On Extended Model Order Reduction for Linear Time Delay Systems

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Abstract This chapter presents a so-called extended model-reduction technique for linear delay differential equations. The presented technique preserves the infinite-dimensional nature of the system and facilitates the preservation of properties such as system parameterizations (uncertainties). It is proved in this chapter that the extended model-reduction technique also preserves stability properties and provides a guaranteed a-priori bound on the reduction error. The reduction technique relies on the solution of matrix inequalities that characterize controllability and observability properties for time delay systems. This work presents conditions on the feasibility of these inequalities, and studies the applicability of the extended model reduction to a spatio-temporal model of neuronal activity, known as delay neural fields. Lastly, it discusses the relevance of this technique in the scope of model reduction of uncertain time delay systems, which is supported by a numerical example.

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1 Introduction

Models in terms of delay differential equations have extensively been used to describe engineering systems such as, e.g., mechanical and electric/electronic systems [1, 25]. Systems of delay differential equations have also been used to model phenomena in, for instance, economics and biology [18]. Such models can, however, be complex in the sense that they consist of a large number of delay equations. This complexity can handicap simulation, analysis, or controller synthesis and implementation. This work presents a model order reduction technique to address the issue of model complexity of time delay systems.

In the course of the past four decades, a myriad of model order reduction techniques have been proposed for linear delay-free systems. Balanced truncation [22] is probably the most popular of these (see [15] for an overview). Parallel to these efforts, the model-reduction problem of time delay systems has also been studied, though to a much lesser extent. A common approach in the model complexity reduction of time delay systems is approximating the time delay system by a finite-dimensional model of, potentially, low order [2, 19, 20]. This approach has been motivated by the fact that currently analysis and design based on finite-dimensional models is in general more appealing, as it allows for the use of well-developed classical systems and control theory. Nonetheless, delay-structure preserving methods, i.e., methods that preserve the infinite-dimensional nature of the time delay system during model reduction, have also gained considerable attention [3–5, 16, 23, 28, 32, 33]. This attention is because reliable analysis and controller synthesis techniques are available today also for time delay systems [11, 21]. In addition, for a particular order of the reduced model, a reduced model in terms of delay differential equations has the potential to be more accurate than a finite-dimensional approximation of the same order [26]. In addition to the delay structure, in many cases, it is beneficial to preserve other desirable properties of the original model in the reduced-order model. Important examples are stability properties, structures of physical interconnections (e.g., the interconnection of a system and a controller) and the presence of uncertainties and model parameters. This chapter presents such a robust/parameterized model-reduction techniques for linear time delay systems.

This chapter is an extension of the work in [24], which introduced a so-called *extended* balanced truncation procedure for time delay systems. This procedure was motivated by the technique of extended-balanced truncation for finite-dimensional systems in [27, 29]. Following [4, 23], the work [24] defined bounds on the controllability and observability energy functionals of time delay systems, and constructed a model-reduction procedure based upon those. These bounds were characterized by matrices which are solutions to a set of matrix inequalities. Compared to the results in [23], extended balanced truncation comes with additional degrees of freedom in the computation of (bounds on) these functionals through the use of slack variables. It has been shown that the proposed technique is useful for the structured model reduction of closed-loop time delay systems and also for delay systems with polytopic parametrizations/uncertainties. It preserves both asymptotic stability and the

infinite-dimensional nature of the time delay system, while also providing an a-priori computable, guaranteed, delay-dependent error bound.

The contributions of this chapter are fourfold. First, the feasibility of the matrix inequalities in [24] is studied in detail by presenting necessary and sufficient conditions on the feasibility of those matrix inequalities. Crucial results in [24] on the error bound and preservation of stability were lacking mathematical proofs. As a second contribution, this work presents the missing proofs for those results. Third, it studies and numerically illustrates the effectiveness of the extended balancing approach for parameterized/robust model reduction of time delay systems. Lastly, this work studies the applicability of the extended model-reduction technique to models in neuroscience. Namely, a method for dropping the spatial dependency in a particular model of neural fields is presented. This leads to a high-order time delay system, and the extended model-reduction technique is then applied to reduce the order of the resulting neural model without spatial dependency. This contribution is presented as a numerical example.

Outline. After introducing notation, a problem statement is given in Sect. 2. Section 3 introduces and gives a characterization of the observability and controllability energy functionals of a time delay system. Section 4 recapitulates the proposed model order reduction procedure in [24] and provides novel detailed proofs, and easy-to-check feasibility conditions for it are discussed in Sect. 5. The application of this technique to delay neural fields and robust/parameterized model reduction of delay systems is elaborated on in Sects. 6, and 7, respectively, and conclusions are presented in Sect. 8.

Notation. The set of real (non-negative) numbers is indicated by $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$), and the Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$, which is defined as $|x| := \sqrt{x^T x}$. The notation $\mathcal{L}_2([a,b],\mathbb{R}^n)$ is the space of functions $x : [a,b] \to \mathbb{R}^n$ which have a bounded norm $\|x\|_2 = (\int_a^b |x(t)|^2\, dt)^{1/2}$, whereas $\mathcal{L}_\infty([a,b],\mathbb{R}^n)$ is the space of bounded, piecewise continuous functions mapping $[a,b]$ onto $\mathbb{R}^n$. The Banach space of absolutely continuous functions which map the interval $[-\tau, 0]$ onto $\mathbb{R}^n$ is indicated by $\mathcal{C}_n = \mathcal{C}([-\tau,0],\mathbb{R}^n)$. Furthermore, $\mathcal{W}_n = \mathcal{W}([-\tau,0],\mathbb{R}^n)$ refers to the space of bounded functions $\varphi \in \mathcal{C}_n$ with square-integrable derivative in a weak sense, i.e., $\dot{\varphi} \in \mathcal{L}_2([-\tau,0],\mathbb{R}^n)$ for $\varphi \in \mathcal{W}_n$. [12, 18]. A block-diagonal matrix with A_1, ..., A_m on the diagonal is represented as $\text{blkdiag}\{A_1, \cdots, A_m\}$, and I_m is the $m \times m$ identity matrix. The notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$, means that P is a symmetric, positive definite matrix. Matrix transposition and conjugate transposition are shown by the superscripts T and H, respectively. A star $*$ in a symmetric matrix represents a symmetric term.

2 Problem Statement

In this chapter, we consider a time delay system Ω of the form

$$\Omega : \begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bu(t), \\ y(t) = Cx(t) + C_d x(t-\tau) + Du(t), \\ \quad x_0 = \varphi. \end{cases} \tag{1}$$

Here, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the external input and the output, respectively, while τ is a constant time delay. We assume that for all $\tau \in [0, \bar{\tau}]$, with a constant $\bar{\tau} > 0$, the system is asymptotically stable for zero input. For $t \in \mathbb{R}$, the function segment $x_t : [-\tau, 0] \to \mathbb{R}^n$ denotes the state of Ω at the time instance t, where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0]$. The initial condition of the system is given by $\varphi \in \mathcal{C}_n$, such that $x(t) = \varphi(t)$, $t \in [-\tau, 0]$.

The objective is to approximate Ω by an asymptotically stable model $\hat{\Omega}$ of order $k < n$ which has the same delay structure as Ω. Moreover, the input-output behavior of $\hat{\Omega}$ should be close enough, in some measurable sense, to that of Ω. In addition, the model-reduction procedure itself should be applicable to time delay systems with polytopic uncertainties/parameterizations and it should facilitate structured model order reduction (that is, a model order reduction procedure which preserves physical interconnection structures in a system) for time delay systems.

It is noted that since the state of Ω belongs to $\mathcal{C}_n$, it has an infinite-dimensional nature in addition to the, potentially large, finite number of dynamical equations (i.e., state equations) describing it. In this chapter, model order reduction is pursued with respect to only the latter aspect.

3 Observability and Controllability Inequalities

Following [23, 24], we will discuss a model-reduction procedure for time delay systems based on so-called energy functionals.

First, the observability energy functional characterizes the output energy of (1) for a non-zero initial condition and zero input, and it can thus be regarded as a measure of observability. More precisely, we have the following definition taken from [4] (see [16] for a similar definition).

Definition 1 The observability functional of the system (1) is the functional $L_o : \mathcal{C}_n \to \mathbb{R}_{\geq 0}$ defined as

$$L_o(\varphi) = \int_0^\infty |y(t)|^2 dt, \tag{2}$$

where $y(\cdot)$ is the output of the system (1) for the initial condition $x_0 = \varphi$ and zero input $u = 0$.

In addition to the observability functional, the development of a balancing-based model-reduction procedure requires information on the controllability properties of the time delay system. In this regard, we consider the following definition of the controllability functional as a measure of controllability, see again [4] (and [16]).

Definition 2 The controllability functional of the system (1) is the functional $L_c : \mathcal{D}_n \to \mathbb{R}_{\geq 0}$ defined as

$$L_c(\varphi) = \inf\left\{ \int_{-\infty}^{0} |u(t)|^2 dt \,\middle|\, u \in \mathcal{L}_2 \cap \mathcal{L}_\infty\big((-\infty, 0], \mathbb{R}^m\big),\ \lim_{T\to\infty} x_{-T} = 0,\ x_0 = \varphi \right\}, \tag{3}$$

where x_t is the solution of (1) for u that satisfies the above and $\mathcal{D}_n \subset \mathcal{C}_n$ is the domain of L_c, that is the space of function segments φ for which $L_c(\varphi)$ is well defined.

Generally, the a-priori computation of the observability and controllability functionals (2) and (3) is a challenging task [16]. The following lemmas from [24] present quadratic functionals characterized by computable matrices which can provide a tight upper and lower bound of $L_o(\varphi)$ and $L_c(\varphi)$, respectively.

Lemma 1 *Consider the asymptotically stable system Ω in (1). Let there exist matrices $Q > 0$, $Q_d > 0$, $\bar{Q} > 0$ and $S > 0$, and a scalar α_o for which*

$$M_o = \begin{bmatrix} SA + A^T S + Q_d - \bar{Q} & \bar{Q} + SA_d & Q - S + \alpha_o A^T S & C^T \\ * & Q_d - \bar{Q} & \alpha_o A_d^T S & C_d^T \\ * & * & -2\alpha_o S + \tau^2 \bar{Q} & 0 \\ * & * & * & -I_p \end{bmatrix} < 0 \tag{4}$$

holds. Then the functional $E_o : \mathcal{W}_n \times \mathcal{L}_2([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ given by

$$E_o(\varphi, \dot{\varphi}) = \varphi^T(0) Q \varphi(0) + \int_{-\tau}^{0} \varphi^T(s) Q_d \varphi(s)\, ds + \tau \int_{-\tau}^{0} \int_{\theta}^{0} \dot{\varphi}^T(s) \bar{Q} \dot{\varphi}(s)\, ds d\theta, \tag{5}$$

satisfies

$$E_o(\varphi, \dot{\varphi}) \geq L_o(\varphi), \tag{6}$$

for each $\varphi \in \mathcal{W}_n$ and with the functional L_o as in Definition 1.

Proof The proof of this lemma can be found in [24]. □

Lemma 2 *Consider the time delay system in (1). Let there exist matrices $P > 0$, $P_d > 0$, $\bar{P} > 0$ and $R > 0$, and a positive scalar α_c which satisfy*

$$M_c = \begin{bmatrix} AR + RA^T + P_d - \bar{P} & \bar{P} + A_d R & P - R + \alpha_c R A^T & B \\ * & -P_d - \bar{P} & \alpha_c R A_d^T & 0 \\ * & * & -2\alpha_c R + \tau^2 \bar{P} & \alpha_c B \\ * & * & * & -I_m \end{bmatrix} < 0. \tag{7}$$

Then the functional $E_c : \mathcal{W}_n \times \mathcal{L}_2([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ *given by*

$$E_c(\varphi, \dot{\varphi}) = \varphi^T(0)U\varphi(0) + \int_{-\tau}^{0} \varphi^T(s)U_d\varphi(s)\,ds + \tau\int_{-\tau}^{0}\int_{\theta}^{0} \dot{\varphi}^T(s)\bar{U}\dot{\varphi}(s)\,ds d\theta, \quad (8)$$

with $U = R^{-1}PR^{-1}$, $U_d = R^{-1}P_dR^{-1}$, $\bar{U} = R^{-1}\bar{P}R^{-1}$, *satisfies*

$$E_c(\varphi, \dot{\varphi}) \leq L_c(\varphi), \quad (9)$$

for all $\varphi \in \mathcal{D}_n \cap \mathcal{W}_n$ *and* L_c *as in Definition 2.*

Proof The proof has been omitted for the sake of brevity.

Remark 1 The variables S, α_o in (2), and R, α_c in (7) are referred to as the slack variables. By contrast, Q, Q_d, $\bar{Q}$ and U, U_d, $\bar{U}$ (also P, P_d, $\bar{P}$) which characterize the energy functionals (5) and (8), respectively, are referred to as the main decision variables.

The next section recalls the proposed model-reduction procedure in [24] and provides proofs for the technical results not provided in [24].

4 Model order reduction by truncation

Consider a partitioning of $x(t)$ and x_t (and φ) as

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \; x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}, \; \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \quad (10)$$

where $x_1(t) \in \mathbb{R}^k$ and $\varphi_1 \in \mathcal{W}_k$, with $k < n$ and together with the corresponding partitioning of the system matrices

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; A_d = \begin{bmatrix} A_{d,11} & A_{d,12} \\ A_{d,21} & A_{d,22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \; C_d = \begin{bmatrix} C_{d,1} & C_{d,2} \end{bmatrix}. \end{aligned} \quad (11)$$

A reduced-order approximation of (1), denoted by $\hat{\Omega}$, is obtained by truncation of the dynamics that correspond to x_2, leading to

$$\hat{\Omega} : \begin{cases} \dot{\zeta}(t) = A_{11}\zeta(t) + A_{d,11}\zeta(t-\tau) + B_1u(t), \\ \hat{y}(t) = C_1\zeta(t) + C_{d,1}\zeta(t-\tau) + Du(t), \\ \quad \zeta_0 = \hat{\varphi}, \end{cases} \quad (12)$$

where $\zeta(t) \in \mathbb{R}^k$ and $\hat{y}(t) \in \mathbb{R}^p$ approximates $y(t)$, and $\hat{\varphi} \in \mathcal{W}_k$ is the initial condition of $\hat{\Omega}$.

The system $\hat{\Omega}$ approximates x_1 in the partitioned coordinates, and it clearly captures the delay structure of the original system Ω. In the sequel, it is shown that this type of model approximation can preserve other properties of the original model in the reduced-order model provided that the matrices S and R have a certain structure. First, we define an extended-balanced realization of Ω.

Definition 3 A realization as in (1) is said to be extended balanced if there exist matrices $S > 0$, $Q > 0$, $Q_d > 0$, $\bar{Q} > 0$, and a scalar α_o satisfying (4), matrices $R > 0$, $P > 0$, $P_d > 0$, $\bar{P} > 0$, and a scalar α_c satisfying (7), and, additionally, S and R are such that

$$S = R = \Sigma = \text{blkdiag}\{\sigma_1 I_{m_1}, \sigma_2 I_{m_2}, \cdots, \sigma_q I_{m_q}\}. \tag{13}$$

Here, the constants $\sigma_i > 0$, which satisfy $\sigma_i > \sigma_{i+1}$, $i \in \{1, ..., q-1\}$, are extended singular values of multiplicities m_i and $\Sigma_{i=1}^{q} m_i = n$.

Since S and R are symmetric, positive definite matrices, the system (1) can always be transformed into an extended-balanced form by exploiting the standard balancing transformation [10].

Lemma 3 *Let there exist symmetric matrices $S > 0$, $Q > 0$, $Q_d > 0$ and $\bar{Q} > 0$, and a scalar α_o satisfying (4), and symmetric matrices $R > 0$, $P > 0$, $P_d > 0$ and $\bar{P} > 0$, and a scalar α_c satisfying (7). Then, there exists a coordinate transformation $x(t) = Tz(t)$, with $T \in \mathbb{R}^{n\times n}$, such that the realization in the new coordinates is extended balanced.*

An interesting feature of the presented model order reduction is that it guarantees the preservation of stability properties, as stated in the following theorem.

Theorem 1 *Let the system (1), which is asymptotically stable for zero input, be in an extended-balanced realization and consider the reduced-order system (12) obtained by truncation for $k \geq 1$. Then, the reduced-order system $\hat{\Omega}$ is asymptotically stable for zero input.*

Proof As the system (1) is an extended-balanced realization, there exist a diagonal matrix $S > 0$, and matrices $Q > 0$, $Q_d > 0$ and $\bar{Q} > 0$, and a scalar α_o such that (4) holds. Thus, for any full-column rank matrix Ψ of appropriate dimensions it holds that

$$\Psi^T M_o \Psi < 0 \tag{14}$$

with M_o as in (4). Since S is diagonal (recall Definition 3), we can write it in a block-diagonal form as $S = \text{blkdiag}\{S_1, S_2\}$, where $S_1 \in \mathbb{R}^{k\times k}$ corresponds to the reduced model $\hat{\Omega}$ and S_2 to the truncated dynamics. Now, we choose $\Psi = \text{blkdiag}\{\psi, \psi, \psi\}$, with $\psi = [I_k \;\; 0_{k\times(n-k)}]^T$. With this choice of Ψ and exploiting the block-diagonal structure of S, (14) implies that

$$\Psi^T M_o \Psi = \begin{bmatrix} A_{11}^T S_1 + S_1 A_{11} + Q_{d,11} - \bar{Q}_{11} & * & * \\ A_{d,11}^T S_1 + \bar{Q}_{11} & -Q_{d,11} - \bar{Q}_{11} & * \\ Q_{11} - S_1 + \alpha_o S_1 A_{11} & \alpha_o S_1 A_{d,11} & -2\alpha_o S_1 + \tau^2 \bar{Q}_{11} \end{bmatrix} < 0, \quad (15)$$

where $Q_{11} > 0$, $Q_{d,11} > 0$, $\bar{Q}_{11} > 0$ are the upper left $k \times k$ blocks of $Q > 0$, $Q_d > 0$ and $\bar{Q} > 0$, respectively. Now, using results from [11, Chapter 3], it is easily verified that (15) is a sufficient condition for the asymptotic stability of the reduced-order system for all time delays in the interval $[0, \tau]$. It should be mentioned that one may use the inequality (7) to prove this theorem in a similar way. □

The availability of an a-priori computable error bound is an appealing property of the presented model order reduction technique. The next theorem presents this property.

Theorem 2 *Let the asymptotically stable system Ω as in (1) be in an extended-balanced realization, as defined in Definition 3, and consider the reduced-order system $\hat{\Omega}$, as in (12), obtained by truncation for $k = \Sigma_{i=1}^{r} m_i$ for some $r > 0$. Moreover, let $\alpha_o = \alpha_c = \alpha$. Then, for any common input function $u \in \mathcal{L}_2 \cap \mathcal{L}_\infty([0, T], \mathbb{R}^m)$ and initial conditions $\varphi = 0$ and $\hat{\varphi} = 0$ for (1) and (12), respectively,*

$$\int_0^T \left| y(t) - \hat{y}(t) \right|^2 dt \leq \varepsilon^2 \int_0^T |u(t)|^2 \, dt,$$

for all $T \geq 0$ and where the error bound ε is given as

$$\varepsilon = 2 \sum_{i=r+1}^{q} \sigma_i, \quad (16)$$

with σ_i as in (13).

Before presenting a proof for this theorem, we give a technical lemma which can be proved based on results in [9].

Lemma 4 *Consider a system of the form (1). If $x_{t_0} \in \mathcal{W}_n$ at $t_0 \in \mathbb{R}_{\geq 0}$ and $u \in \mathcal{L}_\infty([t_0, t_1], \mathbb{R}^m)$ for $t_1 \geq t_0$, then $x_t \in \mathcal{W}_n$ for all $t \in [t_0, t_1]$.*

Now, we prove Theorem 2.

Proof To prove this theorem, we take a one-step reduction approach. To this end, we first take a reduced-order system of the form (12) which is obtained by truncating the states corresponding to the final extended singular value σ_q, leading to a reduced-order model with $k = n - m_q$. Next, we define auxiliary states

$$z(t) := \begin{bmatrix} x_1(t) - \zeta(t) \\ x_2(t) \end{bmatrix}, \quad w(t) := \begin{bmatrix} x_1(t) + \zeta(t) \\ x_2(t) \end{bmatrix}. \quad (17)$$

Using (1) and (12) for zero initial conditions, the definitions in (17) lead to the dynamics

$$\begin{aligned}\dot{z}(t) &= Az(t) + A_d z(t-\tau) + \bar{B}\bar{u}(t),\\ \delta y(t) &= Cz(t) + C_d z(t-\tau),\end{aligned} \tag{18}$$

and

$$\dot{w}(t) = Aw(t) + A_d w(t-\tau) + 2Bu(t) - \bar{B}\bar{u}(t), \tag{19}$$

where $\bar{u}^T(t) = [\,\zeta^T(t)\ \zeta^T(t-\tau)\ u^T(t)\,]$, $\bar{B}^T = [\,0\ \bar{B}_2^T\,]$, with $\bar{B}_2 = [\,A_{21}\ A_{d,21}\ B_2\,]$, and $\delta y(t) = y(t) - \hat{y}(t)$ is the output of the error system. Now, based on the auxiliary dynamics and the observability and controllability functionals in (5) and (8), a functional is introduced as

$$V(z_t, w_t, \dot{z}_t, \dot{w}_t) = E_o(z_t, \dot{z}_t) + \sigma_q^2 E_c(w_t, \dot{w}_t), \tag{20}$$

which is well defined as $z, w \in \mathcal{W}_n$ (u is assumed to be piecewise continuous and bounded) due to Lemma 4. Similar to the proof of Lemma 1 in [24], it can be shown that the time-derivative of V along the trajectories of (18) and (19) is upper bounded by

$$\begin{aligned}\dot{V}(z_t, w_t, \dot{z}_t, \dot{w}_t) \le\ & \xi_z^T(t)\bar{M}_o\xi_z(t) + \sigma_q^2\xi_w^T(t)\bar{M}_c\xi_w(t) - |\delta y(t)|^2\\ &+ \left(2\sigma_q\right)^2|u(t)|^2 + 2\left(z^T(t) + \alpha_o\dot{z}^T(t)\right)S\bar{B}\bar{u}(t)\\ &- 2\sigma_q^2\left(w^T(t) + \alpha_c\dot{w}^T(t)\right)R^{-1}\bar{B}\bar{u}(t),\end{aligned} \tag{21}$$

where $\bar{M}_o$ is obtained by applying a Schur complement to M_o defined in (4) and

$$\bar{M}_c := \text{blkdiag}\{R, R, R, I_m\}^{-T} M_c \text{blkdiag}\{R, R, R, I_m\}^{-1},$$

with M_c and R as in (7), and

$$\begin{aligned}\xi_z^T(t) &:= [\,z^T(t)\ z^T(t-\tau)\ \dot{z}^T(t)\,],\\ \xi_w^T(t) &:= [\,w^T(t)\ w^T(t-\tau)\ \dot{w}^T(t)\ u^T(t)\,].\end{aligned}$$

Given that $\bar{M}_o < 0$ and $\bar{M}_c < 0$ due to (4) and (7), (21) further implies that

$$\begin{aligned}\dot{V}(z_t, w_t, \dot{z}_t, \dot{w}_t) \le\ & -|\delta y(t)|^2 + \left(2\sigma_q\right)^2|u(t)|^2 + 2\left(z^T(t) + \alpha_o\dot{z}^T(t)\right)S\bar{B}\bar{u}(t)\\ &- 2\sigma_q^2\left(w^T(t) + \alpha_c\dot{w}^T(t)\right)R^{-1}\bar{B}\bar{u}(t).\end{aligned} \tag{22}$$

Also, recalling that S and R have diagonal structures due to the extended-balanced form of the high-order system (see Definition 3), the time-derivative of V in (22) satisfies

$$\begin{aligned}\dot{V}(z_t, w_t, \dot{z}_t, \dot{w}_t) \le\ & -|\delta y(t)|^2 + \left(2\sigma_q\right)^2|u(t)|^2\\ &+ 2\left(z_2^T(t) + \alpha_o\dot{z}_2^T(t)\right)S_2\bar{B}_2\bar{u}(t) - 2\sigma_q^2\left(w_2^T(t) + \alpha_c\dot{w}_2^T(t)\right)R_2^{-1}\bar{B}_2\bar{u}(t),\end{aligned} \tag{23}$$

where S_2 and R_2 are the lower right $m_q \times m_q$ blocks of S and R, respectively.

Next, using the facts that $S_2 - \sigma_q^2 R_2^{-1} = 0$, $w_2 = z_2 = x_2$, for $\alpha_o = \alpha_c = \alpha$, we obtain

$$\dot{V}(z_t, w_t, \dot{z}_t, \dot{w}_t) \leq -|\delta y(t)|^2 + (2\sigma_q)^2 |u(t)|^2,$$

. Now, integrating the above over the interval $[0, T]$ gives

$$V(z_T, w_T, \dot{z}_T, \dot{w}_T) - V(z_0, w_0, \dot{z}_0, \dot{w}_0) \leq -\int_0^T |\delta y(t)|^2 \, dt + \left(2\sigma_q\right)^2 \int_0^T |u(t)|^2 \, dt.$$

The asymptotic stability of the original system implies that $0 \leq V(z_T, w_T, \dot{z}_T, \dot{w}_T) < \infty$. Moreover, $V(z_0, w_0, \dot{z}_0, \dot{w}_0) = 0$, because of the zero initial condition. Therefore the left-hand side of the above inequality exists and it is positive for all $T \geq 0$, thus

$$\int_0^T \left|y(t) - \hat{y}(t)\right|^2 dt \leq \left(2\sigma_q\right)^2 \int_0^T |u(t)|^2 \, dt.$$

As a result, the one-step reduction error bound is

$$\varepsilon = 2\sigma_q. \tag{24}$$

Next, following an analysis similar to the one presented in [13], which is based on the triangle inequality, it can be shown that extending the above to multiple one-step reductions leads to (16). □

The next section studies the feasibility of the matrix inequalities (4) and (7).

5 Feasibility of the Matrix Inequalities

In this section, we discuss feasibility conditions for the proposed model order reduction method. As this method relies on the matrix inequalities (4) and (7), we give easy-to-check conditions (both necessary and sufficient) for existence of solutions to these matrix inequalities for a common scalar $\alpha_c = \alpha_o = \alpha$, as required for the application of Theorem 2.

First, the following lemma shows that the feasibility of the inequalities is always guaranteed for sufficiently small delays provided $A + A_d$ is Hurwitz.

Lemma 5 *Let* (1) *be asymptotically stable for* $\tau = 0$. *Then, there exists a positive scalar* ϵ *for which the matrix inequalities in* (4) *and* (7) *are feasible for all* $\tau \in [0, \epsilon)$.

Proof The fact that the system (1) is asymptotically stable for $\tau = 0$ implies that $A_c := A + A_d$ is Hurwitz. Therefore, there exists a matrix $Q = Q^T > 0$ such that

$$A_c^T Q + Q A_c + C_c^T C_c < 0, \tag{25}$$

where $C_c = C + C_d$. The strict inequality in (25) guarantees the existence of a (large) $\bar{\alpha} > 0$ such that

$$\begin{aligned} QA_c + A_c^T Q + C_c^T C_c + \left(QA_d - Q_d + C_c^T C_d\right) \\ \times \left(\bar{\alpha} Q + Q_d - C_d^T C_d\right)^{-1} \left(QA_d - Q_d + C_c^T C_d\right)^T < 0. \end{aligned} \tag{26}$$

Following a Schur complement, this inequality implies that

$$\begin{bmatrix} QA_c + A_c^T Q + C_c^T C_c & QA_d - Q_d + C_c^T C_d \\ * & -\bar{\alpha} Q - Q_d + C_d^T C_d \end{bmatrix} + \tau^2 \bar{\alpha} \begin{bmatrix} A_c^T \\ A_d^T \end{bmatrix} Q \begin{bmatrix} A_c & A_d \end{bmatrix} < 0 \tag{27}$$

for all $\tau \in [0, \epsilon_o)$ provided ϵ_o is sufficiently small. It can be shown that this inequality is equivalent to (4) for $S = Q$, $\alpha = \tau^2 \bar{\alpha}$ and $\bar{Q} = \bar{\alpha} Q$. Thus, inequality (4) also holds for all $\tau \in [0, \epsilon_o)$. A similar argument can be performed about the feasibility of (7), i.e., we can show that there exists a sufficiently small ϵ_c such that (7) becomes feasible for all $\tau \in [0, \epsilon_c)$. The definition $\epsilon := \min\{\epsilon_o, \epsilon_c\}$ completes the proof of Lemma 5. □

Next, we present *necessary* conditions for the feasibility of (4) and (7) in terms of upper bounds on the delay τ.

Lemma 6 *Let $A_m := A - A_d$ be a non-Hurwitz matrix and $\bar{\lambda}_m$ be an eigenvalue of A_m which has the largest modulus in the right-half complex plane. Then, a necessary condition for* (4) *and* (7) *to hold is that*

$$\tau < \frac{2}{|\bar{\lambda}_m|}. \tag{28}$$

Lemma 7 *Let A in* (1) *be a non-Hurwitz matrix and $\bar{\lambda}$ be an eigenvalue of A which has the largest modulus in the closed right-half complex plane. Then, a necessary condition for* (4) *and* (7) *to hold is that*

$$\tau < \sqrt{\frac{2}{|\bar{\lambda}|^2 + \underline{\sigma}_d^2}}, \tag{29}$$

where $\underline{\sigma}_d$ is the smallest singular value of A_d.

Proof We present proofs for Lemmas 6 and 7 jointly, and based only on (4). First, we eliminate the slack variables from (4) by multiplying it from the left and right by

$$\begin{bmatrix} I_n & 0 & A^T & 0 \\ 0 & I_n & A_d^T & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} I_n & 0 & A^T & 0 \\ 0 & I_n & A_d^T & 0 \end{bmatrix}^T,$$

respectively. This procedure results in

$$\begin{bmatrix} QA + A^T Q - \bar{Q} + Q_d + \tau^2 A^T \bar{Q} A & QA_d + \bar{Q} + \tau^2 A^T \bar{Q} A_d \\ * & -\bar{Q} - Q_d + \tau^2 A_d^T \bar{Q} A_d \end{bmatrix} < 0. \tag{30}$$

This inequality further implies that

$$\begin{bmatrix} A_m^T Q + QA_m - 4\bar{Q} + \tau^2 A_m^T \bar{Q} A_m & QA_d + 2\bar{Q} + Q_d + \tau^2 A_m^T \bar{Q} A_d \\ * & -\bar{Q} - Q_d + \tau^2 A_d^T \bar{Q} A_d \end{bmatrix} < 0. \tag{31}$$

Namely, this can be shown by the left and right multiplication of (30) by

$$\begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix}, \text{ and } \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix}^T,$$

respectively. Considering its upper left block, the inequality in (31) now implies that

$$A_m^T Q + QA_m - 4\bar{Q} + \tau^2 A_m^T \bar{Q} A_m < 0.$$

Let v be an eigenvector of A_m for the eigenvalue $\lambda_m = \mu_m + j\omega_m$. Then, left and right multiplication of this inequality by v^H and v implies

$$2\mu_m v^H Q v + (\tau^2 |\lambda_m|^2 - 4) v^H \bar{Q} v < 0. \tag{32}$$

Now, we consider only eigenvalues in the right-half complex plane. Namely, if $\mu_m \geq 0$, the satisfaction of (32) requires that $\tau < 2/|\lambda_m|$ and $\bar{Q} > 0$. This result establishes (28).

Next, we prove Lemma 7. The feasibility of (30) implies that

$$A^T Q + QA - \bar{Q} + Q_d + \tau^2 A^T \bar{Q} A < 0, \tag{33}$$

$$-\bar{Q} - Q_d + \tau^2 A_d^T \bar{Q} A_d < 0, \tag{34}$$

respectively, as follows from considering the block-diagonal elements. From (34), we obtain that $-\bar{Q} + \tau^2 A_d^T \bar{Q} A_d < Q_d$. Using this result in (33), we conclude the necessity of the following inequality:

$$A^T Q + QA - 2\bar{Q} + \tau^2 A^T \bar{Q} A + \tau^2 A_d^T \bar{Q} A_d < 0. \tag{35}$$

Now, if we take v as an eigenvector of A corresponding to an eigenvalue λ which lies in the closed right-half complex plane, (35) implies that

$$2\mathrm{Re}(\lambda) v^H Q v + \left(\tau^2 |\lambda|^2 + \tau^2 \underline{\sigma}_d^2 - 2\right) v^H \bar{Q} v < 0.$$

Since $\mathrm{Re}(\lambda) \geq 0$, this relation cannot be feasible without the satisfaction of (29). □

Remark 2 We note that the conditions provided by Lemmas 6 and 7 are only necessary conditions and not sufficient, i.e., they imply the infeasibility of the matrix inequalities if those conditions do not hold.

Remark 3 The condition of Lemmas 6 and 7 are more beneficial and practical when the model order reduction problem of feedback control systems with delays in the feedback channel is concerned, especially for systems with an unstable plant, leading to a non-Hurwitz A. In these system, the matrix $A - A_d$ is often non-Hurwitz.

Next, we present a result that is helpful in solving the matrix inequalities. Namely, given the couplings among α and the slack matrices in (4) and (7) (assuming that $\alpha_o = \alpha_c = \alpha$, in view of Theorem 2), these inequalities are nonlinear. To still enable solving these inequalities by using existing techniques for linear matrix inequalities, we perform a line search over α. For the line search to become more efficient, bounds on the search space for α should be provided. The following lemma provides such lower bound.

Lemma 8 *Consider A, and define $A_m := A - A_d$ and let λ and λ_m be arbitrary eigenvalues of A and A_m, respectively. Then, a necessary condition for the matrix inequalities* (4) *and* (7) *to hold is that*

$$\alpha > \max\{\tau^2 \mathrm{Re}(\lambda), \frac{\tau^2}{4}\mathrm{Re}(\lambda_m)\}. \tag{36}$$

Proof Here, we use only (4) to derive this inequality. The term $(M_o)_{33}$ (the (3,3) component of M_o) implies that

$$\bar{Q} < \frac{2\alpha}{\tau^2} S, \tag{37}$$

which follows from the fact that $(M_o)_{33}$ is a diagonal element. Using this result along with the fact that $(M_o)_{11} < 0$, we can conclude that

$$SA + A^T S + Q_d - \frac{2\alpha}{\tau^2} S < 0. \tag{38}$$

Let $Av = \lambda v$, i.e., v is an eigenvector corresponding to the eigenvalue λ of A. Then, left and right multiplication of the above inequality with v and v^H, respectively, implies that

$$v^H SAv + v^H A^T Sv - \frac{2\alpha}{\tau^2} v^H Sv + v^H Q_d v < 0.$$

This, in turn, leads to

$$\left(2\mathrm{Re}(\lambda) - \frac{2\alpha}{\tau^2}\right) v^H Sv < 0.$$

Since $S > 0$, this inequality holds only for $\alpha > \mathrm{Re}(\lambda)\tau^2$. Following a similar procedure, it can be shown that the satisfaction of (4) also requires $\alpha > \mathrm{Re}(\lambda_m)\tau^2/4$,

with λ_m an eigenvalue of A_m. The fact that these hold for all eigenvalues of A and A_m leads to (36). □

Remark 4 Clearly, the lower bound in (36) becomes zero when A and $A - A_d$ are both Hurwitz, given the fact that $\alpha < 0$ is not allowed because of the fact that $(M_o)_{33}$ must be negative definite.

6 Example: Delay Neural Fields

This section presents a numerical example. The involved matrix inequalities are solved using the software CVX [14].

In this example, we study the application of the extended model-reduction technique to a model which describes the spatio-temporal interactions between neural populations in the brain. For comparison, we have also applied the position balancing technique in [16] to this model. Contrary to the bounds on the energy functions used in this chapter, position balancing relies on matrices that characterize the exact observability and energy functionals for a restricted class of functionals. These matrices represent the solution to a set of differential equations which are solved approximately [17].

Consider the delayed-neural fields model (see [6] for a survey) in the form of integro-differential equations:

$$l_i \dot{x}_i(r,t) = -x_i(r,t) + s_i\left(\sum_{j=1}^{n} \int_{\mathcal{R}} w_{ij}(r,r') x_j(r', t - \tau_{ij}(r,r'))\, dr' + I_i(r,t)\right), \tag{39}$$

for $i = 1, ..., q$, where q is the number of considered neuronal populations. The compact set $\mathcal{R} \subset \mathbb{R}$ describes the spatial domain containing the neuronal populations; it is assumed here to be uni-dimensional for simplicity. Moreover, $r \in \mathcal{R}$ is the spatial variable and $x_i(r,t)$ represents the neuronal activity of population i at time $t \geq 0$ and position $r \in \mathcal{R}$; $w_{ij} : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ is a bounded function such that $w_{ij}(r,r')$ describes the synaptic strength between location r' in population j and location r in population i. The constant $l_i > 0$ is the time decay constant of population i; $I_i : \mathcal{R} \times \mathbb{R} \to \mathbb{R}$ denotes the exogenous input to population i; $\tau_{ij} : \mathcal{R} \times \mathcal{R} \to [0, \bar{\tau}]$, $\bar{\tau} \geq 0$, is the self (for $i = j$) or mutual (for $i \neq j$) time delay resulting from the non-instantaneous communication between neurons, due to the finite velocity of signals along the axons. The continuously differentiable function $s_i : \mathbb{R} \to \mathbb{R}$ describes the excitability of population i.

To be able to rewrite (39) in the form (1), we first assume that the self time delays are zero ($\tau_{ii} = 0$) and the mutual delays are all fixed and equal, i.e., $\tau_{ij}(r,r') = \tau$ for all $i \neq j$ and all $r, r' \in \mathcal{R}$. With this assumption, and after linearizing the system around an operating profile $x_i^*(r)$ for the input $I_i(r,t) = I_i^*(r)$ (see [7] for details), the approximate model has the form

$$L\dot{\tilde{x}}(r,t) = -\tilde{x}(r,t) + S\int_{\mathcal{R}} \left(W_1(r,r')\tilde{x}(r',t) + W_2(r,r')\tilde{x}(r',t-\tau)\right) dr' + S\tilde{I}(r,t), \tag{40}$$

where $\tilde{x}^T := [\tilde{x}_1, ..., \tilde{x}_q]$ with $\tilde{x}_i = x_i - x_i^*$, $\tilde{I}^T := [\tilde{I}_1, ..., \tilde{I}_q]$ with $\tilde{I}_i = I_i - I_i^*$, $L = \text{diag}\{l_1, ..., l_q\}$ and $W_1 = \text{diag}\{w_{ii}\}$, for $i = 1, ..., q$, and $W_2 = [w_{ij}] - W_1$, for all $i, j = 1, ..., q$. Finally, $S = \text{diag}\{\bar{s}_1, ..., \bar{s}_q\}$, where $\bar{s}_i$ results from the linearization of the function s_i.

In the absence of delays ($\tau_{ij}(r,r') = 0$), an approach was proposed in [30] to analytically reduce the dynamics of the infinite-dimensional dynamics (39) to a finite-dimensional differential equation by assuming that the kernels w_{ij} can be decomposed on a finite basis of spatial functions. Following this idea, we assume that $W_i(r,r')$, $i = 1, 2$, is a so-called Pincherle-Goursat Kernel, i.e., there exist $X_i(r) \in \mathbb{R}^{q\times N_i}$ and $Y_i(r) \in \mathbb{R}^{q\times N_i}$, $N_i \in \mathbb{N}$, such that

$$W_i(r,r') = X_i(r)Y_i^T(r'). \tag{41}$$

We note that $X_i(r)$ contains the basis vectors of W_i. We further assume that there exists $\tilde{i}(t) \in \mathbb{R}^{N_1+N_2}$, for which the decomposition $\tilde{I}(r,t) = X(r)\tilde{i}(t)$, with $X = [X_1, X_2]$, holds. Given the structure of W_i in (41) and of $\tilde{I}$, we approximate the solution $\tilde{x}$ as $\tilde{x}(r,t) = \tilde{X}(r)v(t)$, where

$$\tilde{X}(r) = \left[\, X_1(r)\ X_2(r)\ X_e(r) \,\right] \tag{42}$$

can be regarded as a reduction basis (albeit depending on the spatial variable). In (42), $X_e(r) \in \mathbb{R}^{q\times N_e}$ denotes a potential enrichment of this reduction basis over the elements X_1 and X_2, which result from the structure of W_i. Moreover, $v(t) \in \mathbb{R}^n$, $n = N_1 + N_2 + N_e$, is an unknown vector the driving dynamics which is yet to be obtained.

Remark 5 We note that the structure of this approximation separates the effect of the spatial and temporal variables.

Then, the substitution of (41) and the approximation (42) into (40) leads to

$$L\tilde{X}(r)\dot{v}(t) = -\tilde{X}(r)v(t) + SX_1(r)K_1v(t) + SX_2(r)K_2v(t-\tau) + S\tilde{X}(r)\tilde{i}(t), \tag{43}$$

where $K_i = \int_{\mathcal{R}} Y_i^T(r')\tilde{X}(r')\,dr'$, $i = 1, 2$. This equation holds for every r, so we can multiply both sides of (43) by $\tilde{X}^T(r)$ from the left. Then, integration of both sides of the resulting equation over $\mathcal{R}$ leads to

$$M_l\dot{v}(t) = (M_1K_1 - M)v(t) + M_2K_2v(t-\tau) + M_s\tilde{i}(t),$$

where

Table 1 Parameters of the neural field

Parameter	Value	Parameter	Value
L	diag{10, 20}	w_{11}	0
S	diag{20, 20}	w_{12}	$-30\exp\left(-\frac{\lvert r-r'-1.32\times10^{-2}\rvert^2}{0.06}\right)$
$\mathcal{R}$	$[0, 2.5]\cup[12.5, 15]\times10^{-3}$	w_{21}	$38\exp\left(-\frac{\lvert r-r'-1.25\times10^{-3}\rvert^2}{0.06}\right)$
τ	0.03 sec	w_{22}	$-2.55\exp\left(-\frac{\lvert r-r'\rvert^2}{0.03}\right)$

$$M_l = \int_{\mathcal{R}} \tilde{X}^T(r)L\tilde{X}(r)dr, \quad M_1 = \int_{\mathcal{R}} \tilde{X}^T(r)SX_1(r)dr, \quad M_2 = \int_{\mathcal{R}} \tilde{X}^T(r)SX_2(r)dr,$$
$$M_s = \int_{\mathcal{R}} \tilde{X}^T(r)SX(r)dr, \quad M = \int_{\mathcal{R}} \tilde{X}^T(r)\tilde{X}(r)dr. \tag{44}$$

Clearly, if M_l is invertible, this equation can be written in the form (1) by defining

$$A = M_l^{-1}\left(M_1K_1 - M\right), \quad A_d = M_l^{-1}M_2K_2, \quad B = M_l^{-1}M_sF,$$
$$C = \int_{\mathcal{R}} \bar{C}(r)\tilde{X}(r)\,dr, \quad C_d = 0, \quad D = 0.$$

Here, we have considered $\tilde{i}(t) = Fu(t)$ with $F \in \mathbb{R}^{(N_1+N_2)\times m}$ and $u(t) \in \mathbb{R}^m$ as the input. We note that F is defined such that the elements of u are independent. Moreover, $\bar{C}(r) \in \mathbb{R}^p$ is the distributed output matrix. Namely, we consider outputs of the form $y(t) = \int_{\mathcal{R}} \bar{C}(r)\tilde{x}(r,t)\,dr$. Given the complexity of w_{ij} and the enrichment basis X_e, the dimension of $\tilde{X}(r)$ and, subsequently, the order n of the time delay system describing the dynamics of $v(t)$ can be large.

In this example, we consider a neural field with the parameters reported in Table 1. The input is given by $\tilde{I}_1(r,t) = 0$ and $\tilde{I}_2(r,t) = (1+r)\exp(-r^2/0.03)u(t)$ and the output is characterized by $\bar{C}(r) = [1, 0.1]$. After computing $X_1(r)$ and $X_2(r)$, where a truncated Taylor series expansion has been exploited (for details, see Appendix A) and considering $X_e = 0$, we obtain a system of the form (1) of order $n = 9$, and $F^T = [1, 1, 0, ..., 0]$. The frequency response function of this system between the input u and the output y is represented by $G_v(j\omega)$.

The corresponding singular values resulting from the application of the extended model order reduction technique in comparison to those from the position balancing technique are plotted in Fig. 1. In the same figure, we have reported the reduction error ε, for the extended technique, as a function of the reduction order k. It is observed that the singular values from the position balancing technique are smaller than those from the extended method. However, we note that the position balancing technique does not provide an a-priori error bound, neither does it guarantee the stability of the reduced system. We observe a quick decay in the singular values from the extended technique after $k = 2$. Thus, we may approximate the dynamics of $v(t)$ by a model, with the frequency response function represented by $\hat{G}_v(j\omega)$, of order $k = 2$ and expect an accurate model approximation. In Fig. 2, the frequency

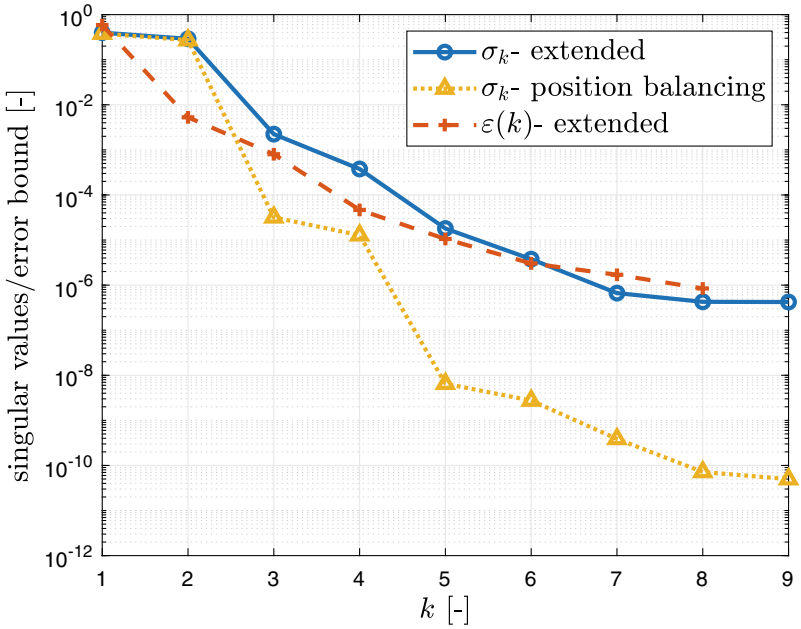

Fig. 1 The singular values σ_k and the error bound ε as a function of the reduction order k

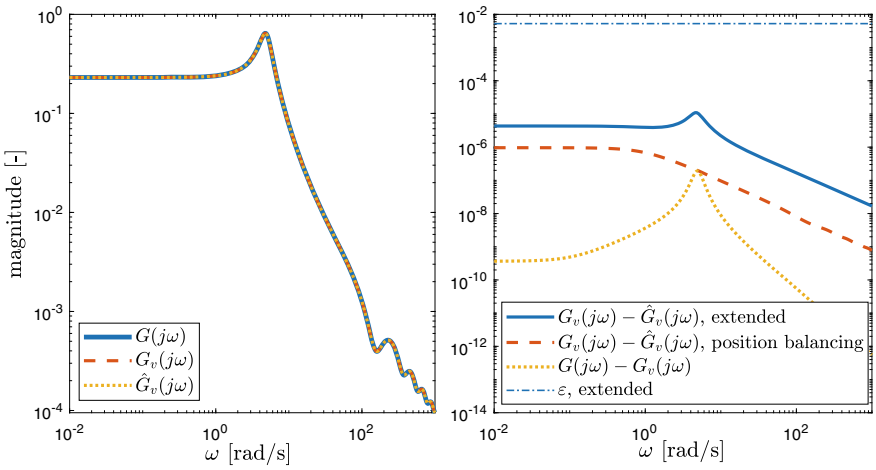

Fig. 2 Comparison between the transfer functions of the original, reduced and error systems in the neural field example

response function $G(j\omega)$ of the original (linearized) model in (40) is compared to $G_v(j\omega)$ and $\hat{G}_v(j\omega)$. In the same figure, we have presented transfer functions of the error systems $G(j\omega) - G_v(j\omega)$ and $G_v(j\omega) - \hat{G}_v(j\omega)$, of both techniques. From this figure, we can clearly observe the high accuracy of the approximation from the extended technique. The approximate model from the position balancing method is slightly more accurate. We note that $G(j\omega)$ is obtained by performing a spatial discretization over a grid of 200 cells, and the same grid has been used to numerically compute the matrices in (44). The error between $G(j\omega)$ and $G_v(j\omega)$ stems from the limited resolution of the discretization and also the Taylor series expansion.

Remark 6 In addition to slightly outperforming the presented model-reduction technique in terms of accuracy, position balancing relies on the computation of delay Lyapunov equations which only require asymptotic stability of the model (instead of solutions to matrix inequalities as in (4) and (7)). Nonetheless, we stress that position balancing does neither provide guarantees on stability preservation nor gives an a-priori bound on the reduction error.

We stress that the assumption made here requires a strong separation between spatial and temporal evolution of (40) as well as a spatially uniform delays. Further work is needed to relax these requirements.

7 Application to Parameterized Model Reduction

An extended model-reduction procedure as presented in the previous sections is particularly suited for system-theoretic applications such as structured and parameterized model reduction. In this chapter, we focus on the latter application and refer to [24] for a detailed discussion on the former.

Namely, a large class of parameterized time delay systems can be written in the form of time delay systems with a polytopic parameterization of the form

$$\Omega_\delta : \begin{cases} \dot{x}(t) = A_\delta x(t) + A_{d\delta} x(t-\tau) + B_\delta u(t), \\ y(t) = C_\delta x(t) + C_{d\delta} x(t-\tau) + D_\delta u(t), \\ x_0 = \varphi. \end{cases} \tag{45}$$

where the subscript δ denotes a polytopic parameterization such that a parameterized matrix M_δ is defined as $M_\delta := \sum_{i=1}^{d} \delta_i M_i$, where M_i, $i = 1, ..., d$, is a given matrix and $\delta \in \Delta$ with $\Delta = \{\delta \in \mathbb{R}^d | \delta_i \geq 0, \sum_{i=1}^{d} \delta_i = 1\}$. It is assumed that for all $\delta \in \Delta$, this system has the same stability properties as the system in (1).

Although the methods in [23] and [4] can be generalized to enable the reduction of this type of systems, those can result in low-quality model approximations and conservative error bounds, if not infeasible. On the other hand, the extended model reduction improves both the feasibility and the accuracy of model approximation for this type of systems. This is due to the fact that in an extended model-reduction

method, we can assign a polytopic structure to the main decision variables to increase the degrees of freedom in the model-reduction procedure. In conventional methods, such as those in [23] and [4], the main decision variables Q and P are directly used in computing the balancing transformation, and assigning a polytopic structure to those complicates the reduction procedure (see [31], for parameterized model reduction of delay-free systems to get an idea about complexities that can arise when assigning parametric structures to P and Q).

In the extended technique, for the parameterized system in (45), the inequality (4) is adapted to the following form:

$$M_{o\delta} = \begin{bmatrix} SA_\delta + A_\delta^T S + Q_{d\delta} - \bar{Q}_\delta & \bar{Q}_\delta + SA_{d\delta} & Q_\delta - S + \alpha_o A_\delta^T S & C_\delta^T \\ * & -Q_{d\delta} - \bar{Q}_\delta & \alpha_o A_{d\delta}^T S & C_{d\delta}^T \\ * & * & -2\alpha_o S + \tau^2 \bar{Q}_\delta & 0 \\ * & * & * & -I_p \end{bmatrix} < 0. \tag{46}$$

By virtue of the properties of the polytopic uncertainty/parameterization, it can be shown that $M_{o\delta} = \sum_{i=1}^{d} \delta_i M_{oi}$ (note that $S = \sum_{i=1}^{d} \delta_i S$) with

$$M_{oi} = \begin{bmatrix} SA_i + A_i^T S + Q_{di} - \bar{Q}_i & \bar{Q}_i + SA_{di} & Q_i - S + \alpha_o A_i^T S & C_i^T \\ * & -Q_{di} - \bar{Q}_i & \alpha_o A_{di}^T S & C_{di}^T \\ * & * & -2\alpha_o S + \tau^2 \bar{Q}_i & 0 \\ * & * & * & -I_p \end{bmatrix}, i = 1, ..., d. \tag{47}$$

This implies that if there exist matrices $Q_i > 0$, $\bar{Q}_i > 0$, $Q_{di} > 0$, $i = 1, ..., d$, and $S > 0$, and a scalar α_o such that $M_{oi} < 0$ for $i = 1, ..., d$, then $M_{o\delta} < 0$. This result together with a similarly adapted inequality $M_{c\delta} < 0$ (an adaption to the inequality (7)) provides matrices S and R required for reducing (45) by pursuing the same procedure as in Sect. 4.

Remark 7 It is noted that in this parameterized model order reduction technique, S and α_o must satisfy d (the number of parameters) inequalities of the form (47) simultaneously.

Remark 8 The error bound obtained from the parameterized technique is robust in the sense that it holds for all $\delta \in \Delta$. Moreover, it can be shown that the reduced system is asymptotically stable and it has the same parameterization as the original one.

7.1 Example

Next, we present an example. In this example, we consider a wave equation which has a damping factor in the forward direction. The wave equation, together with the

considered boundary conditions and the initial condition, is given by

$$\frac{\partial}{\partial t}q_1(t,\xi) + c\frac{\partial}{\partial x}q_1(t,\xi) = 0.025 f q_1(t,\xi), \tag{48}$$

$$\frac{\partial}{\partial t}q_2(t,\xi) - c\frac{\partial}{\partial x}q_2(t,\xi) = 0, \tag{49}$$

$$q_1(t,0) = \beta_1 q_2(t,0) + u(t), \tag{50}$$

$$q_2(t,l) = \beta_2 q_1(t,l), \tag{51}$$

$$q_1(0,\xi) = 0, \tag{52}$$

$$q_2(0,\xi) = 0, \tag{53}$$

where $t \geq 0$ and $\xi \in [0, l]$ are the temporal and spatial variables, respectively. Here, $l = 1000$ m is the length of the spatial domain. Moreover, $q_i(t,\xi) \in \mathbb{R}$, $i = 1, 2$, are the distributed variables, $c = 1000$ m/s is the speed of the traveling wave components, and f is a damping factor. We take f to be uncertain, but we assume that the upper and lower bounds of it are known as $f \in [0.5, 10.5]$. Moreover, $\beta_1 = 1$ and $\beta_2 = 0.7$, and $u(t)$ is the input. The output is given by

$$y(t) = q_1(t,l). \tag{54}$$

From the literature, it is know that this system can be modeled by delay-difference equations [8]. However, in this study, for the sake of illustration, we discretize the first PDE describing q_1 (48) to obtain an approximative model of it in terms of ODEs, whereas we write the other PDE (49) in terms of an equivalent delay equation, that is, we can show that $q_2(t,0) = q_2(t-\tau, l)$, with $\tau = l/c$.

To perform the discretization, the spatial domain of the first PDE is discretized into n cells of length $\Delta\xi$. In the discretization scheme, $Q_i(t)$, for $i = 1, 2, \ldots, n$, approximates the spatial average of $q_1(t,\xi)$ over the ith cell and satisfies

$$\dot{Q}_i(t) = \gamma_1 Q_{i-1}(t) - \gamma_2 Q_i(t), \quad i = 1, 2, \ldots, n \tag{55}$$

with $\gamma_1 = c/\Delta\xi$ and $\gamma_2 = c/\Delta\xi - 0.025 f$. In this formulation, we approximate $Q_0(t) \approx q_1(t,0)$. Following the fact that $q_2(t,0) = q_2(t-\tau,l)$, and by using the boundary conditions (50) and (51), we can further write $Q_0(t) \approx \beta_1\beta_2 Q_n(t-\tau) + u(t)$, where the approximation $q_1(t,l) \approx Q_n(t)$ has been used. Finally, using (55) together with these relations and the approximation $y(t) \approx Q_n(t)$, we obtain a model of the form (1) with $C = [0, 0, \cdots, 1]$, $C_d = 0$ and

$$A = \begin{bmatrix} -\gamma_2 & 0 & & 0 \\ \gamma_1 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & \gamma_1 & -\gamma_2 \end{bmatrix}, \; A_d = \begin{bmatrix} 0 & \gamma_1\beta_1\beta_2 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} \gamma_1 \\ 0 \end{bmatrix}.$$

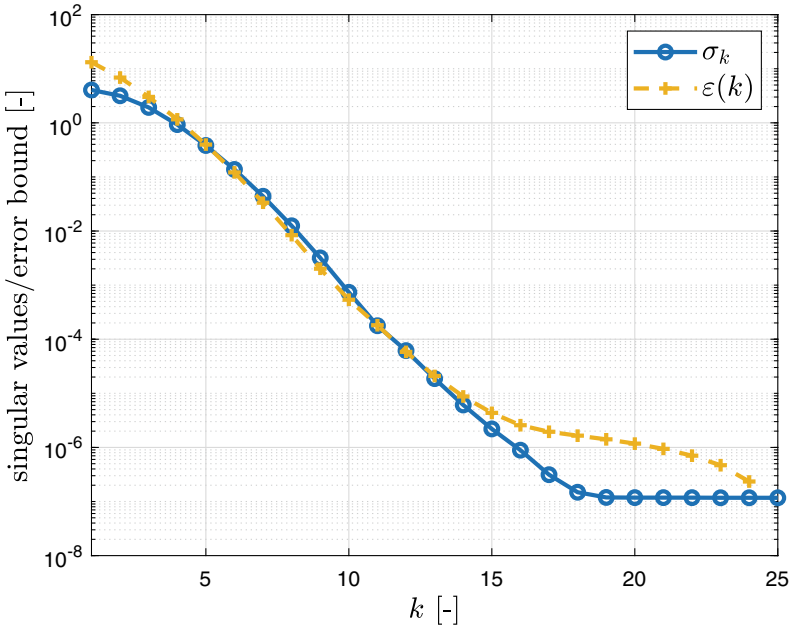

Fig. 3 The singular values σ_k and the error bound ε as a function of the reduction order k, for the robust model reduction

We can then write this model as a time delay system with polytopic uncertainties, due to uncertainties in f, in the form of (45) with $d = 2$. The order of this model, determined by the resolution of the discretization, is chosen to be $n = 25$. The frequency response function of the discretized model between input u and output y is denoted by $G(j\omega)$.

The presented robust/parameterized model order reduction method has been applied to this model. Figure 3 presents the resulting extended singular values σ_i in comparison to the error bound ε as a function of the order k of the reduced system. Based on this figure, we choose $k = 4$. Note that σ_i and ε are independent of the uncertain variable. Figure 4 reports the frequency response function of the original model $G(j\omega)$ of order $n = 25$ in comparison to the reduced-order model $\hat{G}(j\omega)$ of order $k = 4$ for the extremal values $f = 0.5$ and $f = 10.5$. We observe that for both extremal values of f, the model-reduction results are quite accurate. We also observe, in the subfigure on the right-hand side of Fig. 4, that in both cases, the $\mathcal{H}_\infty$-norm of the error system $G(j\omega) - \hat{G}(j\omega)$ is smaller than the a-priori obtained error bound, as expected.

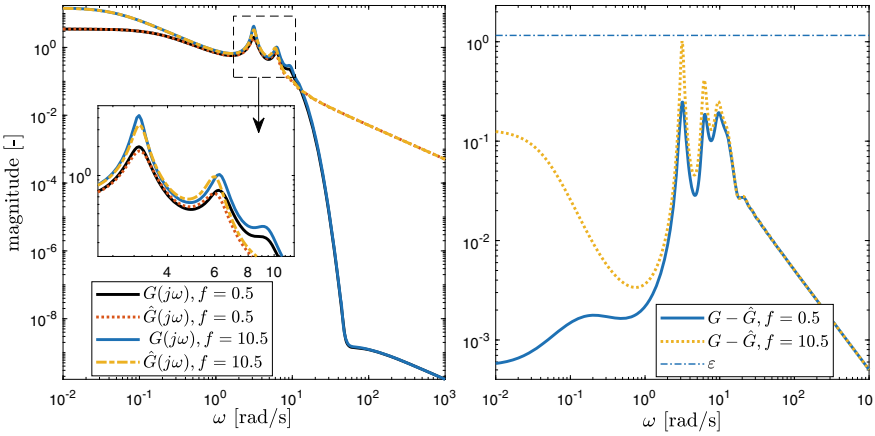

Fig. 4 (Left) comparison between the frequency response function of the original system G and the reduced-order one $\hat{G}$, and (right) error bound in comparison to the frequency response function of the error system $G - \hat{G}$ for the extremal values of the uncertain parameter f

8 Conclusions

In this chapter, by introducing slack variables in the computation of bounds on the energy functionals, we have obtained an extended model-reduction technique for linear time delay systems. This technique exhibits more flexibility compared to its existing counterparts, making it interesting for purposes such as parameterized and structured model reduction. Moreover, the proposed technique preserves stability properties and also provides a computable error bound. We have numerically evaluated the performance of the proposed method by applying it to a model of neural fields in the brain and to a model with polytopic uncertainties.

Appendix A. Derivation of $X(r)$

We consider $w_{ij}(r, r')$, for $i, j = 1, 2$. This function can be written in the following general form

$$W_{ij}(r, r') = k_{ij} \exp\left(-\frac{|r - r' - \mu_{ij}|^2}{2\sigma_{ij}}\right)$$
$$= k_{ij} \exp\left(-\frac{|r|^2}{2\sigma_{ij}}\right) \exp\left(-\frac{|r' + \mu_{ij}|^2}{2\sigma_{ij}}\right) \exp\left(\frac{r(r' + \mu_{ij})}{\sigma_{ij}}\right)$$

for some constants k_{ij}, σ_{ij} and μ_{ij}. We wish to decompose $w_{ij}(r, r')$ into a multiplication of only-r and only-r' dependent functions. However, the term $\exp(r(r' + \mu_{ij})/\sigma_{ij})$ cannot be directly decomposed into such a desirable form. To cope with this issue, we use the Taylor series approximation of order ρ of this term to obtain

$$\exp\left(\frac{r(r'+\mu_{ij})}{\sigma_{ij}}\right) \approx \begin{bmatrix} 1 & r & r^2 & \dots & r^\rho \end{bmatrix} \begin{bmatrix} 1 & \frac{(r'+\mu_{ij})}{\sigma_{ij}} & \frac{(r'+\mu_{ij})^2}{2\sigma_{ij}^2} & \dots & \frac{(r'+\mu_{ij})^\rho}{\rho!\sigma_{ij}^\rho} \end{bmatrix}^T,$$

where ρ is the order of approximation. With this approximation, we can now write

$$w_{ij}(r, r') \approx f_{ij}(r) g_{ij}^T(r')$$

where

$$f_{ij}(r) = \begin{bmatrix} f_{ij,0}(r) & \cdots & f_{ij,\rho}(r) \end{bmatrix},$$
$$g_{ij}(r) = \begin{bmatrix} g_{ij,0}(r) & \cdots & g_{ij,\rho}(r) \end{bmatrix}$$

with

$$f_{ij,m}(r) = r^m \exp\left(-\frac{|r|^2}{2\sigma_{ij}}\right),$$
$$g_{ij,m}(r') = k_{ij} \frac{(r'+\mu_{ij})^m}{m!\sigma_{ij}^m} \exp\left(-\frac{|r'+\mu_{ij}|^2}{2\sigma_{ij}}\right), \quad m = 0, 2, ..., \rho.$$

With this representation of $w(r, r')$, we may choose

$$X_1 = \begin{bmatrix} 0 \\ f_{22} \end{bmatrix}, \quad X_2 = \begin{bmatrix} f_{12} & 0 \\ 0 & f_{21} \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} 0 \\ g_{22} \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 & g_{21} \\ g_{12} & 0 \end{bmatrix}.$$

With this choice of X_1 and X_2, we obtain $N_1 = \rho$ and $N_2 = 2\rho$. We also note that this choice of X_1 and X_2 leads to $w_{11} = 0$.

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