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A Volterra Series Approach to the Approximation of Stochastic Nonlinear Dynamics

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Abstract. A response approximation method for stochastically excited, nonlinear, dynamic systems is presented. Herein, the output of the nonlinear system is approximated by a finite-order Volterra series. The original nonlinear system is replaced by a bilinear system in order to determine the kernels of this series. The parameters of the bilinear system are determined by minimizing, in a statistical sense, the difference between the original system and the bilinear system. Application to a piecewise linear model of a beam with a nonlinear one-sided support illustrates the effectiveness of this approach in approximating truly nonlinear, stochastic response phenomena in both the statistical moments and the power spectral density of the response of this system in case of a white noise excitation.

Keywords: Stochastic excitations, response approximation, bilinearization.

1. Introduction

Stochastically excited nonlinear dynamic systems are often encountered in practice. Examples are nonlinear suspensions in vehicles on random road surfaces, high-rise buildings forced by wind or earthquakes, and offshore structures excited by wave motions at sea.

The simulation of the stochastic response of such systems [1] is, in general, very time-consuming since accurate estimates of the response statistics require the simulation of long (or many) time-series. Therefore, response approximation methods are needed. For strongly nonlinear systems the statistical linearization technique [2] generally fails to provide accurate results.

To overcome these problems, here a nonlinear approximation method, called statistical bilinearization, is presented. Herein, the input-output relation of the nonlinear system is described using a Volterra series [3]. Of course, the computational efficiency using this description should be significantly larger than that of the simulation of the response of the original nonlinear system. Therefore, a finite-order Volterra series will be used to describe the input-output relation of the nonlinear system. Finite-order Volterra systems are systems with polynomial nonlinearities. Such a nonlinear approximation technique using polynomial nonlinearities can be seen as a natural extension of linearization. In order to determine the Volterra kernels in the Volterra series, the original, nonlinear system is replaced by a bilinear system which has the same Volterra kernels up to a certain order [4]. The parameters of this bilinear system are determined to ensure that it describes the original system optimally in a statistical sense. Besides the fact that the replacement of the original system by a finite-order Volterra model gives us the advantage of computational efficiency, the gradual extension of

the replacing models from linear towards polynomial will enhance our understanding of the nonlinear response phenomena of the original system.

The method of statistical bilinearization is applied to a piecewise linear model of a beam with a nonlinear one-sided support. The piecewise linear model can represent many systems with one-sided stiffness phenomena. Practical examples are elastic stops in vehicle suspensions, snubbers on solar arrays connected to satellites [5], suspension bridges or models used in the offshore industry [6].

The basic ideas behind the use of an input-output description for the nonlinear system based on Volterra series are briefly described in Section 2. In Section 3, a technique called bilinearization (or Carleman linearization) is used to construct a finite-order Volterra model. In Section 4, the statistical bilinearization technique is proposed and applied to the piecewise linear system. In Section 5, some results of the application of the statistical bilinearization technique to the piecewise linear model of a beam with a nonlinear one-sided support are discussed and compared to those of simulation and statistical linearization. Conclusions will be presented in Section 6.

2. Problem Definition

Consider an affine, nonlinear system with the following state equations:

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{a}(t, \underline{x}(t)) + \underline{b}(t, \underline{x}(t))\, u(t), \\ y(t) &= \underline{c}(t)\, \underline{x}(t), \quad t \geq 0,\ \underline{x}(0) = \underline{0}, \end{aligned} \tag{1}$$

where $\underline{x}(t)$ is an n-dimensional state column-vector, while $u(t)$ is a scalar input and $y(t)$ a scalar output. Furthermore, $\underline{a}(t, \underline{x}(t))$ and $\underline{b}(t, \underline{x}(t))$ are time-dependent vectorfields on $\mathbb{R}^n$, with $\underline{a}(t, \underline{0}) = \underline{0}\, \forall\, t$, and $\underline{c}(t)$ is an n-dimensional row-vector. It is assumed that $\underline{a}$, $\underline{b}$ and $\underline{c}$ are analytic functions in $\underline{x}$ and continuous in t.

In [4], it is stated that, when a solution of the state Equation (1) exists for $u(t) = 0$ ($t \in [0, T]$) and initial condition $\underline{x}(0) = \underline{0}$, there is a Volterra system representation [3] for (1):

$$\begin{aligned} y(t) = &\int_0^t h_1(t-\tau_1)\, u(\tau_1)\, \mathrm{d}\tau_1 + \int_0^t \int_0^{\tau_1} h_2(t-\tau_1, t-\tau_2)\, u(\tau_1)\, u(\tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 + \cdots \\ &+ \int_0^t \cdots \int_0^{\tau_{k-1}} h_k(t-\tau_1, \ldots, t-\tau_k)\, u(\tau_1) \ldots u(\tau_k)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_k + \cdots . \end{aligned} \tag{2}$$

Herein, $h_k(t-\tau_1, t-\tau_2, \ldots, t-\tau_k)$, $k = 1, 2, \ldots$, is called the kth-order Volterra kernel. The first term in (2) corresponds to the well-known convolution representation of linear input-output systems. The subsequent terms in (2), for $k = 2, 3, \ldots$, represent natural extensions of the linear system using polynomial, nonlinear terms. Note that the polynomial nature of this representation enters through the polynomial structure in inputs in the integrand. The Volterra series (2) converges on some interval $t \in [0, T]$ when $|u(t)| < \epsilon$ for some sufficiently small $\epsilon > 0$. So, this means that the convergence of the Volterra representation, which is in general not exponential, is only guaranteed on a bounded time interval and for an input signal which is

sufficiently small. Note that causality is implied through the integration limits of the multiple integrals in (2); for each t, $y(t)$ merely depends on $\{u(\nu),\ \nu \in [0, t]\}$.

Since in general solving (1) for a given arbitrary input function $u(t)$ is a difficult or even impossible task, many people have tried some kind of approximation technique to describe the input-output behaviour of this system in an approximate way. In this perspective, we will aim to determine a polynomial input-output expression for (1) up to order p by truncating the series in (2):

$$\begin{aligned} y(t) &= \int_0^t h_1(t-\tau_1)u(\tau_1)\,\mathrm{d}\tau_1 \\ &\quad + \sum_{k=2}^{p} \int_0^t \cdots \int_0^{\tau_{k-1}} h_k(t-\tau_1,\ldots,t-\tau_k)u(\tau_1)\ldots u(\tau_k)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_k, \end{aligned} \tag{3}$$

which should approximate (1) sufficiently close at least on a finite time interval and for small enough inputs. Of course, in general, the accuracy of the approximation using (3) depending on a specific choice for the order of the approximation p is very difficult to determine due to the fact that one has little insight into the rate of convergence of the Volterra series. The main issue in the characterization of such system is the determination of $h_k(t-\tau_1, t-\tau_2, \ldots, t-\tau_k)$, $k = 1, 2, \ldots, p$. Hereto, the bilinearization procedure, as described in Section 3, will be used.

3. Bilinearization

In this section, a method called bilinearization or Carleman linearization [4] is described. The idea is that an affine, nonlinear system with analytic nonlinearities, as in (1), can be approximated, on a finite time interval and for small inputs, by a system with bilinear state equations of the form [7]:

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{A}(t)\,\underline{x}(t) + \underline{D}(t)\,\underline{x}(t)\,u(t) + \underline{e}(t)\,u(t), \\ y(t) &= \underline{C}(t)\,\underline{x}(t), \quad t \geq 0,\ \underline{x}(0) = \underline{0}, \end{aligned} \tag{4}$$

where $\underline{x}(t)$ is an n_b-dimensional column-vector with state variables ($n_b = \sum_{l=1}^{p} n^l$), while $u(t)$ and $y(t)$ are scalar inputs and outputs, respectively. Furthermore, $\underline{A}(t)$ and $\underline{D}(t)$ are $n_b \times n_b$ matrices, $\underline{e}(t)$ is an n_b-dimensional column-vector and $\underline{C}(t)$ is an n_b-dimensional row-vector. Moreover, it is important to note that analytical expressions for the Volterra kernels of such a bilinear system are available [8]. One can truncate the resulting Volterra system at a specific order to obtain a finite-order Volterra system description as in (3). This system description can then be used to approximate the response statistics of the bilinear system and, thus, to approximate the response statistics of the original, nonlinear system (1).

3.1. The Bilinearization Technique

Here, the bilinearization technique will be described briefly. We aim to determine a polynomial input-output expression for (1) up to order p, as in (3), which approximates (1) sufficiently close. To do so, first, bilinear state equations, as in (4), have to be constructed in such a way that these can be represented by the same Volterra kernels (up to order p) as

the system in (1). Next, the input-output relation for that bilinear system can be determined. Then, an approximation for the input-output relation of (1) is available and can be used to approximate the response statistics of this system.

In order to find an approximate description of (1) in terms of a system with bilinear state equations, the right-hand side of (1) can be replaced by a power series representation:

$$\begin{aligned} \underline{a}(t, \underline{x}(t)) &= \underline{A}_1(t)\, \underline{x}(t) + \underline{A}_2(t)\, \underline{x}^{(2)}(t) + \cdots + \underline{A}_p(t)\, \underline{x}^{(p)}(t) + \cdots, \\ \underline{b}(t, \underline{x}(t)) &= \underline{B}_0(t) + \underline{B}_1(t)\, \underline{x}(t) + \cdots + \underline{B}_{p-1}(t)\, \underline{x}^{(p-1)}(t) + \cdots, \end{aligned} \tag{5}$$

where the superscript notation (p) refers to the repetitive application of the Kronecker product:

$$\underline{x}^{(2)}(t) = \underline{x}(t) \otimes \underline{x}(t). \tag{6}$$

For matrices $\underline{P}$ and $\underline{Q}$ with dimensions $n_p \times m_p$ and $n_q \times m_q$, respectively, the Kronecker product is defined as a $(n_p n_q) \times (m_p m_q)$ matrix:

$$\underline{P} \otimes \underline{Q} = \begin{bmatrix} P_{11}\underline{Q} & \dots & P_{1m_p}\underline{Q} \\ \vdots & \vdots & \vdots \\ P_{n_p 1}\underline{Q} & \dots & P_{n_p m_p}\underline{Q} \end{bmatrix}. \tag{7}$$

Using (7), (6) can be expressed as

$$\underline{x}^{(2)} = \begin{bmatrix} x_1\underline{x} \\ x_2\underline{x} \\ \vdots \\ x_n\underline{x} \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 & \dots & x_1x_n & x_2x_1 & x_2^2 & \dots & x_2x_n & \dots & x_nx_1 & \dots & x_n^2 \end{bmatrix}^T \tag{8}$$

for an n-dimensional column-vector $\underline{x}$. Note that $\underline{x}^{(p)}(t)$ is an n^p-dimensional column-vector and $\underline{A}_p(t)$ is an $n \times n^p$ matrix. Using (5), (1) can be rewritten as

$$\begin{aligned} \dot{\underline{x}}(t) &= \sum_{k=1}^{p} \underline{A}_k(t)\, \underline{x}^{(k)}(t) + \sum_{k=0}^{p-1} \underline{B}_k(t)\, \underline{x}^{(k)}(t)\, u(t) + \cdots, \\ y(t) &= \underline{c}(t)\, \underline{x}(t), \quad \underline{x}^{(k)}(0) = \underline{0}, \quad t \geq 0. \end{aligned} \tag{9}$$

In order to determine the first p kernels corresponding to (9), differential equations are developed for $\underline{x}^{(j)}(t)$ [4]:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\underline{x}^{(j)}(t)] = \sum_{k=1}^{p-j+1} \underline{A}_{j,k}(t)\, \underline{x}^{(k+j-1)}(t) + \sum_{k=0}^{p-j} \underline{B}_{j,k}(t)\, \underline{x}^{(k+j-1)}(t) + \cdots, \tag{10}$$

with $\underline{x}^{(j)}(0) = \underline{0}$ (for $j = 1, \dots, p$), $\underline{A}_{1,k} = \underline{A}_k$ and, for $j > 1$,

$$\begin{aligned} \underline{A}_{j,k}(t) &= \underline{A}_k(t) \otimes \underline{I}_n \otimes \cdots \otimes \underline{I}_n + \underline{I}_n \otimes \underline{A}_k(t) \otimes \cdots \otimes \underline{I}_n \\ &\quad + \cdots + \underline{I}_n \otimes \cdots \otimes \underline{I}_n \otimes \underline{A}_k(t). \end{aligned} \tag{11}$$

It should be noted that there are $j-1$ Kronecker products in each term and j terms. The notation for $\underline{B}_{j,k}(t)$ is likewise. Note that $\underline{I}_n$ represents an $n \times n$ identity matrix. Now, by setting

$$\underline{x}^{\otimes}(t) = \begin{bmatrix} \underline{x}^{(1)}(t) \\ \underline{x}^{(2)}(t) \\ \vdots \\ \underline{x}^{(p)}(t) \end{bmatrix}, \tag{12}$$

(10) can be written as a bilinear state equation neglecting terms of order larger than p, i.e. $\underline{x}^{(p+i)}(t),\ i > 0$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\underline{x}^{\otimes}(t) \;=\; & \begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \cdots & \underline{A}_{1p} \\ \underline{0} & \underline{A}_{21} & \cdots & \underline{A}_{2,p-1} \\ \underline{0} & \underline{0} & \cdots & \underline{A}_{3,p-1} \\ \vdots & \vdots & \vdots & \vdots \\ \underline{0} & \underline{0} & \cdots & \underline{A}_{p1} \end{bmatrix} \underline{x}^{\otimes}(t) \\ & + \begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} & \cdots & \underline{B}_{1,p-1} & \underline{0} \\ \underline{B}_{20} & \underline{B}_{21} & \cdots & \underline{B}_{2,p-2} & \underline{0} \\ \underline{0} & \underline{B}_{30} & \cdots & \underline{B}_{3,p-3} & \underline{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \underline{0} & \underline{0} & \cdots & \underline{B}_{p0} & \underline{0} \end{bmatrix} \underline{x}^{\otimes}(t)u(t) + \begin{bmatrix} \underline{B}_{10} \\ \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \end{bmatrix} u(t), \end{aligned}$$

$$y(t) = \left[\, \underline{c}(t) \;\; \underline{0} \;\; \ldots \;\; \underline{0} \,\right] \underline{x}^{\otimes}(t), \qquad \underline{x}^{\otimes}(0) = \underline{0}, \tag{13}$$

where $\underline{x}^{\otimes}(t)$ is an n_b-dimensional column-vector of state variables ($n_b = \sum_{l=1}^{p} n^l$). This equation is called a Carleman linearization or the bilinearization of the linear-analytic state equation (1).

3.2. Input-Output Relation for Bilinear State Equations

Here, the Volterra representation of (13) will be described. Since the Volterra representation of (13) coincides with that of (1) up to order p, we can now use (13) to evaluate the input-output behaviour of (1). Note that (13) is a bilinear system as in (4).

It can be shown that the input-output relation of the bilinear state equations (4) can be written as [4]:

$$\begin{aligned} y(t) \;=\; & \sum_{k=1}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} \underline{C}(t)\, \underline{\Phi}(t, \tau_1)\, \underline{D}(\tau_1)\, \underline{\Phi}(\tau_1, \tau_2)\, \underline{D}(\tau_2) \\ & \ldots \underline{D}(\tau_{k-1})\, \underline{\Phi}(\tau_{k-1}, \tau_k)\, \underline{e}(\tau_k)\, u(\tau_1) \ldots u(\tau_k)\, \mathrm{d}\tau_k \ldots \mathrm{d}\tau_1, \end{aligned} \tag{14}$$

where $\underline{\Phi}(t, \tau)$ is the transition matrix of $\underline{A}(t)$ defined by the Peano–Baker series:

$$\underline{\Phi}(t, \tau) \;=\; \underline{I} + \int_{\tau}^{t} \underline{A}(\tau_1)\, \mathrm{d}\tau_1 + \int_{\tau}^{t} \underline{A}(\tau_1) \int_{\tau}^{\tau_1} \underline{A}(\tau_2)\, \mathrm{d}\tau_2\, \mathrm{d}\tau_1$$

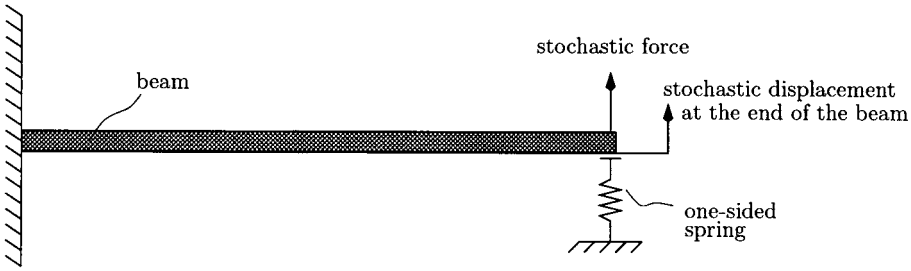

Figure 1. Beam system with a one-sided elastic support.

$$+\cdots+\int_{\tau}^{t}\underline{A}(\tau_1)\int_{\tau}^{\tau_1}\underline{A}(\tau_2)\ldots\int_{\tau}^{\tau_{k-1}}\underline{A}(\tau_k)\,\mathrm{d}\tau_k\ldots\mathrm{d}\tau_1+\cdots. \tag{15}$$

Note that in (14) there is no contribution coming from the initial condition $\underline{x}(0)$ since in (4) it was assumed that $\underline{x}(0)=\underline{0}$. For a stationary system $\underline{A}(t)$ is a constant matrix. Consequently, it can be shown that in that case

$$\underline{\Phi}(t_1,t_2):=\underline{\Phi}(t_1-t_2)=\mathrm{e}^{\underline{A}(t_1-t_2)}. \tag{16}$$

Combining (14), (13), (4) and (3) gives the Volterra kernels of the bilinear system (13). The kernels up to order p also represent the kernels of (1).

4. Statistical Bilinearization: Application to the Piecewise Linear System

In this section, the bilinearization technique will be used within a technique that will be termed statistical bilinearization. The statistical bilinearization technique will be applied to a beam with a nonlinear (one-sided) support, as described in [9] and [10] and depicted in Figure 1. Here, a single-degree-of-freedom, piecewise linear model for this system will be used. In a dimensionless form, this model can be described by the following piecewise linear differential equation:

$$\ddot{x}+2\zeta\,\dot{x}+x+\alpha\,\epsilon(x)\,x=u,\quad \text{where}\ \ \epsilon(x)=\begin{cases}0 & \text{if } x\geq 0,\\ 1 & \text{if } x<0,\end{cases} \tag{17}$$

x is a dimensionless measure for the displacement at the end of the beam, and u is a stationary, random, Gaussian, zero-mean process representing the force on the beam. Moreover, ζ represents a dimensionless damping parameter representing material damping of the beam and α is a nonlinearity parameter, which is the ratio of the stiffness of the one-sided spring and the stiffness of the beam. This model will be approximated using polynomial nonlinearities. Here, only terms up to order two ($p=2$) will be used. The approximating system can, therefore, be written as

$$\ddot{x}+2\zeta\,\dot{x}+\beta_1\,x_E+\beta_2\,x_E^2-\beta_2\,E\{x_E^2\}=u, \tag{18}$$

where $x_E=x-E\{x\}$ and $\ddot{x}_E=\ddot{x}$, $\dot{x}_E=\dot{x}$ due to stationarity. Equivalent to the procedure followed in statistical linearization [2], in which the approximating system is described by (18) with $\beta_2=0$, an error is defined and minimized in the mean-square sense in order to find the

optimal parameters β_1 and β_2 of the replacing system. This error embodies the difference between the piecewise linear system and the quadratic system (18):

$$\varepsilon_{\text{bilin}} = (x + \alpha\ \epsilon(x)\ x) - \beta_1\ x_E - \beta_2\ x_E^2 + \beta_2\ E\{x_E^2\}, \tag{19}$$

for given β_1 and β_2. Subsequently, our goal is to minimize $E\{\varepsilon_{\text{bilin}}^2\}$ with respect to β_1 and β_2. This results in the following equations:

$$\beta_1 = \frac{E\{x_E(x + \alpha\ \epsilon(x)\ x)\}}{\sigma_x^2}; \quad \beta_2 = \frac{E\{x_E^2(x + \alpha\ \epsilon(x)\ x)\} - E\{x + \alpha\ \epsilon(x)\ x\}\sigma_x^2}{E\{x_E^4\} - \sigma_x^4}. \tag{20}$$

At this point we have four unknown quantities (μ_x, σ_x, β_1 and β_2) and two equations. A third equation can be found through the averaging of Equation (17):

$$E\{x + \alpha\ \epsilon(x)\ x\} = 0. \tag{21}$$

In order to find a necessary fourth equation to solve for the unknowns, the bilinearization procedure will be applied. This will yield an expression for σ_x^2 for given values of β_1 and β_2.

By choosing an approximating Volterra system as in (18), the power series representation of the original, nonlinear system as required in (5) is readily defined. Consequently, the matrices in Equation (13) can be determined. Since this equation is a bilinear state equation of the form of (4), the matrices of (4) are also known:

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\beta_1 & -2\zeta & -\beta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -\beta_1 & -2\zeta & 0 & 1 \\ 0 & 0 & -\beta_1 & 0 & -2\zeta & 1 \\ 0 & 0 & 0 & -\beta_1 & -\beta_1 & -4\zeta \end{bmatrix}, \tag{22}$$

$$\underline{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \underline{e} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T. \tag{23}$$

Next, (14) can be used to compute the kernels of this bilinear system. However, first $\underline{\Phi}(t-\tau) = \mathrm{e}^{\underline{A}(t-\tau)}$ is computed. This can be done using the relation

$$\mathrm{e}^{\underline{A}t} = \mathcal{L}^{-1}\left\{\left(s\underline{I}_6 - \underline{A}\right)^{-1}\right\}, \tag{24}$$

where $\mathcal{L}$ is the Laplace operator and $s \in \mathbb{C}$. This expression is very lengthy due to the fact that the matrix entries are functions of the parameters ζ, β_1 and β_2. Therefore, it is not given here and the elements of $\underline{\Phi}(t-\tau)$ will be denoted by $\Phi_{jk}(t-\tau)$, $j, k = 1, \ldots, 6$. Now, the Volterra kernels of the bilinear system can be evaluated. Firstly, the first-order (linear) kernel can be determined from (14):

$$h_1(t, \tau_1) = \underline{c}\ \underline{\Phi}(t, \tau_1)\underline{e} = \Phi_{12}(t, \tau_1) = h_1(t - \tau_1). \tag{25}$$

Clearly, due to the stationarity of $\underline{A}(t)$, $h_1(t, \tau_1)$ can be written as $h_1(t - \tau_1) = \Phi_{12}(t - \tau_1)$. Secondly, the observation of (14) admits the determination of the second-order kernel:

$$\begin{aligned} h_{2tri}(t, \tau_1, \tau_2) &= [\Phi_{12}(\tau_1 - \tau_2)\,(\Phi_{14}(t - \tau_1) + \Phi_{15}(t - \tau_1)) \\ &\quad + 2\,\Phi_{16}(t - \tau_1)\,\Phi_{22}(\tau_1 - \tau_2)]\,\theta(t - \tau_1)\,\theta(\tau_1 - \tau_2), \end{aligned} \tag{26}$$

where

$$\theta(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{27}$$

The presence of the θ terms in (26) implies that $t > \tau_1 > \tau_2$, which means that $h_{2\text{tri}}$ is a triangular kernel, indicated by the subscript tri. The fact that $h_{2\text{tri}}$ is a triangular kernel follows from the integration limits in (14). Since $h_{2\text{tri}}(t, \tau_1, \tau_2) = h_{2\text{tri}}(t + \Delta t, \tau_1 + \Delta t, \tau_2 + \Delta t)$, $h_{2\text{tri}}$ is a stationary kernel. In case of stationarity, the kernel $h_{2\text{tri}}$ can be written as

$$\begin{aligned} h_{2\text{tri}}(\tau_1, \tau_2) &:= h_{2\text{tri}}(0, -\tau_1, -\tau_2) \\ &= [\Phi_{12}(\tau_2 - \tau_1)\,(\Phi_{14}(\tau_1) + \Phi_{15}(\tau_1)) \\ &\quad + 2\Phi_{16}(\tau_1)\,\Phi_{22}(\tau_2 - \tau_1)]\,\theta(\tau_1)\,\theta(\tau_2 - \tau_1). \end{aligned} \tag{28}$$

At this point, we have information on the first-order and second-order Volterra kernels of the bilinear system.

This information can be used to compute the variance of the output of the bilinear system $\sigma_y^2 (= \sigma_x^2)$ using

$$\sigma_y^2 = \int_{-\infty}^{\infty} S_{yy}(\omega)\,\mathrm{d}\omega. \tag{29}$$

For a second-order Volterra system, such as (18), the power spectral density $S_{yy}(\omega)$ obeys

$$\begin{aligned} S_{yy}(\omega) &= |H_1(i\omega)|^2\,S_{uu}(\omega) + 2\int_{-\infty}^{\infty} H_{2\text{symm}}(i(\omega - \gamma), i\gamma) \\ &\quad \times\; H_{2\text{symm}}(i(-\omega + \gamma), -i\gamma)\,S_{uu}(\gamma)\,S_{uu}(\omega - \gamma)\,\mathrm{d}\gamma, \end{aligned} \tag{30}$$

where $H_1(s)$, $s \in \mathbb{C}$ is the first-order transfer function, which can be determined by taking the one-dimensional Laplace transform of $h_1(t)$. Moreover, $H_2(s_1, s_2)$, with $s_1 \in \mathbb{C}$ and $s_2 \in \mathbb{C}$, is the second-order, symmetric transfer function, which can be found by, firstly, performing a two-dimensional Laplace transform on $h_{2\text{tri}}(\tau_1, \tau_2)$ and, secondly, performing a symmetrizing operation on the result. It should be noted that a term with a Dirac delta pulse at $\omega = 0$ is omitted in expression (30) for $S_{yy}(\omega)$, since that term does not contribute to σ_y^2 but relates to μ_y^2. Now, using (30), the power spectral density of the output can be computed. Consequently, the variance of the output can be evaluated through (29) for specific values of β_1 and β_2 and, thus, we have defined the fourth equation needed in the statistical bilinearization technique.

Since $\sigma_y^2 = \sigma_x^2$ is now known, a new estimate for the mean of the response of the piecewise linear system can be determined using (21). New values for β_1 and β_2 can be computed by solving the equations in (20). In order to be able to evaluate the expected values in these

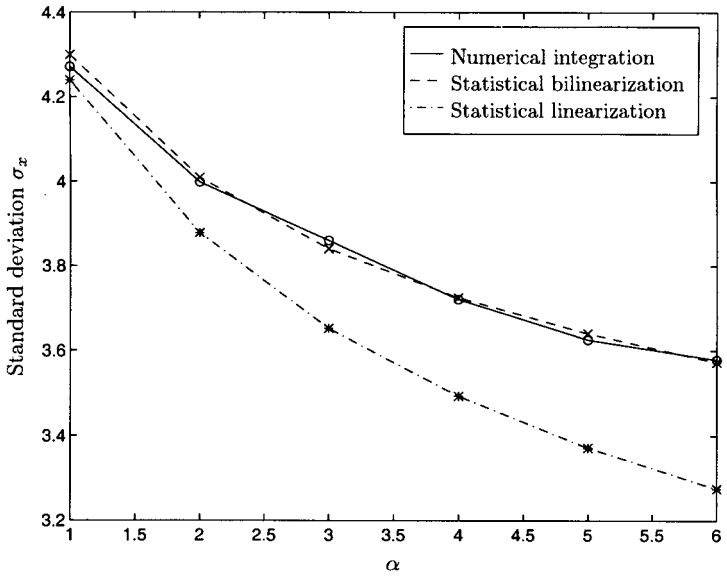

Figure 2. Estimation of the standard deviation σ_x for $\zeta = 0.01$.

equations, a functional form for the probability density function of the response has to be chosen. Here, for the sake of efficiency, a Gaussian probability density function is used.

The procedure, described before, has to be applied recursively in an optimization loop, which consists of the following steps:

1. choose initial values for β_1 and β_2;
2. use Equations (21), (29) and (30) to compute estimates for μ_x and σ_x, given β_1 and β_2;
3. compute new values for β_1 and β_2 using the information gained in step 2 and the equations in (20);
4. return to step 2 until both β_1 and β_2 have converged.

For the example described in the next section, the convergence of the parameters β_1 and β_2 was attained by performing the optimization loop, as described above, up to a maximum number of 5 times while the demanded relative convergence (defined by $\sum_{j=1}^{2} |(\beta_{j_{k+1}} - \beta_{j_k})/\beta_{j_k}|$ for the kth optimization step) of these parameters was set at 0.1%. In general, no predictions can be made on the rate of convergence of these parameters for different inputs or different parameter values of the original, nonlinear system.

5. Results

The statistical bilinearization technique is applied to the piecewise linear system. Hereby, we investigate the white-noise excited case with $S_{uu}(\omega) = 1/2\pi$. Of course, cases involving non-white excitations can be investigated as well.

In Figure 2, the estimates for the standard deviation of the response of the piecewise linear system, obtained by application of the statistical bilinearization technique, are displayed. In this figure, these results are compared to the results of the statistical linearization technique and simulation (using numerical integration techniques for stochastic differential equations

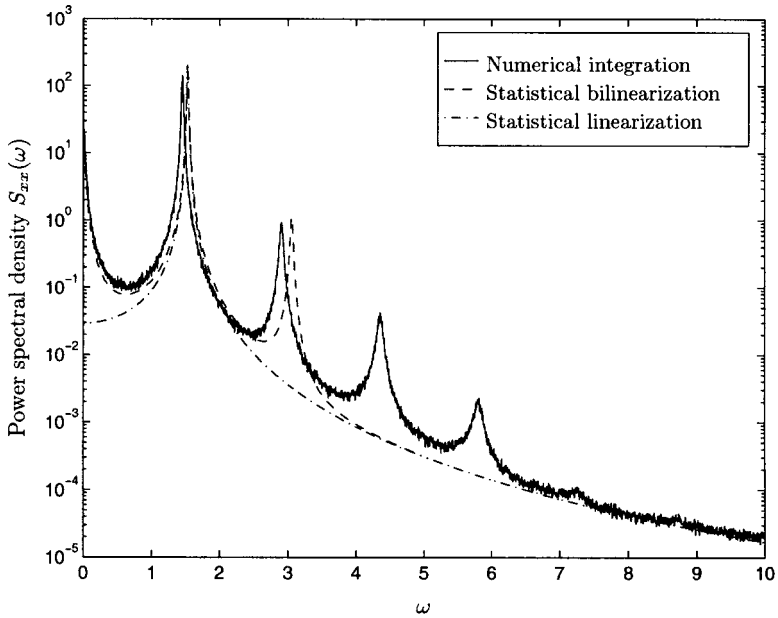

Figure 3. Estimation of the power spectral density $S_{XX}(\omega)$ for $\alpha = 6$ and $\zeta = 0.01$.

[1, 11]) for varying nonlinearity α and $\zeta = 0.01$. It should be noted that the statistical linearization technique is equivalent to the statistical bilinearization technique when the quadratic terms in (18) are omitted. Note that, due to the fact that the piecewise linear system only exhibits a stiffness nonlinearity, the determination of an equivalent damping parameter is not required in the statistical linearization technique nor in the statistical bilinearization technique. Clearly, the statistical bilinearization technique estimates the standard deviation of the response very accurately, in contradiction to the statistical linearization technique. The source of this accurate approximation can be found by observing the frequency domain information (see Figure 3). This figure shows that two important nonlinear frequency domain phenomena, namely, the multiple resonance peaks (two in this case) and the high-energy, low-frequency spectral content, are modeled very well by the bilinearization procedure, whereas the statistical linearization technique fails to model these specifically nonlinear response phenomena. These phenomena represent important contributions to the energy in the response. As a consequence, the variance of the response can only be estimated accurately when these phenomena are modeled. Clearly, only the second resonance frequency appears, whereas higher resonances are absent in the output of the Volterra model. This is a consequence of the fact that we only incorporated a second-order polynomial nonlinearity in our nonlinear model. Higher resonances could be approximated by including higher-order polynomial terms in our Volterra model. It should be noted that the high-energy, spectral content of the response is due to the stiffness asymmetry in the piecewise linear system. The presence of the (asymmetric) quadratic nonlinearity of the bilinear system ensures accurate approximation of the phenomenon.

In this form, the statistical bilinearization technique is computationally very efficient (for this specific example the computation times are even comparable to those of the statistical linearization technique). Moreover, the bilinearization approach provides much more accurate results than the statistical linearization technique (in the application to the piecewise linear

system). This is a consequence of the fact that in the statistical bilinearization technique the most important, nonlinear, frequency-domain response phenomena are modeled. The statistical bilinearization procedure is so efficient because it can provide very accurate results using only a second-order Volterra system. In this respect it distincts itself from the bilinearization procedure as proposed in [7]. Namely, in the bilinearization procedure a bilinear system is pursued, whose output $y_b(t)$ converges to the output of the original, nonlinear system $y(t)$. The system parameters of the bilinear system are determined by minimizing the error on the output through $|y_b(t) - y(t)|$ for all t in the time interval of interest. In the statistical bilinearization procedure the parameters of the bilinear system are determined by minimizing an error that represents a weighted closeness of the original, nonlinear system and the approximating bilinear system; namely, the parameters are determined by minimizing $E\{\varepsilon_{\text{bilin}}^2\}$ with $\varepsilon_{\text{bilin}}$ given in (19). As a consequence, accurate results can be obtained using a low-order Volterra model.

The method of statistical bilinearization could be extended by choosing another form of the probability density function of the response. In [12], a method called quadratization is proposed. In [13], the method was generalized to multi-degree-of-freedom systems. The statistical quadratization method is in essence the same as the statistical bilinearization technique, when a second-order polynomial model is used in the bilinearization procedure. Application of this technique to a tension leg platform is discussed in [14, 15]. In [12, 13], a non-Gaussian, truncated Gram–Charlier expansion is used for the probability density function of the response. In such a model, higher-order statistical moments play a role. Non-Gaussian characteristics of the response, such as the skewness, can then be approximated. More recently, in [16] Volterra series up to second order were used to model the non-Gaussian features of the response of wind-excited structures. In the application to the piecewise linear system, the statistical bilinearization technique was also extended using such a non-Gaussian form for the probability density function. However, the variance estimates were not improved due to this extension and the skewness was not estimated accurately. Another drawback of this extension is the fact that the estimation of the third-order moment of the output of the bilinear system involves the numerical evaluation of triple integrals. This dramatically reduces the computational efficiency of the method.

Moreover, the third resonance peak could be predicted by extending the polynomial model (18) with a third-order nonlinearity (in this case a cubic stiffness nonlinearity). However, this would result in an expression for the power spectral density of the output (see (30) for the second-order system), which includes double integrals. The computation of the variance would then include the numerical evaluation of triple integrals and would, therefore, be very laborious. Consequently, the computational efficiency of the method would be seriously compromised. Furthermore, due to the fact that the second-order model already provides highly accurate variance estimates for the response of the piecewise linear system, these variance estimates can hardly be improved by this extension. Moreover, an extension of the model with one order does not automatically lead to an improvement of the prediction of all response characteristics. This is due to the fact that one has little insight into the rate of convergence of the Volterra series of the bilinear system [4, 17]. Furthermore, when the statistical bilinearization procedure would be applied to the piecewise linear system using a third-order polynomial model, the resulting parameter estimates β_1 and β_2 will (most likely) differ from those of the second-order model. Predictions on accuracy improvements of response characteristics related to model extensions to higher orders are, therefore, difficult to make. However, it should be

noted that these extensions can be made and that the computational efficiency of the resulting response approximation will still exceed that of simulation considerably.

In this context, it should also be remarked that the piecewise linear system is not analytic everywhere. Therefore, the Volterra series (up to any order) of the bilinear system will not exactly represent that of the piecewise linear system. However, it will represent the best finite-order, polynomial model in some statistical sense (as defined before).

Finally, it should be noted that the statistical quadratization method was extended to incorporate a third-order term in the approximation of the original nonlinearity. Application of this so-called statistical cubicization method to offshore structures is discussed in [18, 19], where it can provide superior results over the quadratization approach. Moreover, in [20] is stated that the statistical quadratization approach provides an accurate response approximation for systems with asymmetrical nonlinearities, whereas the statistical cubicization approach is better suited for application in case of symmetrical nonlinearities.

6. Conclusions

In this paper, a method called statistical bilinearization was developed. The strength of the method can be recognized in the combination of two features. Firstly, the response statistics of the bilinear model can be computed very efficiently (as long as the order of the polynomial model is low). Secondly, a truly nonlinear approximation approach is followed, which makes it possible to accurately approximate typically nonlinear phenomena in the original, nonlinear system in accordance to (the nonlinear) reality.

The statistical bilinearization technique was applied successfully to the piecewise linear system. This application resulted in very accurate variance estimates for the response. Furthermore, typically nonlinear, frequency-domain response phenomena, such as multiple resonance peaks and high-energy, low-frequency spectral content, are modeled correctly. Moreover, it should be noted that the method is numerically far more efficient than simulation and can even compete with the statistical linearization technique in this respect, as long as the polynomial model used in the bilinearization technique is of a low order. Such a low-order Volterra model can provide accurate results because its parameters are determined according to a statistical criterion.

References

1. Kloeden, P. E. and Platen, E., *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin, 1992.
2. Roberts, J. B. and Spanos, P. D., *Random Vibration and Statistical Linearization*, Wiley, New York, 1990.
3. Volterra, V., *Theory of Functionals and of Integral and Integro-Differential Equations*, Dover, New York, 1959.
4. Rugh, W. J., *Nonlinear System Theory*, John Hopkins University Press, London, 1981.
5. Van Campen, D. H., Fey, R. H. B., Van Liempt, F. P. H., and De Kraker, A., 'Steady-state behaviour of a solar array system with elastic stops', in *Proceedings IUTAM Symposium on New Applications of Nonlinear and Chaotic Dynamics in Mechanics*, F. C. Moon (ed.), Kluwer, Dordrecht, 1997, pp. 303–312.
6. Thompson, J. M. T. and Stewart, H. B., *Nonlinear Dynamics and Chaos; Geometrical Methods for Engineers and Scientists*, Wiley, New York, 1986.
7. Lesiak, C. and Krener, A. J., 'The existence and uniqueness of Volterra series for nonlinear systems', *IEEE Transactions on Automatic Control* **AC-23**(6), 1978, 1090–1095.
8. Bruni, C., Di Pillo, G., and Koch, G., 'On the mathematical models of bilinear systems', *Richerche di Automatica* **2**(1), 1971, 11–26.

9. Fey, R. H. B., 'Steady-state behaviour of reduced dynamic systems with local nonlinearities', Ph.D. Thesis, Eindhoven University of Technology, The Netherlands, 1992.
10. Van de Vorst, E. L. B., 'Long term dynamics and stabilization of nonlinear mechanical systems', Ph.D. Thesis, Eindhoven University of Technology, The Netherlands, 1996.
11. Van de Wouw, N., 'Steady state behaviour of stochastically excited nonlinear dynamic systems', Ph.D. Thesis, Eindhoven University of Technology, The Netherlands, 1999.
12. Spanos, P. D. and Donley, M. G., 'Equivalent statistical quadratization for nonlinear systems', *Journal of Engineering Mechanics* **117**(6), 1991, 1289–1310.
13. Spanos, P. D. and Donley, M. G., 'Non-linear multi-degree-of-freedom system random vibration by equivalent statistical quadratization', *International Journal of Non-Linear Mechanics* **27**(5), 1992, 735–748.
14. Kareem, A., Zhao, J., and Tognarelli, M. A., 'Surge response statistics of tension leg platforms under wind and wave loads: A statistical quadratization approach', *Probabilistic Engineering Mechanics* **10**(4), 1995, 225–240.
15. Donley, M. G. and Spanos, P. D., 'Stochastic response of a tension leg platform to viscous and potential drift forces', in *Safety and Reliability Proceedings International Offshore Mechanics and Arctic Engineering Symposium*, Vol. 2, C. B. Soares (ed.), ASME, New York, 1992, pp. 325–334.
16. Benfratello, S., Di Paolo, M., and Spanos, P. D., 'Stochastic response of MDOF wind-excited structures by means of Volterra series approach', *Journal of Wind Engineering and Industrial Aerodynamics* **74–76**, 1998, 1135–1145.
17. Schetzen, M., *The Volterra and Wiener Theories of Nonlinear Systems*, Wiley, New York, 1980.
18. Li, X., Quek, S., and Koh, C., 'Stochastic response of offshore platforms by statistical cubicization', *Journal of Engineering Mechanics* **121**(10), 1995, 1056–1068.
19. Tognarelli, M. A., Kareem, A., Zhao, J., and Rao, K. B., 'Quadratization and cubicization: analysis tools for offshore engineering', in *Proceedings of Engineering Mechanics*, Vol. 2, S. Sture (ed.), ASCE, Reston, VA, 1995, pp. 1175–1178.
20. Kareem, A. and Zhao, J., 'Stochastic response analysis of tension leg platform: A statistical quadratization and cubicization approach', in *Proceedings of the International Conference on Offshore Mechanics and Arctic Engineering*, Vol. 1, S. K. Chakrabarti (ed.), ASME, New York, 1994, pp. 281–292.