Nonlinear Parametric Identification Using Chaotic Data

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Abstract: The subject of this paper is the development of a nonlinear parametric identification method using chaotic data. In former research, the main problem in using chaotic data in parameter estimation appeared to be the numerical computation of the chaotic trajectories. This computational problem is due to the highly unstable character of the chaotic orbits. The method proposed in this paper is based on assumed physical models and has two important components. First, the chaotic time series is characterized by a "skeleton" of unstable periodic orbits. Second, these unstable periodic orbits are used as the input information for a nonlinear parametric identification method using periodic data. As a consequence, problems concerning the numerical computation of chaotic trajectories are avoided. The identifiability of the system is optimized by using the *structure* of the phase space instead of a single physical trajectory in the estimation process. Furthermore, before starting the estimation process, a huge data reduction has been accomplished by extracting the unstable periodic orbits from the long chaotic time series. The method is validated by application to a parametrically excited pendulum, which is an experimental nonlinear dynamical system in which transient chaos occurs.

Key Words: Chaotic data, identification, nonlinear dynamics

NOTATION

- \dot{z} total derivative of z with respect to time t
- \hat{z} estimate of z
- z_{exp} value of z in the experiment
- \hat{z}_0 initial estimate of z
- *a* parameters
- $A_{\rm exc}$ amplitude of excitation
- A^p Jacobian matrix for a cycle with period p (local linear approximation of the nonlinear dynamics around the periodic point x_0)
- A_t^p Jacobian matrix for a cycle with period p over a single time step t, t + 1
- *F* functional form
- g gravitational acceleration
- k_i friction parameters corresponding to friction effect *i*
- *l* length of the pendulum
- *m* mass of the pendulum
- *p* period of the cycle

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NONLINEAR PARAMETRIC IDENTIFICATION USING CHAOTIC DATA 293

- q(k) degrees of freedom on discrete time instant k
- t time
- T period time of the excitation
- x_0 estimated periodic point (position in phase space)
- x_t^p reference point in the local linear approximation A^p
- y_t neighbors of x_t^p
- ϵ ratio of the lengths of the two arms of the driving crank mechanism
- ε linear box size in periodic orbit analysis
- ε_n measurement noise level
- ϕ angular position
- ω angular frequency of the excitation
- ω_0 angular eigenfrequency of the pendulum

1. INTRODUCTION

For the identification of nonlinear dynamical systems, often transient trajectories or periodic equilibrium trajectories are used as input information (Verbeek, 1993; Verbeek, de Kraker, and van Campen, 1995). The deterministic character of chaos implies the theoretical possibility of using chaotic data in parameter estimation methods. The following describes a nonlinear parametric identification method using chaotic data. The assumption is made that the deterministic mathematical model—in this case a set of differential equations—can be derived using physical laws. As a consequence, the identification goal is reduced to estimating the unknown parameters in the differential equation. Fitting the parameters to a chaotic time series has the advantage that the phase-space is filled better by a chaotic time series than, for example, by just one periodic orbit. The better the data covers the phase space, the smaller the probability that essential parts of the dynamical behavior of the system are missed. This means that the identifiability of the dynamic system gets better and the accuracy of the obtained model will be higher.

Earlier attempts in modeling nonlinear dynamical systems using chaotic data can be roughly divided into two classes:

- 1. The first class consists of the parameter estimation in nonlinear maps (Abarbanel, Brown, Sidorowich, and Tsimring, 1992). It is assumed that information on the temporal evolution of orbits q(k)—that lie on a compact attractor—and their neighborhoods in the phase space is available. This information can be used to estimate parameters a in nonlinear maps: $q \rightarrow F(q, a)$, which evolve each $q(k) \rightarrow q(k + 1)$. After a metric, which measures the quality of the fit, has been defined, the problem is reduced to determining the parameters a in the map given a class of functional forms F (chosen from physical reasoning). The nonlinear maps are discrete and often local models. For an obvious reason, continuous global models, for example differential equations, are preferred above discrete local maps.
- 2. The second class deals with a method that fits ordinary differential equations to chaotic data (Baake, Baake, Bock, and Briggs, 1992). The main problem of this approach is the numerical computation of the chaotic trajectory. To tackle this problem, the time



Figure 1. Estimation strategy in Baake et al. (1992)

interval is split into small subintervals. On each subinterval, an initial value problem is defined. The numerical solution of the corresponding initial value problems, computed by integration, should be spoiled by error-propagation, due to positive Lyapunov exponents. In Baake et al. (1992) it is stated that this error-propagation problem can be adequately tackled by the boundary-value-problem methods for parameter estimation in ordinary differential equations—see Bock and Schlöder (1986) and Bock (1987)—by choosing the subintervals sufficiently small.

In Figure 1, the method is illustrated by means of a flowchart. This flowchart makes clear that the estimation process consists of minimizing the discontinuities in the numerical solution (of the multiple initial value problem) as well as the difference between the measurements and the continuous numerical solution simultaneously.

Some problems that remained are the applicability to real-world systems with dimensions higher than three, the treatment of very long time series, and the applicability to real experimental data.

In the following, a different approach will be presented and discussed. In this approach, chaos is described by a skeleton of unstable periodic orbits (see Van de Water, Hoppenbrouwers, and Christiansen, 1991). These unstable periodic orbits are used as the input information for the estimation of the parameters of the differential equation describing the system. The developed methodology will be illuminated through application to a specific system, that is, a parametrically excited pendulum.

2. CHAOS AND UNSTABLE PERIODIC ORBITS

2.1. Introductory Remarks

In Cvitanović (1988) and Van de Water et al. (1991), it is stated that low-dimensional chaos can be described by its underlying skeleton of unstable periodic orbits. A chaotic

NONLINEAR PARAMETRIC IDENTIFICATION USING CHAOTIC DATA 295

294 N. VAN DE WOUW, G. VERBEEK, and D. H. VAN CAMPEN

time series will contain many points that are close to an unstable periodic orbit. After some time, these points will return close to themselves in the phase space and can be used to estimate the position of the periodic orbit. The evolution of the neighborhoods of the points that almost return can be used to estimate the stable and unstable eigenvalues of the unstable periodic orbits. In systems for which a one- or two-dimensional mapping exists, the skeleton of periodic points can be constructed hierarchically and then provides a complete description of chaos (see Cvitanović, 1988). An obstacle in the application of the periodic orbit analysis is the necessity of finding unstable periodic orbits with arbitrary long periods. With a finite number of data points, this is virtually impossible. However, it has been shown that in many cases long cycles are shadowed by nearby short ones. This shadowing means that the orbit of a period p cycle is very close to orbits of cycles with lower periods. So, by determining only the short cycles in an experiment, the dynamics can be characterized at much longer times. The error made by leaving out the true long cycles is expected to diminish rapidly with the increasing length of included orbits (see Cvitanović, 1988, and Artuso, Aurell, and Cvitanović, 1990).

Summarizing the preceding, it can be said that for deterministic dynamical systems of low dimension and smooth dynamics, the cycles provide a detailed invariant characterization, whose virtues are the following:

- 1. Cycle symbol sequences are topological invariants: They give the spatial layout of a strange set.
- 2. Cycle eigenvalues are metric invariants: They give the scale of each piece of a strange set.
- 3. Cycles are ordered hierarchically: Short cycles give good approximations to a strange set, and the errors due to neglecting long cycles can be bounded.
- 4. Periodic points are skeletal in the sense that even though they are determined at finite times, they remain there forever.
- 5. Cycles are robust: Eigenvalues of short cycles vary slowly with smooth parameter changes.
- 6. Short cycles can be extracted accurately from experimental data.

Two kinds of information concerning the unstable periodic orbits can be distinguished: (a) information on the position of the unstable periodic orbits in the phase space and (b) information on the stability of the unstable periodic orbits. Both could be used as input information for an estimation process. In the research covered by this paper, only the position of the periodic orbits in the phase space was used.

2.2. Parametrically Excited Pendulum

To illustrate the principal line of thought, the periodic orbit analysis from chaotic data will be demonstrated for the parametrically excited pendulum developed by Van de Water et al. (1991). In Figure 2, the pendulum is drawn schematically. The support of the pendulum is driven by a crank mechanism. This allows for a stable and simple construction. An optical encoder with an angular resolution of $\frac{2\pi}{4096}$ rad gives information about the instantaneous angular position ϕ of the pendulum. It is interfaced by a logical circuit to a computer that reads ϕ each time the support is in its highest position. A second reading, 15.08 ms later, is used to obtain the angular velocity coordinate $\dot{\phi}$ of the point (ϕ , $\dot{\phi}$) in the Poincaré



Figure 2. The parametrically driven pendulum drawn schematically.

section; phase space points are thus obtained with time intervals of $\frac{2\pi}{\omega}$ s. The noise ϵ_n in the $(\phi, \dot{\phi})$ measurement was estimated from the accuracy of measured returns in the phase plane when the pendulum was in a state of periodic motion: $\epsilon_n = 5.0 \times 10^{-3}$ (relative to the attractor size).

In the following, a suitable mathematical model for the parametrically excited pendulum, in the form of a differential equation, will be discussed.

Ideally, a pendulum, whose mass is concentrated at its end and whose support is lifted periodically, would be described by the well-known differential equation:

$$\ddot{\phi} + \frac{k_2}{ml^2}\dot{\phi} + \left[\frac{g}{l} - \frac{A_{\rm exc}\omega^2}{l}\cos(\omega t)\right]\sin(\phi) = 0.$$
(1)

However, accurate modeling of the actual experiment introduces extra terms in equation (1) that reflect the presence of other damping forces and allow for the noticed peculiarity of the excitation. Besides the friction force represented by the term proportional to $\dot{\phi}$, air resistance introduces a term proportional to $(\dot{\phi})^2 \operatorname{sgn}(\dot{\phi})$, and the bearings will counteract the motion through a Coulomb friction term that only depends on the sign of $\dot{\phi}$. In summary,

296 N. VAN DE WOUW, G. VERBEEK, and D. H. VAN CAMPEN

the original equation of motion has to be augmented to

$$\ddot{\phi} + \frac{k_1}{ml^2} \operatorname{sgn}(\dot{\phi}) + \frac{k_2}{ml^2} \dot{\phi} + \frac{k_3}{ml^2} (\dot{\phi})^2 \operatorname{sgn}(\dot{\phi}) + P_{\text{exc}} = 0$$
(2)

with

$$P_{\text{exc}} = \left[\frac{g}{l} - \frac{A_{\text{exc}}\omega^2}{l} \left[\cos(\omega t) + \epsilon \frac{\cos(2\omega t) + \epsilon^2 \sin^4(\omega t)}{[1 - \epsilon^2 \sin(\omega t)]^{3/2}}\right]\right]\sin(\phi)$$
(3)

where ϵ ($\epsilon < 1$) is the ratio of the lengths of the two rods that make up the driving crank mechanism. Van de Water et al. (1991) estimated the damping constants k_1, k_2 , and k_3 by measuring ϕ in an unexcited, freely swinging pendulum released at $\phi = \pi$ at t = 0 and adjusting k_1, k_2 , and k_3 to obtain the best agreement between measurement and simulation, with equation (2), as far as the maximum values of the excursion are concerned.

At $\omega = 13 \frac{\text{rad}}{\text{s}}$, where ω is the angular frequency of the excitation, the pendulum is in a rotating motion whose frequency is locked to the excitation. When lowering ω , we encounter a series of period-doubling bifurcations that finally lead to a large chaotic attractor at $\omega = 9.09 \frac{\text{rad}}{\text{s}}$, which extends over the full angular range.

Both experiment and simulation exhibit clear fractal behavior. In Figure 3, a simulated time series (\pm 5000 points) is shown. For a picture of an experimental time series, see Van de Water et al. (1991). The time step between successive points in Figure 3 is $T = \frac{2\pi}{\omega}$ s. The actual simulated time series used in the periodic orbit analysis is much longer: about 10⁶ points. It is impossible to realize such a long time series in an experiment because the chaotic attractor is in fact transient. It engulfs the origin that is stable due to the presence



Figure 3. Poincaré section of a simulated time series.

of the hysteretic friction term $k_1 \operatorname{sgn}(\dot{\phi})$. The effect of this term is that the pendulum, after being in a chaotic state for some time, may come to a standstill. To avoid this and to create the longer time series in the simulation, the simulation was performed with $k_1 = 0.0$ Nm. So, in Van de Water et al. (1991) the simulations are performed with the following parameter values:

$$m = 0.0858 \text{ kg} \qquad l = 0.317 \text{ m}$$

$$k_1 = 0.0 \text{ Nm} \qquad k_2 = 4.85 \times 10^{-4} \text{ Nm s}$$

$$k_3 = 1.90 \times 10^{-4} \text{ Nm s}^2 \qquad \omega_0 = 5.57 \frac{\text{rad}}{\text{s}} \qquad (4)$$

$$\omega = 9.09 \frac{\text{rad}}{\text{s}} \qquad A_{\text{exc}} = 0.131 \text{ m}$$

$$\epsilon = 0.173$$

2.3. Periodic Orbit Analysis

For the parametrically excited pendulum, an attempt is made to characterize the structure of the chaotic attractor by a set of unstable periodic orbits (see Van de Water et al., 1991). These periodic orbits are solutions of the differential equation describing the system. Then the nonlinear parametric identification method can be based on these unstable periodic orbits.

In this subsection we describe the way in which the unstable periodic orbits are found from an experimental chaotic time series, for the example of the parametrically excited pendulum. The phase space is partitioned into small boxes of linear size ε (1% of the attractor size). By sorting the points of the chaotic time series with respect to the location of the box to which they belong, neighboring points can be determined. A point that returns to either its own box or to one of the neighboring boxes is called a recurrent point. The time it takes a recurrent point to return gives the length of the candidate cycles. The list of candidate cycles, obtained in this way, may be further reduced by requiring that a box contains at least a few points that ε -return as a p cycle. These points are used to determine a local linear approximation A^p of the dynamical system. In this approximation, the point x_t^p that returns closest is selected as a reference point. A minimum of 32 neighboring points y_t , which were taken from a 2ε neighborhood of x_t^p , is included in the fit.

An estimate of the periodic point is now found in the following way:

1. A least squares fit is used to estimate the Jacobian matrix A_t^p over a single time step t, t + 1:

$$A_t^p (x_t^p - y_t) = x_{t+1}^p - y_{t+1}.$$
 (5)

2. The cycle Jacobian A^p is then composed out of p single-step Jacobians:

$$A^{p} = A^{p}_{t+p-1}A^{p}_{t+p-2}\cdots A^{p}_{t}.$$
 (6)

3. Let x_0 be the estimated periodic point. Then the elements of x_0 can be estimated from

$$A^{p}(x_{t}^{p}-x_{0}) = (x_{t+p}^{p}-x_{0}).$$
⁽⁷⁾



4. This results in

$$x_0 = x_t^p + (I - A^p)^{-1} \left(x_{t+p}^p - x_t^p \right).$$
(8)

The estimated stable and unstable eigenvalues of the periodic orbits are the eigenvalues of A^p . Figure 4 shows the periodic points belonging to the cycles with period times up to length 5 that were found from a simulated time series of 1,048,576 data points. Similar results are obtained by applying the periodic orbit analysis to an experimental time series of 42,754 points (see Van de Water et al., 1991). Note that a huge reduction of data has been obtained, extracting a few periodic points and their eigenvalues from a long chaotic time series. The order of magnitude of the error in the estimated periodic points, computed from the simulated time series, is $O(1.0 \times 10^{-7})$. However, the experimental data are contaminated by noise. The order of magnitude of this noise level $(5.0 \times 10^{-3} \text{ relative to the size of the attractor})$ gives a good indication for the order of magnitude of the relative error that is made in estimating the periodic points for the experimental time series; see equation (8).

In the following section, it will be shown how the unstable periodic points can be utilized to estimate parameters in the differential equation (2) by means of the method of nonlinear parametric identification using periodic data, developed by Verbeek (1993) and Verbeek et al. (1995). Besides the mentioned data reduction, this parameter estimation routine has another advantage.

When the chaotic time series has to be used directly in the parameter estimation (see Baake et al., 1992), the numerical solution, computed by integration, will be spoiled by error propagation, due to positive Lyapunov exponents. In computing periodic solutions by the method described in Verbeek (1993) and Verbeek et al. (1995), no such problems occur.

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3. NONLINEAR PARAMETRIC IDENTIFICATION USING PERIODIC DATA

The unstable periodic orbits extracted from the chaotic time series are the input for the nonlinear parametric identification using periodic data developed by Verbeek (1993) and Verbeek et al. (1995). In principle, each unstable periodic orbit can be used separately because each of them is a solution of the differential equation that describes the system exactly. Therefore, a limited set of unstable periodic orbits, which will be able to represent the chaotic attractor (although not completely), will be used in the parameter estimation program. It seems to be obvious to use as many unstable periodic orbits as is necessary to accomplish a good covering of the phase plane to increase the identifiability of the dynamical system.

In the following, the method of Verbeek et al. (1995) will briefly be discussed. It is assumed that the deterministic mathematical models for structural systems can be derived by using physical laws. Characteristic to estimation is that the problem can be reduced to an optimization problem, depending on a deterministic prediction model for the measured outputs, the measured data, and the amount of prior knowledge available. The parameters of the differential equation are estimated by minimizing the difference between the predicted outputs, computed from the deterministic prediction model, and the measurements in a well-chosen estimator. Verbeek et al. have selected a modified off-line Bayesian estimator as the point estimator, as this parametric estimator is capable of estimating not only the unknown model parameters from the measurement data but the unknown distribution parameters of the errors as well. The method is represented schematically in Figure 5.

The Bayesian estimator is solved by a modified Gauss-Newton iterative solution technique. Further, predicted outputs of the deterministic system for estimated model parameters



Figure 5. Estimation strategy in Verbeek (1993) and Verbeek et al. (1995).

have to be computed. The stable and unstable periodic solutions are calculated by solving a two-point boundary value problem with the finite difference method (see Fey, 1992; Verbeek, 1993; and Verbeek et al., 1995). The local stability of these periodic solutions can be investigated using Floquet theory.

4. NONLINEAR PARAMETRIC IDENTIFICATION USING CHAOTIC DATA

The periodic orbit analysis and the nonlinear parametric identification method using periodic data are coupled to form a nonlinear parametric identification method using chaotic data. The estimation strategy is shown in Figure 6. The measured outputs in Figure 6 consist of the simulated or experimental chaotic time series. These chaotic time series (see Figure 3) form the input for the periodic orbit analysis. This analysis returns the periodic points and their eigenvalues (see Figure 4). For each period p orbit (p = 1, 2, 3, ..., n), there are 2p values (ϕ and ϕ for each periodic phase space point) available for use in the parameter estimation program. In this way, a huge reduction of data is realized. On the other input side of the parameter estimation program, the periodic solver returns the predicted periodic points of the deterministic system for given estimates of the model parameters. In the parameter estimation program, the difference between the periodic points estimated in the periodic orbit analysis and the periodic points computed by the periodic solver is minimized to obtain good estimates of the model parameters.

Two important remarks have to be made concerning the described estimation strategy:

1. The computation of a periodic solution demands a discrete initial guess for that solution. In section 3, it was stated that the different unstable periodic orbits are very close to each other in the phase space. The shadowing mechanism implies that the closeness of the different periodic orbits in the phase space is not a specific property of the parametrically excited pendulum. It is therefore clear that the starting solution for the periodic solver has to be a very good guess. Otherwise, the iterative modified Newton procedure will just converge to another periodic solution than the one that



Figure 6. Estimation strategy in Van de Wouw (1994).

NONLINEAR PARAMETRIC IDENTIFICATION USING CHAOTIC DATA 301

was searched for. So far, only a trial-and-error strategy succeeded in finding suitable starting solutions. This trial-and-error strategy consisted of integrating the differential equation as described, varying the model parameters until a solution was found that did not diverge or lead to a p times period 1 solution. The preceding observation is a widely known difficulty in computing higher period solutions of dynamical systems. Of course, this is a disadvantage of the method.

2. In the current implementation of the described estimation strategy, only the place in the phase space (ϕ and $\dot{\phi}$ values) of the periodic points is used in the object function that is to be minimized in the parameter estimation program. However, the output of the periodic orbit analysis also contains information on the stability of the periodic points. Information on the local stability of the numerically computed periodic solutions can be obtained using Floquet theory. The object function, defined in the parameter estimated in the periodic orbit analysis and eigenvalues computed using Floquet theory. Then not only the position in the phase space but also the eigenvalues of the periodic points form the criteria that measure the quality of a set of given estimates of the model parameters.

5. RESULTS

The estimation strategy visualized by Figure 6 was followed, using a simulated and an experimental chaotic time series as the measured outputs. The results will be discussed in the following subsections.

5.1. Using a Simulated Time Series

The simulation data were produced by integrating equation (2), using the parameters given in subsection 2.2. This resulted in the time series depicted in Figure 3. No noise was added to the data. From this equation, it is clear that the parameters m, l, k_2 , and k_3 cannot be estimated simultaneously, because these parameters do not represent independent terms in the differential equation. The parameters A_{exc}, ω , and g are supposed to be known. Furthermore, it is expected that the estimation of the mass m and the friction parameters k_i (i = 2, 3) is more difficult than the estimation of l, which is the most characteristic feature of the pendulum. The different goals of the estimation process therefore are (a) to obtain quantitative knowledge of the **physical** parameters m, l, k_1, k_2 , and k_3 and (b) to obtain quantitative knowledge of the system. Then knowledge on each physical parameter is not so important. In this case $l, \frac{k_2}{m}$, and $\frac{k_3}{m}$ are estimated.

5.1.1. Estimating the physical parameters. The estimation of the physical parameters was separated into two parts:

1. Estimation of m and l with given values for k_2 and k_3 (used in the simulation): Using all the periodic points depicted in Figure 4, corresponding to the unstable periodic orbits up to period 5, resulted in the estimates given in Table I. The result that m and l can be estimated accurately implies that the periodic orbits contain necessary and sufficient

Table I. Estimates of *m* and *l* using all periodic orbits up to period 5 found in the simulated time series.

Table II. Estimates of k_2 and k_3 using all periodic orbits up to period 5 found in the simulated time series.

Initial deviation	Final deviation	Initial deviation	Final deviation
from exact values	from exact values	from exact values	from exact values
$\frac{\left \frac{\hat{m}_{0}-m}{m}\right = 50\%}{\left \frac{\hat{l}_{0}-l}{l}\right = 20\%}$	$\left \frac{\hat{m}-m}{m}\right = 0.76\%$ $\left \frac{\hat{l}-l}{l}\right = 0.28\%$	$\left \frac{\hat{k}_{2_0} - k_2}{k_2} \right = 100\%$ $\left \frac{\hat{k}_{3_0} - k_3}{k_3} \right = 100\%$	$\left \frac{\hat{k}_2 - k_2}{k_2}\right = 11.3\%$ $\left \frac{\hat{k}_3 - k_3}{k_3}\right = 3.2\%$

information on the nonlinear dynamics of the system in a compact form. Information concerning the influence of using different sets of periodic orbits in the estimation routine is discussed in Van de Wouw (1994).

2. Estimation of k_2 and k_3 with given values for m and l (used in the simulation): The results of estimating the friction parameters k_2 and k_3 , using all the periodic orbits up to period 5, are given in Table II. The friction parameters are estimated reasonably well, regarding the fact that initially there was assumed to be no friction at all. But the relative errors on the estimates of k_2 and k_3 are larger than those on the estimates of m and l. Furthermore, the error on \hat{k}_2 is also much larger than the error on \hat{k}_3 . This could be explained by the fact that the order of magnitude of the term in the differential equation (2) that contains k_2 is lower than the order of magnitude of the term that contains k_3 ($O(\dot{\phi})$ vs. $O(\dot{\phi}^2)$). Therefore, \hat{k}_2 can be adjusted rather easily (in comparison with \hat{k}_3 in the estimation process, without having much effect on the periodic points that will be estimated by the periodic solver. So, errors in the data (that refer to periodic points from the periodic orbit analysis) are likely to be compensated by the adjustment of \hat{k}_2 in the estimation program.

5.1.2. Estimating the mathematical parameters. In this case $\frac{k_2}{m}$, $\frac{k_1}{m}$, and l were estimated simultaneously. The results of estimating these parameters, using all periodic orbits up to period 5, are given in Table III. These results correspond very well with the results in Table II.

5.2. Using an Experimental Time Series

Again, the estimation strategy of Figure 6 was followed, taking the experimental chaotic time series (see Van de Water et al., 1991) as the measured output.

For the experimental system, values for the friction parameters $(k_{1_{exp}}, k_{2_{exp}}, and k_{3_{exp}})$ were estimated by Van de Water et al. (1991), see section 2.2. The values for m and l were simply calculated from geometrical data (m_{exp} and l_{exp}). For the estimation of these parameters, unstable periodic orbits up to period 5, extracted from the experimental time series, were used. For a reason to be explained, only the physical parameters were estimated. The estimation of the physical parameters again was separated into two parts, because it is impossible to estimate m, l, k_1, k_2 , and k_3 simultaneously.

Table III. Estimates of I, $\frac{k_2}{m}$ and $\frac{k_3}{m}$ using all Table IV. Estimates of m and I using all periodic orbits up to period 5 found in the riodic orbits up to period 5 found in the exsimulated time series.

perimental time series.

Initial deviation from exact values	Final deviation from exact values	Initial deviation from m _{exp} and l _{exp}	Final deviation from m _{exp} and l _{exp}
$\left \frac{\hat{l}_0 - l}{l}\right = 18.9\%$	$\left \frac{\hat{l}-l}{l}\right = 0.0\%$	$\left \frac{\hat{m}_0 - m_{\exp}}{m_{\exp}}\right = 51\%$	$\left \frac{\hat{m}-m_{\rm exp}}{m_{\rm exp}}\right = 9.6\%$
$\left \frac{\hat{k}_{2_0}/m - k_2/m}{k_2/m}\right = 33.6\%$	$\left \frac{\hat{k}_2/m - k_2/m}{k_2/m}\right = 11.3\%$	$\left \frac{\hat{l}_0 - l_{\exp}}{l_{\exp}}\right = 18.9\%$	$\left \frac{\hat{l}-l_{\exp}}{l_{\exp}}\right = 0.3\%$
$\left \frac{\hat{k}_{3_0}/m - k_3/m}{k_3/m}\right = 33.4\%$	$\left \frac{\hat{k}_3/m - k_3/m}{k_3/m}\right = 3.3\%$		

5.2.1. Estimation of the parameters m and l. In the estimation process, fixed values for the friction parameters $k_{1_{en}}$, $k_{2_{en}}$, and $k_{3_{en}}$, measured by Van de Water et al. (1991), are used. The results of the estimation of m and l, using all periodic orbits up to period 5, are ordered in Table IV. From this table, it can be concluded that *l* can be estimated very accurately, using the experimental data. However, the estimation of m is less accurate. The result that the estimate \hat{m} is less accurate than it was using the simulated data has several causes:

- 1. The experimental time series is contaminated by noise.
- 2. The experimental time series is substantially shorter than the simulated time series.
- 3. The two previous points obviously lead to less accurate periodic point estimates by the periodic orbit analysis, see subsection 2.3.
- 4. The measured periodic points were picked from a figure such as Figure 4, as the periodic points belonging to the simulation were obtained in numerical form.

In spite of these sources of errors, the estimates of m and l can be considered relatively accurate.

5.2.2. Estimation of the friction parameters k_1, k_2 , and k_3 . In the estimation process, the measured values m_{exp} and l_{exp} are used. The attempt to estimate k_1, k_2 , and k_3 simultaneously failed. The reason for this failure is that the terms of the differential equation (2) concerning k_1 and k_2 ($k_1 \operatorname{sgn}(\phi), k_2 \phi$) are relatively small compared to the term $k_3 \phi^2 \operatorname{sgn}(\phi)$ and the other terms in the differential equation. So, as explained before, the estimates \hat{k}_1 and \hat{k}_2 can be adapted almost arbitrarily. For this reason, it will of course also be impossible to estimate $\frac{k_1}{m}$, $\frac{k_2}{m}$, $\frac{k_3}{m}$, and l or even k_3 and m simultaneously. Therefore, only k_3 was estimated, using the estimates on k_1 and k_2 presented by Van de Water et al. (1991) in their experiment (see subsection 2.2). The results are ordered in Table V. Note that the accuracy of the estimate \hat{k}_3 is about the same as the accuracy of \hat{m} . The accuracy of the estimates is very acceptable given the level of measurement noise on the data. Note that $k_{3_{exp}}$ —estimated value for k_3 of Van de Water et al. (1991)—is only another estimate for the value of k_3 in the experiment; see section 2.2.

304 N. VAN DE WOUW, G. VERBEEK, and D. H. VAN CAMPEN

Table V. Estimates of k_3 using all periodic orbits up to period 5 found in the experimental time series.

Initial deviation	Final deviation
from k _{3exp}	from k _{3exp}
$\left \frac{\hat{k}_{3_0} - k_{3_{\exp}}}{k_{3_{\exp}}}\right = 100\%$	$\left \frac{\hat{k}_3 - k_{3_{\exp}}}{k_{3_{\exp}}}\right = 11.6\%$

6. DISCUSSION

A nonlinear parametric identification method using chaotic data has been developed. It consists of the coupling of a periodic orbit analysis and a nonlinear parametric identification method using periodic data. The method has been validated by estimating the parameters of the differential equation that describes a parametrically excited pendulum. The accuracy of these estimates appeared to be very high in the case of data generated by simulation. The estimation of the model parameters using the experimental data was more difficult, because these data were less accurate than those from the simulation. Still, the accuracy of the estimates is very acceptable.

Some advantages of the method, especially compared to the method of fitting ordinary differential equations directly to chaotic data (see Baake et al., 1992) are important to note:

- 1. The problem of the error propagation in the numerical solution during integration, due to positive Lyapunov exponents, is avoided.
- 2. Before the estimation routine is used a huge data reduction has been accomplished, without losing essential information on the nonlinear dynamics of the system.
- 3. Instead of a set of trajectories, the structure of the phase space is used in the estimation routine as essential information on the nonlinear dynamics of the system.
- 4. The use of very long chaotic time series does not give rise to any problems. Long time series are even required to ensure a certain accuracy of the computed periodic points.

Besides the requirement for long chaotic time series, a second disadvantage should be mentioned: The use of the periodic solver that provides the computation of the periodic solutions requires very good starting solutions as initial guesses on the periodic orbits, because the unstable periodic orbits that characterize the chaotic attractor are so close in phase space. So far only a trial-and-error strategy has been used to find suitable starting solutions. Therefore, a better and more comprehensive method in finding suitable starting solutions is desirable. Furthermore, the use of the eigenvalues of the unstable periodic orbits in the estimation of the model parameters should be incorporated to improve the quality of the estimations.

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