathbf{w}|))\} \leq \alpha_1^{-1}(e^{-\alpha_3 n T} \alpha_2(|\mathbf{x}_m(t_0)|)) + \alpha_1^{-1}(\gamma_w(|\mathbf{w}|))\}$  for  $t_0+nT+(n-1)t_d \leq t \leq t_0+(n+1)T+nt_d$  and based on it, we can find  $\beta(s, t)$  and  $\gamma(s)$  given in Theorem 2.

#### A.3. Proof of Theorem 3

Along the trajectories of the system (25), the timederivative of the Lyapunov candidate function V(t), as in (26), satisfies  $dV(t)/dt = 2\mathbf{z}^{T}\mathbf{P}\dot{\mathbf{z}} - (\mathbf{z} - \mathbf{v}_{2})^{T}\mathbf{X}(\mathbf{z} - \mathbf{v}_{2}) + 2(\rho_{\max} - \rho)(\mathbf{z} - \mathbf{v}_{2})^{T}\mathbf{X}\dot{\mathbf{z}}$ . To satisfy (22) we require that  $dV(t)/dt \le -\alpha |V(t)|^2$  for some  $\alpha > 0$  when  $V(t) \ge \gamma_{v}(V_{m}(t))$ , and  $V(t) \ge \gamma_{w}(|\mathbf{w}(t)|)$  with

$$\gamma_v(s) := ps, \quad \gamma_w(s) := g_w s^2 \tag{A22}$$

where  $0 and <math>g_w > 0$ . We define  $\rho_2(t) := t - s_k$ ,  $t \in [t_k, t_{k+1})$  and  $\overline{\zeta} := [\mathbf{z}^T \mathbf{v}_1^T \mathbf{v}_2^T \mathbf{w}^T]^T$ . Then for any matrix  $\mathbf{N}_1$ ,  $\mathbf{N}_2$  we have

$$2\bar{\boldsymbol{\zeta}}^{\mathrm{T}} \mathbf{N}_{1}(\mathbf{z}-\mathbf{v}_{1})+2\bar{\boldsymbol{\zeta}}^{\mathrm{T}} \mathbf{N}_{2}(\mathbf{z}-\mathbf{v}_{2})$$

$$=2\bar{\boldsymbol{\zeta}}^{\mathrm{T}} (\mathbf{N}_{1}+\mathbf{N}_{2}) \int_{t-\rho}^{t} \dot{\boldsymbol{z}}(s) \, \mathrm{d}s$$

$$+2\bar{\boldsymbol{\zeta}}^{\mathrm{T}} \mathbf{N}_{1} \int_{t-\rho_{2}}^{t-\rho} \dot{\boldsymbol{z}}(s) \, \mathrm{d}s$$

$$=2\bar{\boldsymbol{\zeta}}^{\mathrm{T}} (\mathbf{N}_{1}+\mathbf{N}_{2}) \int_{t-\rho}^{t} (\mathbf{A}\mathbf{z}(s)+\mathbf{B}_{1}\mathbf{v}_{1}(s)$$

$$+\mathbf{B}_{2}\mathbf{w}(s)) \, \mathrm{d}s$$

$$+2\bar{\boldsymbol{\zeta}}^{\mathrm{T}} \mathbf{N}_{1} \int_{t-\rho_{2}}^{t-\rho} (\mathbf{A}\mathbf{z}(s)+\mathbf{B}_{1}\mathbf{v}_{1}(s)$$

$$+\mathbf{B}_{2}\mathbf{w}(s)) \, \mathrm{d}s \qquad (A23)$$

Moreover, using the fact that  $2\mathbf{x}^{\mathrm{T}}\mathbf{y} \leq \mathbf{x}^{\mathrm{T}}\mathbf{G}\mathbf{x} + \mathbf{y}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{y}$ , for any positive definite matrix  $\mathbf{G} = \mathbf{G}^{\mathrm{T}} > 0$ , the following inequalities hold:

$$\begin{aligned} 2\bar{\boldsymbol{\zeta}}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2}) \int_{t-\rho}^{t} (\mathbf{A}\mathbf{z}(s)+\mathbf{B}_{1}\mathbf{v}_{1}(s)+\mathbf{B}_{2}\mathbf{w}(s)) \, \mathrm{d}s \\ &\leqslant \lambda_{1}^{-1}\rho\bar{\boldsymbol{\zeta}}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})\mathbf{A}\mathbf{P}^{-1}\mathbf{A}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})^{\mathrm{T}}\bar{\boldsymbol{\zeta}} \\ &+\lambda_{1}\int_{t-\rho}^{t}\mathbf{z}(s)^{\mathrm{T}}\mathbf{P}\mathbf{z}(s) \, \mathrm{d}s \\ &+2\rho\bar{\boldsymbol{\zeta}}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})\mathbf{B}_{1}\mathbf{v}_{1} \\ &+\lambda_{2}^{-1}\rho\bar{\boldsymbol{\zeta}}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})\mathbf{B}_{2}\mathbf{B}_{2}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})^{\mathrm{T}}\bar{\boldsymbol{\zeta}} \\ &+\lambda_{2}\int_{t-\rho}^{t}\mathbf{w}^{\mathrm{T}}(s)\mathbf{w}(s) \, \mathrm{d}s \end{aligned}$$

$$\begin{aligned} 2\bar{\boldsymbol{\zeta}}^{\mathrm{T}}\mathbf{N}_{1}\int_{t-\rho_{2}}^{t-\rho}(\mathbf{A}\mathbf{z}(s)+\mathbf{B}_{1}\mathbf{v}_{1}(s)+\mathbf{B}_{2}\mathbf{w}(s)) \, \mathrm{d}s \\ &\leqslant \lambda_{1}^{-1}(\rho_{2}-\rho)\bar{\boldsymbol{\zeta}}^{\mathrm{T}}\mathbf{N}_{1}\mathbf{A}\mathbf{P}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{N}_{1}^{\mathrm{T}}\bar{\boldsymbol{\zeta}} \\ &+\lambda_{1}\int_{t-\rho_{2}}^{t-\rho}\mathbf{z}^{\mathrm{T}}(s)\mathbf{P}\mathbf{z}(s) \, \mathrm{d}s \\ &+\lambda_{3}^{-1}(\rho_{2}-\rho)\bar{\boldsymbol{\zeta}}^{\mathrm{T}}\mathbf{N}_{1}\mathbf{B}_{1}\mathbf{P}^{-1}\mathbf{B}_{1}\mathbf{N}_{1}^{\mathrm{T}}\bar{\boldsymbol{\zeta}} \\ &+\lambda_{3}\int_{t-\rho_{2}}^{t-\rho}\mathbf{v}_{1}^{\mathrm{T}}(s)\mathbf{P}\mathbf{v}_{1}(s) \, \mathrm{d}s \\ &+\lambda_{2}^{-1}(\rho_{2}-\rho)\bar{\boldsymbol{\zeta}}^{\mathrm{T}}\mathbf{N}_{1}\mathbf{B}_{2}\mathbf{B}_{2}^{\mathrm{T}}\mathbf{N}_{1}^{\mathrm{T}}\bar{\boldsymbol{\zeta}} \\ &+\lambda_{2}\int_{t-\rho_{2}}^{t-\rho}\mathbf{w}^{\mathrm{T}}(s)\mathbf{w}(s) \, \mathrm{d}s \end{aligned} \tag{A24}$$

for  $\lambda_i > 0, i = 1, 2, 3$ . We require that if

$$V(t) \ge p V_m(t), \quad V(t) \ge g_w |\mathbf{w}(t)|^2$$
 (A25)

then  $dV(t)/dt \le -\alpha |V(t)|^2$ , and consequently the condition (22) holds with  $\gamma_v$ ,  $\gamma_w$  defined in (A22). In other words, we assume  $V(t) \ge pV_m(t)$ , and  $V(t) \ge g_w |\mathbf{w}(t)|^2$  hold and based on these assumptions we would like to find a condition that guarantees that  $dV(t)/dt \le -\alpha |V(t)|^2$ . From the assumption

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 $V(t) \ge pV_m(t)$ , we conclude that  $V(t) \ge pV(s)$  for  $s \in [t - \rho, t]$  and  $V(t - \rho) \ge pV(s)$  for  $s \in [t - \tau_{\max} - \rho_2, t - \rho]$  for  $\forall t \ge 0$ . Moreover,

$$\lambda_1 \int_{t-\rho_2}^t \mathbf{z}^{\mathrm{T}}(s) \mathbf{P} \mathbf{z}(s) \, \mathrm{d} s \leqslant \lambda_1 \int_{t-\rho_2}^t V(s) \, \mathrm{d} s$$
$$\leqslant \lambda_1 \rho_2 p V(t)$$

$$\lambda_{3} \int_{t-\rho_{2}}^{t-\rho} \mathbf{v}_{1}^{\mathrm{T}}(s) \mathbf{P} \mathbf{v}_{1}(s) \,\mathrm{d}s \leqslant \lambda_{3} \int_{t-\rho_{2}}^{t-\rho} V(s) \,\mathrm{d}s$$
$$\leqslant \lambda_{3}(\rho_{2}-\rho) p V(t-\rho)$$
$$= \lambda_{3}(\rho_{2}-\rho) p \mathbf{v}_{2}^{\mathrm{T}} \mathbf{P} \mathbf{v}_{2} \quad (A26)$$

In addition, we have that  $V(t) \ge g_w |\mathbf{w}(t)|^2$  for  $\forall t \ge 0$ , so

$$\lambda_2 \int_{t-\rho_2}^t \mathbf{w}^{\mathrm{T}}(s) \mathbf{w}(s) \,\mathrm{d}s \leq \lambda_2 \int_{t-\rho_2}^t g_w^{-1} V(s) \,\mathrm{d}s$$
$$\leq \lambda_2 \rho_2 g_w^{-1} p V(t) \qquad (A27)$$

We replace (A26) and (A27) in (A24) and then combine with (A23) to conclude  $\Delta_1 \ge 0$  (when (A25) holds) where

$$\Delta_{1} := -2\bar{\boldsymbol{\zeta}}^{\mathrm{T}}\mathbf{N}_{1}(\mathbf{z}-\mathbf{v}_{1}) - 2\bar{\boldsymbol{\zeta}}^{\mathrm{T}}\mathbf{N}_{2}(\mathbf{z}-\mathbf{v}_{2}) +\lambda_{1}^{-1}\rho\bar{\boldsymbol{\zeta}}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})\mathbf{A}\mathbf{P}^{-1}\mathbf{A}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})^{\mathrm{T}}\bar{\boldsymbol{\zeta}} +2\rho\bar{\boldsymbol{\zeta}}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})\mathbf{B}_{1}\mathbf{v}_{1} +\lambda_{2}^{-1}\rho\bar{\boldsymbol{\zeta}}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})\mathbf{B}_{2}\mathbf{B}_{2}^{\mathrm{T}}(\mathbf{N}_{1}+\mathbf{N}_{2})^{\mathrm{T}}\bar{\boldsymbol{\zeta}} +\lambda_{1}^{-1}(\rho_{2}-\rho)\bar{\boldsymbol{\zeta}}^{\mathrm{T}}\mathbf{N}_{1}\mathbf{A}\mathbf{P}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{N}_{1}^{\mathrm{T}}\bar{\boldsymbol{\zeta}} +\lambda_{3}^{-1}(\rho_{2}-\rho)\bar{\boldsymbol{\zeta}}^{\mathrm{T}}\mathbf{N}_{1}\mathbf{B}_{1}\mathbf{P}^{-1}\mathbf{B}_{1}^{\mathrm{T}}\mathbf{N}_{1}^{\mathrm{T}}\bar{\boldsymbol{\zeta}} +\lambda_{2}^{-1}(\rho_{2}-\rho)\bar{\boldsymbol{\zeta}}^{\mathrm{T}}\mathbf{N}_{1}\mathbf{B}_{2}\mathbf{B}_{2}^{\mathrm{T}}\mathbf{N}_{1}^{\mathrm{T}}\bar{\boldsymbol{\zeta}} +(\lambda_{1}\rho_{2}+\lambda_{2}\rho_{2}g_{w}^{-1})pV(\tilde{\boldsymbol{\zeta}}(t),t) +\lambda_{3}(\rho_{2}-\rho)p\mathbf{v}_{2}^{\mathrm{T}}\mathbf{P}\mathbf{v}_{2}$$
(A28)

The condition (22) holds if there exists a  $\lambda_4$ ,  $\alpha \ge 0$  such that

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} + \Delta_1 + \lambda_4 (V(t) - g_w \mathbf{w}^{\mathrm{T}}(t) \mathbf{w}(t))$$
$$\leqslant -\alpha |V(t)|^2 \tag{A29}$$

The term  $\triangle_1$  reduces the conservativeness by exploiting the relationship between **z**, **v**<sub>1</sub> and **v**<sub>2</sub>. The third term is added based on S-procedure [30] and is a crucial element because otherwise the matrix inequalities in Theorem 3 would not be feasible. We replace  $\rho_2 - \rho$  by  $\tau_{\text{max}}$  and  $\rho_2$  by  $\tau_{\text{max}} + \rho$  (note that  $\rho_2 - \rho \leqslant \tau_{\text{max}}$ ) and (A29) holds if

$$\mathbf{M}_1 + (\rho_{\max} - \rho)\mathbf{M}_2 + \rho\mathbf{M}_3 < 0$$
 (A30)

where

$$\begin{split} \bar{\mathbf{M}}_1 &:= \mathbf{M}_1 + \tau_{\max} \lambda_1^{-1} \mathbf{N}_1 \mathbf{A} \mathbf{P}^{-1} \mathbf{A}^{\mathrm{T}} \mathbf{N}_1^{\mathrm{T}} \\ &+ \tau_{\max} \lambda_3^{-1} \mathbf{N}_1 \mathbf{B}_1 \mathbf{P}^{-1} \mathbf{B}_1^{\mathrm{T}} \mathbf{N}_1^{\mathrm{T}} \\ &+ \tau_{\max} \lambda_2^{-1} \mathbf{N}_1 \mathbf{B}_2 \mathbf{B}_2^{\mathrm{T}} \mathbf{N}_1^{\mathrm{T}} \\ \bar{\mathbf{M}}_3 &:= \mathbf{M}_3 + \lambda_1^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{A} \mathbf{P}^{-1} \mathbf{A}^{\mathrm{T}} + (\mathbf{N}_1 + \mathbf{N}_2)^{\mathrm{T}} \\ &+ \lambda_2^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{B}_2 \mathbf{B}_2^{\mathrm{T}} (\mathbf{N}_1 + \mathbf{N}_2)^{\mathrm{T}} \end{split}$$

and  $\mathbf{M}_1$ ,  $\mathbf{M}_2$ ,  $\mathbf{M}_3$  are defined in (28). The condition (A30) is equivalent to (see [31])  $\mathbf{M}_1 + \rho_{\max} \mathbf{M}_2 < 0$ ,  $\mathbf{M}_1 + \rho_{\max} \mathbf{M}_3 < 0$ , which are equivalent to (27a) and (27b) by Schur Lemma. From (A21) we can conclude  $\lambda_{\min}(\mathbf{P})|\mathbf{z}(t)|^2 \leq V(t) \leq |V_m(t)| \leq \max\{\lambda_{\max}(\mathbf{P})|\mathbf{z}_m(t_0)|^2 e^{-\alpha nT}, \gamma_w ||\mathbf{w}||_{t_0}\} \leq \lambda_{\max}(\mathbf{P})|\mathbf{\bar{z}}(t_0)|^2 e^{-\alpha nT} + \gamma_w ||\mathbf{w}||_{t_0}$ , for  $t_0 + nT + (n-1)t_d \leq t \leq t_0 + (n+1)T + nt_d$  where *T* is small enough such that  $p \leq e^{-\alpha T}$ . We pick  $p = e^{-\alpha T}$ and we can show that the system is ISS over class  $\mathscr{S}$ with the functions  $\beta, \gamma$  defined in (16) with  $g_1(t), g_2$ defined in (29).

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