THE GLOBAL OUTPUT REGULATION PROBLEM: AN INCREMENTAL STABILITY APPROACH

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Abstract: We present a global solution to the output regulation problem for a class of nonlinear systems. The solution is based on the incremental stability property. The question of existence of the proposed solution can be answered by checking solvability of the regulator equations and feasibility of certain linear matrix inequalities.

Keywords: Global output regulation, nonlinear systems, incremental stability, LMI.

1. INTRODUCTION

In this paper we address the output regulation problem, which includes the problems of tracking reference signals and rejecting disturbances generated by an external autonomous system (exosystem). For linear systems, this problem was thoroughly investigated in 1970-s by (Francis and Wonham, 1976; Davison, 1976) and others. For nonlinear systems, intensive research started with the papers (Isidori and Byrnes, 1990) and (Huang and Rugh, 1990), which provided a solution to the local output regulation problem for general nonlinear systems. These papers were followed by a number of results dealing with different aspects of the output regulation problem for nonlinear systems: approximate, robust, adaptive output regulation, see (Byrnes *et al.*, 1997; Byrnes and Isidori, 2000) and references therein.

Most of the results are based on the assumption of solvability of the so-called regulator equations, which provide existence of a controlled-invariant manifold on which the regulated output equals to zero. Existence of such (locally defined) manifold proved to be necessary at least in the case of neutrally stable exosystems (Isidori and Byrnes, 1990). If this assumption holds, the problem reduces to finding a controller that would make this manifold invariant and asymptotically stable. Depending on the region of attraction of this manifold, one can distinguish local, semiglobal and global solutions. In the local case, the dynamics of the system near an equilibrium point is similar to the dynamics of its linearization at this point and the invariant output-zeroing manifold can be *locally* stabilized with a linear controller. Non-local results require more sophisticated approaches. Most of the semiglobal results make use of certain types of high-gain controllers or observers, see e.g. (Khalil, 2000). Only a few global results exist and are mostly limited to systems which are linear in the unmeasured variables (Serrani and Isidori, 2000; Ding, 2001). A recent paper (Marconi and Serrani, 2002) addresses the global robust output regulation problem for systems in lower-triangular form.

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In this paper, we provide a global solution to the output regulation problem based on the so-called incremental stability property (Demidovich, 1961; Angeli, 2002; Fromion *et al.*, 1999). Roughly speaking, every solution of a system with this property is globally asymptotically stable. Relations between incremental stability and the output regulation problem were outlined in (Pavlov *et al.*, 2002). In the present paper, we aim at finding a controller that makes the output-zeroing manifold invariant and the closed-loop system incrementally stable, which together imply global asymptotic stability of the output-zeroing manifold.

The paper is organized as follows. In Section 2, we formulate the global output regulation problem and introduce some preliminary notions and results. The main results on controller design are given in Section 3. Section 4 contains an example and Section 5 contains conclusions. All proofs are given in the Appendix.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider systems modelled by equations of the form

$$\dot{x} = f(x, u, w) \tag{1}$$

$$e = h_r(x, w) \tag{2}$$

$$y = h_m(x, w), \tag{3}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, regulated output $e \in \mathbb{R}^l$ and measured output $y \in \mathbb{R}^k$. The exogenous signal $w(t)$, which can be viewed as a disturbance in equation (1) or as a reference signal in (2), is generated by the exosystem

$$\dot{w} = s(w), \quad w \in \mathcal{W}, \tag{4}$$

where $\mathcal{W} \subset \mathbb{R}^m$ is an open positively invariant set. It is assumed that

A1 every trajectory of (4) starting in $\mathcal{W}$ belongs to some compact subset of $\mathcal{W}$. For example, if $\mathcal{W} = \mathbb{R}^m$ then the last assumption implies that all trajectories of (4) are bounded on $t \in [0, +\infty)$.

The functions $f(x, u, w)$, $h_m(x, w)$ and $s(w)$ are assumed to be continuously differentiable.

The global output regulation problem is formulated in the following way: find, if possible, a feedback of the form

$$\dot{\xi} = \eta(\xi, y) \tag{5}$$

$$u = \theta(\xi, y) \tag{6}$$

such that all solutions of the system

$$\dot{x} = f(x, \theta(\xi, h_m(x, w)), w) \tag{7}$$

$$\dot{\xi} = \eta(\xi, h_m(x, w)) \tag{8}$$

$$\dot{w} = s(w); \tag{9}$$

are bounded and ensure that $e(t) = h_r(x(t), w(t)) \to 0$ as $t \to \infty$.

Quadratic stability

Prior to solving the problem, we give the following definitions.

Definition 1. A matrix function $\mathcal{A}(z) \in \mathbb{R}^{n\times n}$ is called quadratically stable over $\mathcal{Z}$ if for some $\mathcal{P} = \mathcal{P}^T > 0$ and $\mathcal{Q} = \mathcal{Q}^T > 0$

$$\mathcal{P}\mathcal{A}(z) + \mathcal{A}(z)^T\mathcal{P} \le -\mathcal{Q} \quad \forall z \in \mathcal{Z}. \tag{10}$$

Definition 2. A pair of matrix functions $\mathcal{A}(z) \in \mathbb{R}^{n\times n}$ and $\mathcal{B}(z) \in \mathbb{R}^{n\times p}$ is said to be quadratically stabilizable over $\mathcal{Z}$ if there exist a matrix $K \in \mathbb{R}^{p\times n}$ such that $\mathcal{A}(z) + \mathcal{B}(z)K$ is quadratically stable over $\mathcal{Z}$.

Definition 3. A pair of matrix functions $\mathcal{A}(z) \in \mathbb{R}^{n\times n}$ and $\mathcal{C}(z) \in \mathbb{R}^{k\times n}$ is said to be quadratically detectable over $\mathcal{Z}$ if there exist a matrix $L \in \mathbb{R}^{n\times k}$ such that $\mathcal{A}(z)+L\mathcal{C}(z)$ is quadratically stable over $\mathcal{Z}$.

Notice, that if $\mathcal{A}(z) \equiv \mathcal{A}$ is constant, then quadratic stability of $\mathcal{A}$ is equivalent to matrix $\mathcal{A}$ being Hurwitz; quadratic stabilizability of $(\mathcal{A}, \mathcal{B})$ and quadratic detectability of $(\mathcal{A}, \mathcal{C})$ are equivalent to conventional stabilizability and detectability of the pairs of constant matrices $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$, respectively. Similar to the case of constant matrices, the pair $(\mathcal{A}(z), \mathcal{B}(z))$ is quadratically stabilizable over $\mathcal{Z}$ if and only if $(\mathcal{A}^T(z), \mathcal{B}^T(z))$ is quadratically detectable over $\mathcal{Z}$. This follows from pre- and post-multiplication by $\mathcal{P}^{-1}$ of the inequality

$$\mathcal{P}(\mathcal{A}(z) + \mathcal{B}(z)K) + (\mathcal{A}(z) + \mathcal{B}(z)K)^T\mathcal{P} \le -\mathcal{Q}.$$

The purpose of the notions introduced above becomes clear from the following lemma, see (Demidovich, 1961; Fromion *et al.*, 1999).

Lemma 1. Consider the system

$$\dot{z} = F(z, v(t)), \tag{11}$$

where $z \in \mathbb{R}^n$, $v(t)$ is a continuous input defined on $t \in [0, +\infty)$ and taking its values in some set $\mathcal{V} \subset \mathbb{R}^k$, and F is C^1 with respect to z and continuous with respect to v. Suppose $\frac{\partial F}{\partial z}(z, v)$ is quadratically stable over $(z, v) \in \mathbb{R}^n \times \mathcal{V}$. Then, for every continuous $v(t) \in \mathcal{V}$ every solution of system (11) is globally exponentially stable.

The proof of this lemma is based on the algebraic fact that under the conditions of Lemma 1 the following relation holds (Demidovich, 1961):

$$(\Delta z)^T\mathcal{P}(F(z + \Delta z, v) - F(z, v)) \le -\varepsilon|\Delta z|^2 \tag{12}$$

for some $\mathcal{P} = \mathcal{P}^T > 0$, $\varepsilon > 0$ and all $z,\ \Delta z \in \mathbb{R}^n$, $v \in \mathcal{V}$.

In general, it is not an easy task to check quadratic stability, stabilizability or detectability. Yet, in some particular cases, this can be done efficiently, as follows from the following lemma.

Lemma 2. Consider the matrix functions $\mathcal{A}(z) \in \mathbb{R}^{n\times n}$, $\mathcal{B}(z) \in \mathbb{R}^{n\times p}$ and $\mathcal{C}(z) \in \mathbb{R}^{k\times n}$.
i) Suppose, there exist matrices $\mathcal{A}_1, \dots \mathcal{A}_s$ such that

$$\mathcal{A}(z) \in \operatorname{co}\{\mathcal{A}_1, \dots \mathcal{A}_s\}, \quad \forall z \in \mathcal{Z},$$

where $\operatorname{co}\{\dots\}$ denotes a convex hull, and the linear matrix inequality (LMI)

$$\mathcal{P} = \mathcal{P}^T > 0, \quad \mathcal{P}\mathcal{A}_i + \mathcal{A}_i^T\mathcal{P} < 0, \quad i = 1, \dots s, \tag{13}$$

is feasible. Then, $\mathcal{A}(z)$ is quadratically stable over $\mathcal{Z}$.
ii) Suppose, there exist matrices $\mathcal{A}_1, \dots \mathcal{A}_s$ and $\mathcal{B}_1, \dots \mathcal{B}_s$ such that

$$[\mathcal{A}(z), B(z)] \in \operatorname{co}\{[\mathcal{A}_1, \mathcal{B}_1], \dots [\mathcal{A}_s, \mathcal{B}_s]\}, \quad \forall z \in \mathcal{Z}$$

and the LMI

$$\begin{aligned} \mathcal{A}_i\mathcal{P} + \mathcal{P}\mathcal{A}_i^T + \mathcal{B}_i\mathcal{Y} + \mathcal{Y}^T\mathcal{B}_i^T < 0, \quad i = 1, \dots s, \\ \mathcal{P} = \mathcal{P}^T > 0. \end{aligned} \tag{14}$$

is feasible. Then, the pair $\mathcal{A}(z), \mathcal{B}(z)$ is quadratically stabilizable over $\mathcal{Z}$ with the matrix $K = \mathcal{Y}\mathcal{P}^{-1}$, where $\mathcal{Y}$ and $\mathcal{P}$ satisfy (14).
iii) Suppose, there exist matrices $\mathcal{A}_1, \dots \mathcal{A}_s$ and $\mathcal{C}_1, \dots \mathcal{C}_s$ such that

$$[\mathcal{A}(z), C(z)] \in \operatorname{co}\{[\mathcal{A}_1, \mathcal{C}_1], \dots [\mathcal{A}_s, \mathcal{C}_s]\}, \quad \forall z \in \mathcal{Z}$$

and the LMI

$$\begin{aligned} \mathcal{P}\mathcal{A}_i + \mathcal{A}_i^T\mathcal{P} + \mathcal{X}\mathcal{C}_i + \mathcal{C}_i^T\mathcal{X}^T < 0, \quad i = 1, \dots s, \\ \mathcal{P} = \mathcal{P}^T > 0. \end{aligned} \tag{15}$$

is feasible. Then, the pair $\mathcal{A}(z), \mathcal{C}(z)$ is quadratically detectable over $\mathcal{Z}$ with the matrix $L = \mathcal{P}^{-1}\mathcal{X}$, where $\mathcal{X}$ and $\mathcal{P}$ satisfy (15).

Lemma 2 is a compilation of some standard results from LMI applications to control (see, e.g. (Boyd *et al.*, 1993)). Another standard result that can be useful for checking quadratic stability is based on the circle criterion, see e.g. (Khalil, 1996).

Lemma 3. Consider a triple of matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. Suppose, the pair (A, B) is controllable, (A, C) is detectable and for some $\mu > 0$, $\mu < +\infty$

$$\operatorname{Re}\left(C(i\omega I - A)^{-1}B\right) > -\frac{1}{\mu}, \quad \forall w \in \mathbb{R}.$$

Then, any matrix function $\mathcal{A}(z)$ defined on some set $\mathcal{Z}$ and satisfying $\mathcal{A}(z) \in \operatorname{co}\{A, A - \mu BC\}$ for all $z \in \mathcal{Z}$ is quadratically stable over $\mathcal{Z}$.

3. MAIN RESULTS

The results in this section are formulated in terms of Jacobians of the functions of system (1)-(3) and exosystem (4). In the sequel, we will use the following notations: $z := (x, u, w) \in \mathbb{R}^{n+p} \times \mathcal{W}$,

$$A(z) := \frac{\partial f}{\partial x}(x, u, w), \; B(z) := \frac{\partial f}{\partial u}(x, u, w),$$
$$P(z) := \frac{\partial f}{\partial w}(x, u, w), \; C(z) := \frac{\partial h_m}{\partial x}(x, w)$$
$$Q(z) := \frac{\partial h_m}{\partial w}(x, w), \; S(z) := \frac{\partial s}{\partial w}(w).$$

The basic assumption that we use is solvability of the so-called regulator equations:
A2 There exist locally Lipschitz mappings $\pi(w)$ and $c(w)$ defined on $\mathcal{W}$ and satisfying the relations

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \tag{16}$$

for all solutions $w(t) \in \mathcal{W}$ of the exosystem (4) and

$$h_r(\pi(w), w) = 0, \quad \forall \; w \in \Omega(\mathcal{W}), \tag{17}$$

where $\Omega(\mathcal{W})$ consists of ω-limit points of trajectories $w(t)$ starting in $w(0) \in \mathcal{W}$.
Remark 1. Since the set $\Omega(\mathcal{W})$ attracts all solutions of system (4) starting in $\mathcal{W}$, then (17) implies that $h_r(\pi(w(t)), w(t)) \to 0$ as $t \to +\infty$ along any solution $w(t)$ starting in $w(0) \in \mathcal{W}$.

Let us first consider the state feedback case when the states x and w are available for measurements, i.e. $y = (x, w)$.

Theorem 1. Consider system (1)-(3) with $y = (x, w)$ and exosystem (4). Suppose, assumptions **A1** and **A2** hold and the pair $(A(z), B(z))$ is quadratically stabilizable over $z \in \mathbb{R}^{n+p} \times \mathcal{W}$. Then, the global output regulation problem is solved by a controller of the form

$$u = c(w) + K(x - \pi(w)), \tag{18}$$

where the matrix K is such that the matrix function $A(z)+B(z)K$ is quadratically stable over $z \in \mathbb{R}^{n+p} \times \mathcal{W}$ and $\pi(w)$, $c(w)$ satisfy (16), (17).

Next, we consider the case when only a certain output y is available for feedback. We assume that $\mathcal{W} = \mathbb{R}^m$. This also implies that the mappings $\pi(w)$ and $c(w)$ in Assumption **A2** are globally defined. The following theorem provides conditions for solvability of the output regulation problem in the case of output feedback.

Theorem 2. Consider system (1)-(3) and exosystem (4). Suppose that assumptions **A1** and **A2** hold, the pair $(A(z), B(z))$ is quadratically stabilizable over $z \in \mathbb{R}^{n+p+m}$, the matrix $B(z)$ is uniformly bounded on $z \in \mathbb{R}^{n+p+m}$ and the pair

$$\begin{bmatrix} A(z) & P(z) \\ 0 & S(z) \end{bmatrix}, \quad [C(z)\ Q(z)] \tag{19}$$

is quadratically detectable over $z \in \mathbb{R}^{n+p+m}$. Then, the global output regulation problem is solved by a controller of the form

$$u = c(\hat{w}) + K(\hat{x} - \pi(\hat{w})) \tag{20}$$
$$\dot{\hat{x}} = f(\hat{x}, u, \hat{w}) + L_1(\hat{y} - y) \tag{21}$$
$$\dot{\hat{w}} = s(\hat{w}) + L_2(\hat{y} - y) \tag{22}$$
$$\hat{y} = h_m(\hat{x}, \hat{w}), \tag{23}$$

where the matrices K and $L = [L_1^T, L_2^T]^T$ are such that the matrix functions $A(z) + B(z)K$ and

$$\begin{bmatrix} A(z) & P(z) \\ 0 & S(z) \end{bmatrix} + L[C(z)\ Q(z)]$$

are quadratically stable over $z \in \mathbb{R}^{n+p+m}$.

Remark 2. As follows from the proofs of Theorem 1 and Theorem 2, the solutions $x(t) = \pi(w(t))$, in the state feedback case, and $(x, \hat{x}, \hat{w}) = (\pi(w(t)), \pi(w(t)), w(t))$, in the output feedback case, are uniformly globally asymptotically stable. If the exosystem admits the solution $w(t) \equiv 0$ and $\pi(0) = 0$ and $\{0\} \in \mathcal{W}$, then for $w(t) \equiv 0$ the closed-loop dynamics resulting from the controllers (18) and (20)-(23) are globally asymptotically stable at the origin.

4. EXAMPLE

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 - x_2 - \cos(x_2) \\ \dot{x}_3 &= u \\ e = y &= x_1 - w_1 \end{aligned} \tag{24}$$

and the exosystem

$$\dot{w}_1 = w_2, \quad \dot{w}_2 = -w_1. \tag{25}$$

The control goal is to find an output feedback controller such that all solutions of the closed-loop system and the exosystem are bounded and $e(t) \to 0$, as $t \to +\infty$.

Every solution of (25) is bounded on $t \geq 0$. The regulator equations admit the solution $\pi_1(w) = w_1$, $\pi_2(w) = w_2$, $\pi_3(w) = w_2 - w_1 + \cos(w_2)$, $c(w) = -w_1 - w_2 + w_1 \sin(w_2)$. The mappings $\pi(w)$ and $c(w)$ are defined globally and continuously differentiable. Hence, assumptions **A1** and **A2** hold. Let us apply Theorem 2. In our case, $z = (x_1, x_2, x_3, u, w_1, w_2)^T$,

$$A(z) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & (\sin(x_2) - 1) & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad S(z) \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$B(z) \equiv [0\ 0\ 1]^T$, $P(z) \equiv 0$, $C(z) \equiv [1\ 0\ 0]$, $Q(z) \equiv [-1\ 0]$.

Notice, that $A(z) \in \mathrm{co}\{A_1, A_2\}$, where

$$A_1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, by Lemma 2 we can check quadratic stabilizability of the pair $(A(z), B(z))$ by checking feasibility of the matrix inequality (14) with $\mathcal{A}_i = A_i$ and $\mathcal{B}_i = B$, $i = 1, 2$. In a similar way, quadratic detectability of the pair (19) can be checked by checking feasibility of the LMI (15) with

$$\mathcal{A}_i = \begin{bmatrix} A_i & P \\ 0 & S \end{bmatrix}, \quad \mathcal{C}_i = [C\ Q], \quad i = 1, 2.$$

Numeric computations show that both LMIs (14) and (15) are feasible and, for example, the matrices $K = [-6\ -11,\ -6]^T$ and $L = [-153,\ -78,\ -13,\ -132,\ 52]$ ensure quadratic stability of the matrix functions $A(z) + B(z)K$ and

$$\begin{bmatrix} A(z) & P(z) \\ 0 & S(z) \end{bmatrix} + L[C(z)\ Q(z)]$$

over $z = (x, u, w) \in \mathbb{R}^6$. The fact that these K and L ensure quadratic stability of these matrix functions, can be also checked using the circle criterion from Lemma 3.

Thus, all conditions of Theorem 2 are satisfied. By this theorem, controller (20) with $\pi(w)$, $c(w)$, K and L specified above and the functions $f(x, u, w)$, $h_m(x, w)$ and $s(w)$ corresponding to the system equations (24), (25), solves the output regulation problem globally. In Fig. 1 one can see simulation results of the closed-loop dynamics.

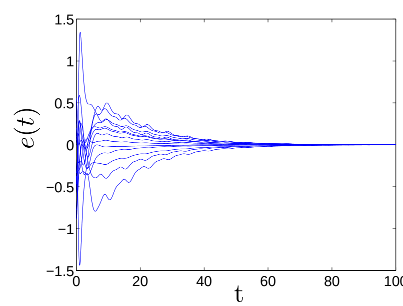

Fig. 1. Closed-loop dynamics: $e(t)$ for different initial conditions of the closed-loop system and the exosystem.

5. CONCLUSIONS

We have presented a state-feedback and an output-feedback solution to the global output regulation problem for a class of nonlinear systems. The controller design is based on the preliminary assumption of the solvability of the regulator equations in the class of locally Lipschitz functions and on the assumption of boundedness of the exosystem's solutions. In the state-feedback case, the proposed controller makes the closed-loop system incrementally stable and guarantees the existence of a solution along which the regulated output tends to zero. This incremental stability, in turn, implies global asymptotic stability of the desired solution. In the output-feedback case, this state-feedback controller is coupled with an observer, which is also incrementally stable. Quadratic stabilizability and detectability conditions, which are sufficient for the existence of the proposed controllers, extend, in a certain way, the conventional stabilizability and detectability conditions to the nonlinear case. In some cases, these conditions can be efficiently checked by solving certain LMIs. This opens a wide range of LMI- and robust control methods for solving the output regulation problem. Robustness issues, which are not addressed in this work, are the next step in our research.

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APPENDIX

Proof of Theorem 1: Consider the closed-loop system

$$\dot{x} = f(x, c(w) + K(x - \pi(w)), w) =: F(x, w). \quad (26)$$

Due to assumption **A2**, for any solution $w(t)$ of the exosystem (4) starting in $w(0) \in \mathcal{W}$, system (26) has a solution $\bar{x}_w(t) := \pi(w(t))$ along which $e(t) \to 0$, as $t \to +\infty$. Since $w(t)$ belongs to a compact subset of $\mathcal{W}$ (due to assumption **A1**) and $\pi(w)$ is continuous on $\mathcal{W}$, $\pi(w(t))$ is bounded for $t \geq 0$. Notice, that by the choice of K the Jacobian $\frac{\partial F}{\partial x}(x, w) = A(x, u(x, w), w) + B(x, u(x, w), w)K$ is quadratically stable over $(x, w) \in \mathbb{R}^n \times \mathcal{W}$. By Lemma 1, the solution $\bar{x}_w(t) = \pi(w(t))$ is globally exponentially stable. Hence, all solutions of (26) are bounded and $x(t) \to \bar{x}_w(t)$ as $t \to +\infty$ and thus

$$e(t) = h_r(x(t), w(t)) \to h_r(\pi(w(t)), w(t)) \equiv 0,$$

as $t \to +\infty$.□

Proof of Theorem 2: Consider the closed-loop system

$$\dot{x} = f(x, u, w), \quad (27)$$
$$\dot{w} = s(w), \quad (28)$$
$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{pmatrix} = \begin{pmatrix} f(\hat{x}, u, \hat{w}) \\ s(\hat{w}), \end{pmatrix} + L(\hat{y} - y), \quad (29)$$
$$\hat{y} = h_m(\hat{x}, \hat{w}), \quad y = h_m(x, w), \quad (30)$$
$$u = c(\hat{w}) + K(\hat{x} - \pi(\hat{w})). \quad (31)$$

Denote $\Delta x = x - \pi(w)$,

$$\xi := \begin{pmatrix} x \\ w \end{pmatrix}, \quad \Delta\xi = \begin{pmatrix} \Delta\xi_x \\ \Delta\xi_w \end{pmatrix} := \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} - \begin{pmatrix} x \\ w \end{pmatrix}.$$

Then, the dynamics of Δx and $\Delta\xi$ can be written as

$$\Delta\dot{x} = F(\pi(w)+\Delta x, w) - F(\pi(w), w) \quad (32)$$
$$+\rho(\xi, \Delta\xi)$$
$$\Delta\dot{\xi} = G(\xi+\Delta\xi, u) - G(\xi, u) \quad (33)$$
$$u = U(\xi+\Delta\xi), \quad (34)$$

where

$$U(\xi) = U(x,w) := c(w) + K(x-\pi(w)),$$
$$F(x,w) := f(x, U(x,w), w),$$
$$\rho(\xi,\Delta\xi) := f(x, U(\xi+\Delta\xi), w) - f(x, U(\xi), w),$$
$$G(\xi,u) := \begin{bmatrix} f(x,u,w) \\ s(w) \end{bmatrix} + Lh_m(x,w).$$

In order to obtain equation (32), we have used (16) and the fact that $U(\pi(w),w) = c(w)$.

Since $\rho(\xi,0) \equiv 0$, $(\Delta x, \Delta\xi) = (0,0)$ is an equilibrium point of (32)-(34). We will show that for every solution $w(t)$ of the exosystem (4), this equilibrium is uniformly globally asymptotically stable (UGAS). To that end, we will show that for every $w(t)$, **a)** $\Delta x = 0$ is a globally exponentially stable equilibrium of (32) with $\Delta\xi \equiv 0$; **b)** $\Delta\xi = 0$ is a globally exponentially stable equilibrium of (33); **c)** system (32) is input to state stable (ISS) with respect to the input $\Delta\xi$. Then, system (32)-(34), treated as a cascade, has a UGAS equilibrium $(\Delta x, \Delta\xi) = (0,0)$ (see Lemma 5.6 in (Khalil, 1996)).

Notice, that due to the choice of the matrices K and L, $\frac{\partial F}{\partial x}(x,w)$ is quadratically stable over $(x,w) \in \mathbb{R}^{n+m}$ and $\frac{\partial G}{\partial \xi}(\xi,u)$ is quadratically stable over $(\xi,u) \in \mathbb{R}^{n+m+p}$. Thus, by virtue of Lemma 1 (see formula (12)), there exist positive definite matrices $\mathcal{P}_c$, $\mathcal{P}_o$ and numbers $\varepsilon_c > 0$, $\varepsilon_o > 0$ such that

$$\Delta x^T \mathcal{P}_c (F(x+\Delta x, w) - F(x,w)) \le -\varepsilon_c |\Delta x|^2, \quad (35)$$
$$\Delta\xi^T \mathcal{P}_o (G(\xi+\Delta\xi, u) - G(\xi,u)) \le -\varepsilon_o |\Delta\xi|^2, \quad (36)$$

for all $x, \Delta x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $\xi, \Delta\xi \in \mathbb{R}^{n+m}$ and $u \in \mathbb{R}^p$. Consider the functions

$$V_c(\Delta x) := \frac{1}{2}(\Delta x)^T \mathcal{P}_c \Delta x,$$
$$V_o(\Delta\xi) := \frac{1}{2}(\Delta\xi)^T \mathcal{P}_o \Delta\xi.$$

Their derivatives along solutions $\Delta x(t)$ and $\Delta\xi(t)$ satisfy

$$\frac{dV_c}{dt} = (\Delta x)^T \mathcal{P}_c (F(\pi(w)+\Delta x, w) - F(\pi(w), w))$$
$$+(\Delta x)^T \mathcal{P}_c \rho(\xi,\Delta\xi)$$
$$\le -\varepsilon_c |\Delta x|^2 + \|\mathcal{P}_c\| |\Delta x| |\rho(\xi,\Delta\xi)|, \quad (37)$$

$$\frac{dV_o}{dt} = (\Delta\xi)^T \mathcal{P}_o (G(\xi+\Delta\xi, w) - G(\xi,w))$$
$$\le -\varepsilon_o |\Delta\xi|^2. \quad (38)$$

In these inequalities, we have used formulas (35) and (36). As follows from (38), the origin $\Delta\xi = 0$ is globally exponentially stable. Since $\rho(\xi,0) \equiv 0$, we conclude from formula (37) that system (32) for $\Delta\xi \equiv 0$ has a globally exponentially stable equilibrium $\Delta x = 0$. Let us show that system (32) is ISS with respect to the input $\Delta\xi$. To that end, we estimate the term $\rho(\xi,\Delta\xi)$. Since $B(z) = \frac{\partial f}{\partial u}(x,u,w)$ is uniformly bounded over $\mathbb{R}^{n+p+m}$, the function $f(x,u,w)$ is globally Lipschitz with respect to u uniformly over $(x,w) \in \mathbb{R}^{n+m}$, i.e. there exists $C_L > 0$ such that

$$|f(x,u_1,w) - f(x,u_2,w)| \le C_L |u_1 - u_2|,$$

for all $(x,w) \in \mathbb{R}^{n+m}$ and $u_1, u_2 \in \mathbb{R}^p$. Thus,

$$|\rho(\xi,\Delta\xi)| \le C_L |U(\xi+\Delta\xi) - U(\xi)|. \quad (39)$$

Notice, that

$$U(\xi+\Delta\xi) - U(\xi)$$
$$= c(\hat{w}) + K(\hat{x} - \pi(\hat{w})) - c(w) - K(x - \pi(w))$$
$$= K\Delta\xi_x + \alpha(w+\Delta\xi_w) - \alpha(w), \quad (40)$$

where $\alpha(w) := c(w) - K\pi(w)$. Hence,

$$|\rho(\xi,\Delta\xi)| \le C_L \|K\| |\Delta\xi_x| \quad (41)$$
$$+ C_L \sup_{t\ge 0} |\alpha(w(t)+\Delta\xi_w) - \alpha(w(t))|$$
$$\le C_L \|K\| |\Delta\xi| + \delta(|\Delta\xi|) =: \sigma(|\Delta\xi|),$$

where

$$\delta(r) := \sup_{t\ge 0,\ |\Delta\xi_w| \le r} |\alpha(w(t)+\Delta\xi_w) - \alpha(w(t))|.$$

Recall, that by the properties of exosystem (4), every solution $w(t)$ belongs to some compact set on $t \in [0,+\infty)$. Thus, the function $\delta(r)$ is a well-defined continuous nondecreasing function satisfying $\delta(0) := 0$. Hence, $\sigma(r) = C_L\|K\| r + \delta(r)$ is a continuous strictly increasing function satisfying $\sigma(0) = 0$, i.e. $\sigma(r)$ is a $\mathcal{K}$-function. Thus, $dV_c/dt \le \varepsilon_c |\Delta x| (-|\Delta x| + \|\mathcal{P}_c\| \sigma(|\Delta\xi|)/\varepsilon_c)$. Hence, for $|\Delta x| > 2\|\mathcal{P}_c\|\sigma(|\Delta\xi|)/\varepsilon_c$ we obtain $dV_c/dt \le -\varepsilon_c/2 |\Delta x|^2$. Application of Theorem 5.2 from (Khalil, 1996) proves ISS stability of (32) with respect to the input $\Delta\xi$.

Thus, application of Lemma 5.6 from (Khalil, 1996) proves that the solution $(\Delta x, \Delta\xi) = (0,0)$ is UGAS equilibrium of system (32)-(34). For the closed-loop system in the original coordinates $(x, \hat{x}, \hat{w})$ this implies that the solution

$$(x(t), \hat{x}(t), \hat{w}(t)) = (\pi(w(t)), \pi(w(t)), w(t))$$

is UGAS. Hence, since $w(t)$ and $\pi(w(t))$ are bounded, every solution of the closed-loop system is also bounded and

$$e(t) = h_r(x(t), w(t)) \to h_r(\pi(w(t)), w(t)) \to 0,$$

as $t \to +\infty$. $\square$