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Fast Identification of Continuous-Time Lur'e-type Systems with Stability Certification

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Abstract: In this paper, we propose an approach for parametric system identification for a class of continuous-time Lur'e-type systems using only steady-state input and output data. Employing a quasi-Newton optimization scheme, we minimize an output error criterion constrained to the set of convergent models, which enforces a stability certificate on the identified model. To compute the steady-state model response efficiently, we adopt the Mixed-Time-Frequency (MTF) algorithm. Furthermore, using the MTF algorithm, we present a method to efficiently compute the gradient of the objective function with any user-defined accuracy. Starting with an initial convergent model estimate, the developed identification algorithm optimizes parameter estimates. The effectiveness of the proposed approach is illustrated in a simulation example.

Keywords: Nonlinear System Identification, Stability Certification.

1. INTRODUCTION

System identification deals with the construction of dynamic models from observed input and output data. Such models are required in model-based controller design or are used to understand the dynamic behavior of complex systems. The identification problem for linear systems has received extensive attention and well-established powerful methods exist, Ljung (1987), together with user-friendly software, e.g., Ljung (1995); Garnier and Gilson (2018). However, many (physical) systems exhibit some form of nonlinearity, motivating the development of identification methods for classes of nonlinear systems.

A practically relevant class of nonlinear systems is the class of Lur'e-type systems, see Khalil (1996). These systems are block structured in the sense that all the linear time-invariant (LTI) dynamics are captured in a LTI block and all the nonlinearities are captured in a static nonlinear block placed in feedback with the LTI block. Both parametric and non-parametric identification methods exist for Lur'e-type systems, e.g., Schoukens and Tiels (2017); Mulders et al. (2014); Vanbeylen (2013); Palanthandalam-Madapusi et al. (2005); Vandersteen and Schoukens (1999).

A common difficulty is the non-convexity of the formulated objective function, where a two step approach is often taken in an attempt to find its global minimum, see Schoukens and Tiels (2017). As a first step, the static nonlinear block is neglected and a Best Linear Approximation (BLA) of the nonlinear system is obtained. As a second step, the non-convex objective function, where all model parameters are involved, is locally minimized using gradient-based optimization techniques starting at the parameter estimates obtained in the first step. In general, no guarantee can be given that this method reaches the global minimum of the objective function.

The second optimization step involves a non-convex objective function, where at each iteration the computation of the response of the nonlinear model with the current parameter estimates is inevitable. For discrete-time systems, the response is obtained in a computationally cheap way by a series of algebraic operations. However, in the case of continuous-time systems, the model response is usually obtained by numerical forward integration of the model dynamics, which is a computationally expensive and time-consuming task. Although the previous references are generally not restricted to discrete-time systems, these do not provide tools to obtain the model response efficiently in the continuous-time case. Furthermore, additional model responses are required to provide numerical estimates for the gradient (and Hessian) of the objective function with respect to the model parameters, see Papalambros and Wilde (2000).

Another challenge is that nonlinear systems can exhibit multiple stable solutions, being attractive for different sets of initial conditions, see Khalil (1996). This is undesired in many situations (Ljung (2010)), e.g., when the model is used for different inputs than those used to fit it. To overcome this problem, Tobenkin et al. (2010, 2017) developed a method imposing incremental stability for a general class of identified nonlinear models. However, the proposed approach is computationally expensive, as it solves a high-dimensional semidefinite programming (SDP) problem. To reduce the computational burden of this method, Umenberger and Manchester (2018) developed a specialized interior point algorithm to solve the formulated SDP problem in a more efficient manner. These approaches perform best if an accurate estimate of the state sequence of the un-

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derlying system generating the observed data is available. Obtaining the state sequence is generally a hard problem for continuous-time systems and rather case specific for nonlinear systems, both in continuous- and discrete-time.

In this paper, we present a *computationally efficient* parametric identification algorithm for a class of continuous-time single-input single-output (SISO) Lur'e-type systems. The proposed approach guarantees that the identified model is exponentially convergent, which is a property of (non)linear systems that guarantees global exponential stability and uniqueness of the steady-state solution, see Pavlov et al. (2004). For continuous-time convergent Lur'e-type systems, the mixed-time-frequency (MTF) algorithm enables fast computation of steady-state solutions under periodic excitations, see Pavlov and van de Wouw (2008). The proposed identification algorithm, therefore, only requires the measured steady-state response of the system to be compared with the steady-state output of the parametric model in an objective function to be minimized. We further present a method, using the MTF algorithm, to compute the gradient of this objective function efficiently and with any user-defined accuracy. This gradient information is required in the gradient-based optimization scheme used to minimize the objective function. The effectiveness of the proposed method is shown in a simulation example. The main benefits of this approach are the computational efficiency of the proposed identification algorithm and the guaranteed stability of the obtained model.

The remainder of this paper is structured as follows. Section 2 introduces the considered class of Lur'e-type systems and recalls sufficient conditions for convergence. Section 3 formulates the identification problem. Section 4 presents the MTF algorithm and also shows its application in the computation of the gradient of the objective function central in the identification problem. Section 5 describes the proposed identification algorithm. Section 6 presents a simulation example. Section 7 closes with concluding remarks and recommendations for future work.

2. CONVERGENT LUR'E-TYPE SYSTEMS

We consider SISO Lur'e-type systems described by

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + \mathcal{H}w(t), \\ y_0(t) &= \mathcal{C}x(t), \\ y(t) &= y_0(t) + e(t) \\ u(t) &= -\phi(y_0(t)), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y_0(t) \in \mathbb{R}$ is the noiseless output, $y(t) \in \mathbb{R}$ is the measured output, $w(t) \in \mathbb{R}$ is the user-defined external input, $\phi(y_0(t)) : \mathbb{R} \to \mathbb{R}$ is a static nonlinear function and $e(t) \in \mathbb{R}$ is additive (sensor) noise. It is assumed that there is no feedthrough from $w(t)$ to $y(t)$, implying that the transfer function $G_{wy}(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{H}$ is strictly proper, where $s \in \mathbb{C}$ is the Laplace variable. Fig. 1 depicts the considered Lur'e-type system schematically.

Stability of forced LTI systems is well-understood; namely, global exponential stability of the origin of the unforced system is sufficient and necessary for globally exponential stability of solutions of the forced system. However, for nonlinear systems, conditions are required to certify the global exponential stability of the solutions of forced nonlinear systems. To this end, we use the notion of convergent dynamics defined as follows.

Definition 1. (Pavlov et al. (2004)). System (1) is said to be globally exponentially convergent if for every input $w(t)$ there exists a solution $\bar{x}(t)$ satisfying the following conditions:

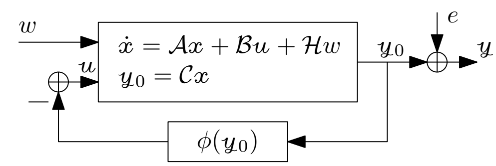

Fig. 1. Schematic representation of Lur'e-type system (1).

- $\bar{x}(t)$ is defined and bounded for all $t \in \mathbb{R}$,
- $\bar{x}(t)$ is globally exponentially stable.

The following theorem provides sufficient conditions for global exponential convergence for system (1).

Theorem 2. (Pavlov et al. (2004); Yakubovich (1964)). Consider system (1). Suppose

C1 The matrix $\mathcal{A}$ is Hurwitz;
C2 There exists a $K > 0$ such that the nonlinearity $\phi(\cdot)$ satisfies $0 \leq \frac{\phi(y_2)-\phi(y_1)}{y_2-y_1} \leq K$ for all $y_1, y_2 \in \mathbb{R}$;
C3 $Re(\mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B}) > -\frac{1}{K}$.

Then, system (1) is globally exponentially convergent according to Definition 1.

Global exponential convergence implies that the effect of initial conditions vanishes and all solutions $x(t)$ converge to the unique globally exponentially stable steady-state solution $\bar{x}(t)$. Furthermore, convergent systems have the property that if the excitation signal $w(t)$ is periodic with period-time T, then the steady-state solution $\bar{x}(t)$ is also periodic with the same period-time T, see Pavlov et al. (2004). In the scope of system identification, this property implies that the (noiseless) steady-state output $\bar{y}_0(t)$ can only contain the same frequencies of the input $w(t)$ and, additionally, higher harmonics of these frequencies. Furthermore, we care to stress that the convergence property formally guarantees the PISPO (periodic input, the same period output) property commonly assumed in literature, e.g., in Schoukens and Tiels (2017).

3. IDENTIFICATION PROBLEM SETTING

The identification problem considered is to estimate the parameters $\theta \in \mathbb{R}^{n_\theta}$ of a Lur'e-type model, that produces the output $y(t,\theta)$ under excitation $w(t)$, which is close to the measured output $y(t)$ quantified by an objective function $J(\theta)$. The available data is the measured steady-state output $\bar{y}(t)$ and the exactly known applied input $w(t)$. We assume that the underlying data-generating system is convergent.

Assumption 1. System (1) satisfies conditions **C1 - C3** of Theorem 2.

Since we only consider steady-state data, being independent of the unobservable and uncontrollable part of the LTI dynamics, we can parametrize the LTI block in the Observability Canonical Form as follows:

$$\begin{aligned} \dot{x}(t,\theta) &= A(\theta)x(t,\theta) + B(\theta)u(t,\theta) + H(\theta)w(t), \\ y(t,\theta) &= Cx(t,\theta), \\ u(t,\theta) &= -\varphi(y(t,\theta),\theta). \end{aligned} \tag{2}$$

The corresponding matrices are

$$A(\theta) = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}, B(\theta) = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, H(\theta) = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix},$$
$$C = [1 \ 0 \ 0 \ \dots \ 0].$$

The total number of parameters n_θ is $3 \times n + n_{NL}$ with n the known state dimension and n_{NL} the number of parameters used to parametrize the nonlinearity φ. The parameters are collected in the vector $\theta^\top = \begin{bmatrix} a_1 \dots a_n \ b_1 \dots b_n \ h_1 \dots h_n \ \theta_{NL}^\top \end{bmatrix}$ with θ_{NL} the parameters defining the static nonlinearity φ. The set $\Theta \subset \mathbb{R}^{n_\theta}$ is defined as the set of parameter vectors θ for which model (2) satisfies conditions **C1 - C3** in Theorem 2 (with $\mathcal{A}, \mathcal{B}, \mathcal{C}, \phi(y)$ replaced by $A(\theta), B(\theta), C, \varphi(y, \theta)$, respectively).

The parametrization of the static nonlinear block is a non-trivial task (Schoukens and Tiels (2017)) where case-specific a priori knowledge can also be employed. One common approach is to parametrize the nonlinearity using basis functions as $\varphi(y(t,\theta),\theta) = \sum_{k=3n+1}^{3n+n_{NL}} \theta_k f_k(y(t,\theta))$, where the basis functions $f_k(y(t,\theta))$ are provided by the user and the coefficients θ_k are the parameters. Such basis functions can be defined in terms of polynomials and trigonometric functions. However, both are unsuitable here since these can not satisfy condition **C2** in Theorem 2, i.e., the nonlinearity will typically not satisfy this incremental sector bound globally. Nonlinearities that are suitable are, e.g., piecewise linear maps, sigmoids, the arctangens function and the hyperbolic tangens function.

Under the assumption of convergence, a periodic excitation $w(t)$ results in a periodic steady-state output $\bar{\underline{y}}(t)$ with the same period-time T. The steady-state error signal over one such period can then be defined as

$$\epsilon(t,\theta) := \bar{y}(t,\theta) - \bar{\underline{y}}(t), \text{ for } t \in [t_0, t_0 + T), \tag{3}$$

where $\bar{y}(t,\theta)$ is the steady-state output of model (2). However, in practice, only a discrete version of the measured $\bar{\underline{y}}(t)$ is available. Therefore, $\epsilon(t,\theta)$ is only defined for $t_k := t_0 + kt_s$ with $k = 0, 1, \dots, N-1$, where t_s is the sampling interval and N is the number of samples in the period. The squared output error is taken as objective function, which is defined as

$$J(\theta) := \frac{1}{N} \sum_{k=0}^{N-1} \epsilon(t_k, \theta)^2. \tag{4}$$

The identification objective is to find the parameter vector $\theta^\star \in \Theta$, which globally minimizes $J(\theta)$ constrained to the set of convergent models characterized by the parameter set Θ, i.e., $\theta^\star = \arg\min_{\theta\in\Theta} J(\theta)$. Since the objective function $J(\theta)$ is highly nonlinear (Schoukens and Tiels (2017)), we employ a gradient-based optimization algorithm to reach a local minimum. In this algorithm, the gradient of the objective function with respect to the parameter vector θ is required, which for (4) reads as

$$\frac{\partial J(\theta)}{\partial \theta} = \frac{2}{N} \sum_{k=0}^{N-1} \epsilon(t_k,\theta) \frac{\partial \epsilon(t_k,\theta)}{\partial \theta}. \tag{5}$$

In (5), the steady-state error $\epsilon(t_k,\theta)$, defined in (3), as well as the gradient of this error $\partial\epsilon(t_k,\theta)/\partial\theta = \partial\bar{y}(t_k,\theta)/\partial\theta$, are required. We will show in the next section that for the considered model structure, we can compute $\epsilon(t_k,\theta)$ and $\partial\epsilon(t_k,\theta)/\partial\theta$ in a computationally efficient and accurate way.

4. COMPUTATION OF STEADY-STATE RESPONSES & GRADIENT INFORMATION

As discussed above, the steady-state output $\bar{y}(t,\theta)$ of model (2) is required at sampling instances t_k in the evaluation of the objective function (4) and the computation of its gradient (5). First, we introduce the MTF algorithm, which enables computation of the steady-state response of convergent Lur'e-type models in a computationally attractive manner. After that, we will show how the MTF algorithm can also be used to compute the gradient of the objective function.

4.1 Mixed-Time-Frequency Algorithm

Pavlov and van de Wouw (2008) developed the MTF algorithm to efficiently compute the steady-state response of convergent Lur'e-type models, which overcomes the drawbacks of numerical forward integration of the dynamics. The algorithm is based on a contraction mapping property that requires the underlying Lur'e-type model to satisfy 'symmetric' sector conditions rather than the conditions stated in Theorem 2, see Pavlov et al. (2013). It is also shown there that any convergent Lur'e-type model of the form (2), satisfying the conditions in Theorem 2 for some K, can be transformed to

$$\begin{aligned} \dot{x}(t,\theta) &= \tilde{A}(\theta)x(t,\theta) + B(\theta)\tilde{u}(t,\theta) + H(\theta)w(t), \\ y(t,\theta) &= Cx(t,\theta), \\ \tilde{u}(t,\theta) &= -\tilde{\varphi}(y(t,\theta),\theta), \end{aligned} \tag{6}$$

where

$$\tilde{A}(\theta) = A(\theta) - \frac{K}{2}B(\theta)C,$$
$$\tilde{\varphi}(y(t,\theta),\theta) = \varphi(y(t,\theta),\theta) - \frac{K}{2}y(t,\theta).$$

The transformed model (6) satisfies the 'symmetric' sector conditions if the original model (2) satisfies the conditions in Theorem 2.

For Lur'e-type models satisfying the 'symmetric' sector conditions, the MTF algorithm computes the steady-state response with any user-defined accuracy. This algorithm computes iteratively the response of the LTI block in frequency-domain, while it computes the response of the static nonlinear block in time-domain, which are both computationally cheap steps. The intermediate signals are transformed between the time- and frequency-domain using the (Inverse) Fast Fourier Transform ((I)FFT).

In a practical implementation, truncation of the number of Fourier coefficients to M frequencies is inevitable. It is shown in Pavlov and van de Wouw (2008) that the MTF algorithm converges for any M and, furthermore, that there exists an upper bound for the mismatch between the 'real' steady-state response $\bar{y}(t,\theta)$ and the response $\bar{y}^M(t,\theta)$ computed using MTF limited to M frequencies. As initial guess for the steady-state solution for this algorithm, *any* response (e.g., the zero response) can be taken. The algorithm terminates when the relative error is smaller than or equal to the user-specified tolerance $Y^\star$.

The MTF algorithm is presented in Algorithm 1. There, we use the transfer functions $G_{yu}(j\omega,\theta) = C(j\omega I - \tilde{A}(\theta))^{-1}B(\theta)$ and $G_{yw}(j\omega,\theta) = C(j\omega I - \tilde{A}(\theta))^{-1}H(\theta)$. Furthermore, we use the ℓ_2 signal norm, which is defined as $||Y||_{\ell_2}^2 := \sum_{m=-\infty}^{\infty} |Y[m]|^2$. Moreover, in Algorithm 1, we dropped the dependency on θ for notational convenience. The computed steady-state response $\bar{y}(t_k,\theta)$ can be used together with the measured response $\bar{y}(t_k)$ to compute the output error (3), which is subsequently used in (4) and (5).

Algorithm 1 Mixed-Time-Frequency Algorithm

1: Calculate $W[m]$ of $w(t)$ for $|m| \leq M$ using FFT.
2: Evaluate LTI dynamics in frequency-domain $Y_0[m] = G_{yw}(jm\omega)W[m]$ for $|m| \leq M$.
3: Compute $y_0(t)$ of $Y_0[m]$ using IFFT.
4: Set iteration counter $i = 0$.
5: **while** $||Y_i - Y_{i-1}||_{\ell_2}/||Y_{i-1}||_{\ell_2} > Y^\star$ **do**
6: Evaluate nonlinearity in time-domain $\tilde{u}_{i+1}(t) = -\tilde{\varphi}(y_i(t))$.
7: Compute $\tilde{U}_{i+1}[m]$ of $\tilde{u}_{i+1}(t)$ using FFT.
8: Evaluate LTI dynamics in frequency-domain $Y_{i+1}[m] = G_{yu}(jm\omega)\tilde{U}_{i+1}[m] + Y_0[m]$ for $|m| \leq M$.
9: Compute $y_{i+1}(t)$ of $Y_{i+1}[m]$ using IFFT.
10: Set $i = i + 1$.

4.2 Gradient Computation

It will be shown next that the gradient $\partial\epsilon(t,\theta)/\partial\theta$, required in (5) at sampling instances t_k, is the steady-state solution of another convergent Lur'e-type model, and, hence, can also be computed using the MTF algorithm. The ideas are based on Pavlov et al. (2013), where a similar method is presented to optimize the steady-state performance of nonlinear controlled systems in Lur'e-type form. However, in that work, the parameters only appeared in the static nonlinear block, whereas in (2), the parameters appear in both the LTI and static nonlinear blocks. Therefore, the following result is a generalization of the result presented in Pavlov et al. (2013).

Theorem 3. Consider system (2). Under the conditions of Theorem 2, if the nonlinearity $\varphi(y,\theta)$ is C^1 in (y,θ), then the corresponding partial derivatives $\partial\bar{x}(t,\theta)/\partial\theta_i$ and $\partial\epsilon(t,\theta)/\partial\theta_i$ are, respectively, the unique T-periodic steady-state solution $\bar{\Psi}_i(t,\theta)$ and the corresponding periodic steady-state output $\bar{\lambda}_i(t,\theta)$ of the following model

$$\begin{aligned} \dot{\Psi}_i(t,\theta) &= A(\theta)\Psi_i(t,\theta) + B(\theta)U_i(t,\theta) + W_i(t,\theta), \\ \lambda_i(t,\theta) &= C\Psi_i(t,\theta), \\ U_i(t,\theta) &= -\frac{\partial\varphi}{\partial y}\left(\bar{y}(t,\theta),\theta\right)\lambda_i(t,\theta), \end{aligned} \tag{7}$$

where

$$\begin{aligned} W_i(t,\theta) =& \frac{\partial A}{\partial\theta_i}(\theta)\bar{x}(t,\theta) - B(\theta)\frac{\partial\varphi}{\partial\theta_i}\left(\bar{y}(t,\theta),\theta\right) \\ &- \frac{\partial B}{\partial\theta_i}(\theta)\varphi(\bar{y}(t,\theta),\theta) + \frac{\partial H}{\partial\theta_i}(\theta)w(t). \end{aligned} \tag{8}$$

Furthermore, model (7) satisfies the conditions of Theorem 2 with $\mathcal{A},\mathcal{B},\mathcal{C},\phi(y)$ replaced by $A(\theta), B(\theta), C$, $\frac{\partial\varphi}{\partial y}\left(\bar{y}(t,\theta),\theta\right)\lambda_i(t,\theta)$, respectively.

Proof. The proof can be found in the Appendix. ■

The above theorem shows that the gradient $\partial\epsilon(t,\theta)/\partial\theta_i$, for $i = 1,2,\ldots,n_\theta$, is the steady-state response of a convergent Lur'e-type model. Hence, after transforming it to the form of (6), its steady-state response, required in (5) at sampling instances t_k, can be computed using the MTF algorithm for every parameter θ_i in θ individually.

Remark 4. Once the steady-state output $\bar{y}(t,\theta)$ of (2) is calculated by the MTF algorithm, the steady-state $\bar{x}(t,\theta)$ required in (8) is calculated in the frequency domain from the linear part of (6).

5. IDENTIFICATION ALGORITHM

The formulated identification problem is a non-convex constrained optimization problem, where the objective function $J(\theta)$ is minimized constrained to the set of convergent models, i.e., $\theta \in \Theta$. To ensure θ staying inside Θ in all optimization iterations, the objective function is modified to include a penalty function $\psi(\theta)$:

$$J_{\text{mod}}(\theta) = J(\theta) + \psi(\theta) \tag{9}$$

with

$$\psi(\theta) = \begin{cases} 0 & \text{if } \theta \in \Theta, \\ \infty & \text{if } \theta \notin \Theta. \end{cases}$$

The modified objective function $J_{\text{mod}}(\theta)$ is unbounded when the parameter vector θ fails to stay in the set Θ, whereas the original objective function $J(\theta)$ might remain bounded as Theorem 2 only expresses sufficient conditions for convergence. Therefore, by evaluating $J_{\text{mod}}(\theta)$, we prevent the updated parameter vector θ_{i+1} to take a value for which it leaves the set Θ and thus guarantee the convergence property in all optimization iterations.

The gradient of the modified objective function $J_{\text{mod}}(\theta)$ equals the gradient of $J(\theta)$ as long as $\theta \in \Theta$. To minimize $J_{mod}(\theta)$, a quasi-Newton scheme is employed in which at iteration i, using gradient and Hessian information, the step-direction d_i is calculated by

$$d_i = -H_i^{-1}\frac{\partial J}{\partial\theta}(\theta_i), \tag{10}$$

where H_i is the Hessian estimated iteratively by the Broyden, Fletcher, Goldfarb, and Shanno (BFGS) method, see Papalambros and Wilde (2000). After that, a line search is performed in the step-direction d_i to find the step-size $\beta_i^\star = \arg\min_\beta J_{mod}(\theta_i + \beta d_i)$ for some range of β. The parameter vector is updated according to $\theta_{i+1} = \theta_i + \beta_i^\star d_i$. The algorithm is terminated when the decrease of the objective function $(J_{i-1} - J_i)$ is smaller than a certain user-defined value $J^\star$ or the number of iterations i exceeds a user-defined threshold $i^{\max}$. Algorithm 2 presents the identification algorithm.

Remark 5. As with any gradient-based optimization routine applied to a non-convex optimization problem, convergence to the global minimum is not guaranteed and depends on the initial parameter estimates.

Algorithm 2 Identification Algorithm

1: Set the counter $i = 0$.
2: Calculate $J(\theta_0)$ via (4).
3: **while** $J_{i-1} - J_i > J^\star$ AND $i \leq i^{\max}$ **do**
4: Calculate gradient $\partial J(\theta)/\partial\theta$ via (5) using (7).
5: Determine step-direction d_i via (10).
6: Perform line search $\beta_i^\star = \arg\min_\beta J_{\text{mod}}(\theta + \beta d_i)$.
7: Update parameter vector $\theta_{i+1} = \theta_i + \beta_i^\star d_i$.
8: Calculate $J(\theta_{i+1})$ via (4).
9: Set $i = i + 1$.

6. SIMULATION EXAMPLE

To illustrate the effectiveness of the identification scheme, we present a simulation example in this section. The system under study consists of second-order LTI dynamics with a static smooth saturation nonlinearity in feedback. Such a nonlinearity is frequently encountered in mechanical systems due to actuator limitation, sensor nonlinearities and other nonlinear phenomena.

The considered system reads as

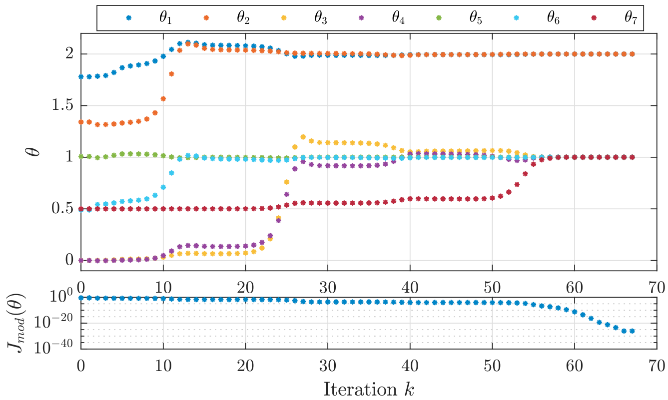

Fig. 2. Iteration history of θ and $J(\theta)$ for $\sigma_e^2 = 0$.

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-\theta_1 & 0\\ -\theta_2 & 1\end{bmatrix} x(t) + \begin{bmatrix}\theta_3\\ \theta_4\end{bmatrix} u(t) + \begin{bmatrix}\theta_5\\ \theta_6\end{bmatrix} w(t),\\ y_0(t) &= [0\ 1]\, x(t), \quad y(t) = y_0(t) + e(t),\\ u(t) &= -\varphi(y_0(t)) = -\tanh\left(\theta_7 y_0(t)\right),\end{aligned} \tag{11}$$

where $e(t)$ is a noise distortion taken from a normal distribution with variance σ_e^2, i.e., $e \sim \mathcal{N}(0, \sigma_e^2)$, and kept constant in between samples. Parameter θ_7 represents the slope of the saturation nonlinearity $\varphi(y_0(t))$ at $y_0(t) = 0$. The true parameter vector $\theta_0 \in \Theta$ is chosen as follows:

$$\theta_0 = [2\ 2\ 1\ 1\ 1\ 1\ 1]^\top. \tag{12}$$

The excitation signal $w(t)$ is a random-phase multisine $w(t) = \sum_{l=1}^{20} A_l \sin(2\pi l f_0 t + \vartheta_l)$ with the base frequency $f_0 = 0.1$ Hz (period time $T = 10$ sec), amplitudes $A_l = 10\,\forall\, l$ and random phases ϑ_l taken in $[0, 2\pi)$. The number of samples $N = 2048$, which implies $M = 1024$ Fourier coefficients taken in the MTF algorithm.

Algorithm 2 minimizes a non-convex function $J_{mod}(\theta)$ defined in (9), which, possibly, contains (many) local minima. Therefore, as a starting point, initial parameter estimates are required. For such initialization, we treat the model as if it contains no nonlinearity in the feedback loop and thus only contains LTI dynamics that are excited in open loop. To this extent, the matrices A, H, and C are initialized using the standard Matlab implementation of N4SID (Van Overschee and De Moor (2012)), and are refined using the prediction-error-method where stability can be enforced (Ljung (1995)). The matrix B is initially set to zero, i.e., $\theta_3 = \theta_4 = 0$. For θ_7, any non-zero initial value can be taken, here 0.5 is chosen. This initialization procedure, where A is enforced to be Hurwitz and $B = 0$ (implying no feedback), guarantees a convergent initial model satisfying the conditions of Theorem 2.

First we discuss the noiseless case $\sigma_e^2 = 0$, for which the results are depicted in Fig. 2. The initial estimates for θ, i.e., at iteration $k = 0$ obtained by the method described above, differ from the true values θ_0. Nevertheless, running Algorithm 2, the parameter estimates converge to their true values θ_0 in (12) in 68 iterations. In every iteration, the objective function $J_{mod}(\theta)$ decreases and, ultimately, reaches (almost) 0. The total time to go through all 68 iterations, in which a total of $68 \times 8 = 544$ model responses are computed, is only 2.7 seconds (on an Intel Core i7-7700HQ, 2.8GHz processor).

Next, results for noise variances $\sigma_e^2 = \{10^{-6}, 10^{-4}, 10^{-2}\}$ are discussed. For each σ_e^2 case, 10 trials are performed. In each such trial, a new noise realization is added to the noiseless output $y_0(t_k)$ to obtain the distorted output $y(t_k)$. The total of 30 trials are presented in Fig. 3, where initial (red dots) and final (black dots) values of the seven parameters are depicted individually and the objective function $J_{mod}(\theta)$ is also shown. It can be seen that the initial estimates are biased with respect to their respective true values depicted in blue. Furthermore, a larger noise variance results in a larger variance of the initial estimates. It can be seen that almost all parameter estimates, resulting from Algorithm 2, are close to their true ones, especially in the cases of $\sigma_e^2 = 10^{-6}$ and $\sigma_e^2 = 10^{-4}$, i.e., the first 20 trials. In the last 10 trials, where a large noise variance of $\sigma_e^2 = 10^{-2}$ is used, in particular, parameter θ_7 has a large variance. This might be due to the product of the output of the static nonlinear function $\varphi(y(t,\theta),\theta)$, parametrized by θ_7, with the matrix $B(\theta)$, parametrized by θ_3 and θ_4, in the dynamics. Additional research is required to explain the specific variances on the final parameter estimates. The minimum of the objective function $J(\theta)$ approximates the noise variance σ_e^2, which is expected since $J(\theta)$ is a measure for the variance of $\epsilon(t_k,\theta)$ in (3). In each trial, approximately 500 model responses are computed on average taking only 2.4 seconds (on a Intel Core i7-7700HQ, 2.8GHz processor), which shows the computational efficiency of the algorithm.

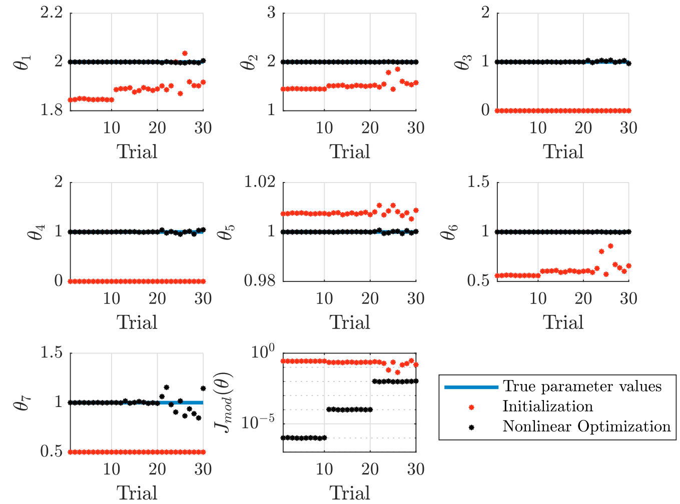

Fig. 3. Simulation results for different noise intensities. The first 10 trials correspond to $\sigma_e^2 = 10^{-6}$, the second 10 to $\sigma_e^2 = 10^{-4}$ and the last 10 to $\sigma_e^2 = 10^{-2}$.

Remark 6. To reduce the effects of $e(t)$ in practice, we can average over several periods of the measured output $y(t)$.

7. CONCLUSION & FUTURE WORK

In this paper, an identification approach has been proposed for convergent continuous-time Lur'e-type systems. The benefit of the proposed approach is that 1) it is computationally attractive and 2) it guarantees that the identified model preserves the convergence property.

Future work includes the development of dedicated initialization methods, extension of the identification scheme to more general forms of nonlinear (feedback) systems and to the Multiple-Input Multiple-Output (MIMO) case.

Appendix A. PROOF OF THEOREM 3

The proof is given for the scalar case of θ. The proof can be repeated analogously for each component of a vector-valued θ. First, consider the following property of Lur'e-type models (2) satisfying the conditions of Theorem 2.

Property 7. (Pavlov et al. (2013)). Consider model (2). Under the conditions of Theorem 2, if $\theta_1(h)$ converges to θ_2 as $h \to 0$, and T-periodic $w_1(t,h)$ converges to T-periodic $w_2(t)$ uniformly in $t \in [t_0, t_0+T)$, then the corresponding steady-state solution $\bar{x}_{w_1(h)}(t, \theta_1(h))$ converges to $\bar{x}_{w_2}(t, \theta_2)$ uniformly in $t \in [t_0, t_0+T)$ as $h \to 0$.

Let us show that for $z(t,h) := \frac{1}{h}(\bar{x}(t,\theta+h) - \bar{x}(t,\theta))$ there exists the limit $\lim_{h\to 0} z(t,h)$. For notational convenience, we drop the argument t from here on and we define $\theta^+ := \theta + h$.

As follows from the definition of the steady-state solution $\bar{x}(\theta)$ of the model (2), for $h \neq 0, z(t,h)$ is a T-periodic function satisfying the dynamics

$$\begin{aligned}\frac{dz(t,h)}{dt} =& \frac{1}{h}\left(A(\theta^+)\bar{x}(\theta^+) - A(\theta)\bar{x}(\theta)\right) + \\ & \frac{1}{h}\left(-B(\theta^+)\varphi(\bar{y}(\theta^+),\theta^+) + B(\theta)\varphi(\bar{y}(\theta),\theta)\right) + \\ & \frac{1}{h}\left(H(\theta^+) - H(\theta)\right)w,\end{aligned}$$

which can be written as

$$\begin{aligned}\frac{dz(t,h)}{dt} =& \frac{1}{h}\left(A(\theta^+)\bar{x}(\theta^+) - A(\theta)\bar{x}(\theta)\right) + \\ & \frac{1}{h}\left(A(\theta^+)\bar{x}(\theta) - A(\theta^+)\bar{x}(\theta)\right) + \\ & \frac{1}{h}\left(-B(\theta^+)\varphi(\bar{y}(\theta^+),\theta^+) + B(\theta)\varphi(\bar{y}(\theta),\theta)\right) + \\ & \frac{1}{h}\left(-B(\theta^+)\varphi(\bar{y}(\theta),\theta) + B(\theta^+)\varphi(\bar{y}(\theta),\theta)\right) + \\ & \frac{1}{h}\left(-B(\theta^+)\varphi(\bar{y}(\theta^+),\theta) + B(\theta^+)\varphi(\bar{y}(\theta^+),\theta)\right) + \\ & \frac{1}{h}\left(H(\theta^+) - H(\theta)\right)w.\end{aligned}$$

Applying the mean value theorem leaves

$$\begin{aligned}\frac{dz(t,h)}{dt} &= A(\theta^+)z + \frac{\partial A}{\partial \theta}(\alpha)\bar{x}(\theta) \\ &+ B(\theta^+)\left(-\frac{\partial \varphi}{\partial \theta}(\bar{y}(\theta),\beta) - \frac{\partial \varphi}{\partial \bar{y}(\theta)}(\gamma,\theta)\frac{\partial \bar{y}}{\partial \theta}(\theta)\right) \\ &- \frac{\partial B}{\partial \varphi}(\zeta)\varphi(\bar{y}(\theta),\theta) + \frac{\partial H}{\partial \theta}(\eta)w,\end{aligned} \tag{A.1}$$

for some $\alpha, \beta, \zeta, \eta \in (\theta, \theta^+)$ and $\gamma \in (\bar{y}(\theta), \bar{y}(\theta^+))$. Thus, we conclude that for a sufficiently small $h \neq 0, z(t,h)$ is a T-periodic solution of the model (A.1) with inputs $w, \bar{x}(\theta)$ and γ. Taking the limit $\lim_{h\to 0} z(h,t)$ and applying Property 7 implies the convergence of $\alpha, \beta, \zeta, \eta$ to θ and γ to $\bar{y}(\theta)$ uniformly in time t, which allows to write (A.1) as (7) with inputs $w, \bar{x}(\theta)$ and $\bar{y}(\theta)$.

Let us next prove that model (7) satisfies conditions **C1-C3** of Theorem 2. Condition **C1** is satisfied since the matrices A, B and C of the LTI block of (7) are the same as those of model (2). Condition **C2** holds with the same K as for the model (2) since $0 \leq \partial\varphi/\partial\bar{y}(\bar{y}(\theta),\theta) \leq K$ for all $\bar{y}(\theta)$ and θ. Condition **C3** holds automatically since A, B, C, and K remain unchanged. Application of Theorem 2 to model (7) concludes that for the T-periodic input $[\bar{x}(\theta), w, \bar{y}(\theta)]$, model (7) has a unique T-periodic solution $\bar{\Psi}(\theta)$. Moreover, the limit $\lim_{h\to 0} z(t,h) = \partial\bar{x}/\partial\theta(\theta) = \bar{\Psi}(\theta)$. Finally, it is straightforward to show that $\partial\epsilon/\partial\theta(\theta) = \partial\bar{y}/\partial\theta(\theta) = \bar{\lambda}(\theta)$, which is the T-periodic output of model (7).

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