Chapter 3
Convergent Systems: Nonlinear Simplicity

Alexey Pavlov and Nathan van de Wouw

Abstract Convergent systems are systems that have a uniquely defined globally asymptotically stable steady-state solution. Asymptotically stable linear systems excited by a bounded time varying signal are convergent. Together with the superposition principle, the convergence property forms a foundation for a large number of analysis and (control) design tools for linear systems. Nonlinear convergent systems are in many ways similar to linear systems and are, therefore, in a certain sense simple, although the superposition principle does not hold. This simplicity allows one to solve a number of analysis and design problems for nonlinear systems and makes the convergence property highly instrumental for practical applications. In this chapter, we review the notion of convergent systems and its applications to various analyses and design problems within the field of systems and control.

3.1 Introduction

In many controller design problems, a controller is designed to ensure that some desired solution of the closed-loop system is asymptotically stable with a desired region of attraction. Traditionally, this is considered as a stabilization problem for the desired *solution*. However, if we take a step back and look at this design problem

A. Pavlov
Statoil ASA, P.b. 1004, 3905 Porsgrunn, Norway
e-mail: alepav@statoil.com

N. van de Wouw (✉)
Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, Noord-Brabant, The Netherlands
e-mail: N.v.d.Wouw@tue.nl

N. van de Wouw
Department of Civil, Environmental and Geo-Engineering, University of Minnesota, Minneapolis, MN, USA

N. van de Wouw
Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands

N. van de Wouw et al. (eds.), *Nonlinear Systems*, Lecture Notes in Control and Information Sciences 470, DOI 10.1007/978-3-319-30357-4_3

from a distance, we can see that the controller actually provides the closed-loop system with a *system* property: the closed-loop system has an asymptotically stable steady-state solution with the given region of attraction (e.g., local, global, or some given set of initial conditions). In addition to that, this steady-state solution has desired properties, e.g., desired output value.

This system point of view on the controller design problem allows one to define an important system property, which is common for linear asymptotically stable systems, but which is far from being straightforward for nonlinear systems: *the convergence property*. A system is called convergent if it has a unique, in a certain sense, globally asymptotically stable solution, called the steady-state solution. Originally, the term "convergence" was coined in the Russian literature in the 1950s. In [41], the notion was defined for nonstationary systems that are periodic in time. In that reference, such a system is called convergent if it has a unique globally asymptotically stable periodic solution. Later, in [10] (see also [34]) this definition was naturally extended to nonlinear systems with arbitrary (not necessarily periodic in time) right-hand sides. These references, together with [56] laid a foundation of basic results for establishing this property for nonlinear systems based on Lyapunov functions, matrix inequalities, and frequency domain methods. Almost 50 years later, notions similar to convergence received significant attention in the literature: contraction analysis, incremental stability and passivity, incremental input-to-state stability, etc. [1–3, 13, 15, 22, 27–29, 32, 44–46, 49, 50, 57]. A comparison establishing differences and similarities between incremental stability on the one hand and convergence on the other hand is provided in [46].

A brief historical overview on convergent systems and subsequent developments of this and similar notions can be found in [34]. Since that paper many new developments on convergent systems have appeared. In particular, sufficient conditions for convergence for different classes of systems have been pursued [8, 9, 25, 26, 35, 37, 39, 42, 43, 54]. Together with theoretical developments on convergent systems and related notions, the benefit of such system-level stability property has been demonstrated by its use to tackle fundamental system-theoretic problems. Further study of convergent systems indeed demonstrated that this notion is very instrumental for a number of design and analysis problems within nonlinear systems and control, such as synchronization, observer design, output tracking and disturbance rejection, the output regulation problem, model reduction, stable inversion of non-minimum phase systems, steady-state performance optimization of control systems, variable gain controller design and tuning and extremum seeking control. For linear systems many of these problems are solved in a relatively simple way. It turns out that this simplicity comes not only from the superposition principle, but also from the convergence property (for linear systems it is equivalent to asymptotic stability of the system with zero input). Unlike the superposition principle, which holds only for linear systems, the convergence property may still hold for a nonlinear system. It appears that convergent nonlinear systems enjoy, to a large extent, the simplicity inherent to linear asymptotically stable systems. This allows one to solve, based on the notion of convergence, a number of challenging nonlinear control and analysis problems.

In this chapter, we will revisit the notion of convergence and review its application to various problems within systems and control. We deliberately omit technical details and generality, keeping focus on the ideas. All technical details and general formulations can still be found in the corresponding references. The chapter is organized as follows. Definitions, sufficient conditions, and basic properties of convergent systems are given in Sect. 3.2. Applications of this notion are reviewed in Sects. 3.3–3.9. Conclusions are given in Sect. 3.10.

3.2 Convergent Systems

Consider the nonlinear system

$$\dot{x} = F(x, t), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{3.1}$$

where $F(x, t)$ is locally Lipschitz in x and piecewise continuous in t.[1]

Definition 3.1 ([10]) System (3.1) is called *convergent* if

(i) there is a unique solution $\bar{x}(t)$ that is defined and bounded for $t \in \mathbb{R}$,
(ii) $\bar{x}(t)$ is globally asymptotically stable.

If $\bar{x}(t)$ is uniformly (exponentially) asymptotically stable, then system (3.1) is called *uniformly (exponentially) convergent*.[2]

Since the time-varying component in a system is usually due to some input, we can define convergence for systems with inputs.

Definition 3.2 System

$$\dot{x} = F(x, w(t)), \quad w(t) \in \mathbb{R}^m, \quad t \in \mathbb{R}, \tag{3.2}$$

is (uniformly, exponentially) convergent for a class of inputs $\mathscr{I}$ if it is convergent for every input $w(t) \in \mathscr{I}$ from that class.

To emphasize the dependence of the steady-state solution on the input $w(t)$, it is denoted by $\bar{x}_w(t)$. Note that any solution of convergent system (3.2) forgets its initial conditions and converges to the steady-state solution, which is determined by the input $w(t)$. Relations between input $w(t)$ and steady-state solution $\bar{x}_w(t)$ can be further characterized by several additional properties.

[1]For simplicity, in this chapter, we consider only continuous-time systems with locally Lipschitz right-hand sides. Definitions and basic results for discrete-time systems and for continuous-time systems with discontinuous right-hand sides can be found in [18, 36, 38, 39, 56].

[2]A more general definition of convergent systems, where the steady-state solution has an arbitrary domain of attraction (not necessarily global as in this chapter) can be found in [35].

Definition 3.3 ([35]) The system (3.2) that is convergent for a class of piecewise continuous bounded inputs is said to have the Uniformly Bounded Steady-State (UBSS) property if for any $r > 0$ there exists $R > 0$ such that if a piecewise continuous input $w(t)$ satisfies $|w(t)| \leq r$ for all $t \in \mathbb{R}$, then the corresponding steady-state solution satisfies $|\bar{x}_w(t)| \leq R$ for all $t \in \mathbb{R}$.

Definition 3.4 ([35]) System (3.2) is called *input-to-state convergent* if it is uniformly convergent for the class of bounded piecewise continuous inputs and, for every such input $w(\cdot)$, system (3.2) is input-to-state stable[3] with respect to the steady-state solution $\bar{x}_w(t)$, i.e., there exist a $\mathscr{K}\mathscr{L}$-function $\beta(r, s)$ and a $\mathscr{K}_\infty$-function $\gamma(r)$ such that any solution $x(t)$ of system (3.2) corresponding to some input $\hat{w}(t) := w(t) + \Delta w(t)$ satisfies

$$|x(t) - \bar{x}_w(t)| \leq \beta(|x(t_0) - \bar{x}_w(t_0)|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} |\Delta w(\tau)|\right). \tag{3.3}$$

In general, the functions $\beta(r, s)$ and $\gamma(r)$ may depend on the particular input $w(\cdot)$.

3.2.1 Conditions for Convergence

Simple sufficient conditions for exponential convergence were given by B.P. Demidovich in [10, 34]. Here we present a slightly modified version of that result, which covers input-to-state convergence.

Theorem 3.1 ([10, 35]) *Consider system (3.2) with the function $F(x, w)$ being C^1 with respect to x and continuous with respect to w. Suppose there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that*

$$P\frac{\partial F}{\partial x}(x, w) + \frac{\partial F}{\partial x}^T(x, w)P \leq -Q, \quad \forall x \in \mathbb{R}^n, \quad w \in \mathbb{R}^m. \tag{3.4}$$

Then, system (3.2) is globally exponentially convergent with the UBSS property and input-to-state convergent for the class of bounded piecewise continuous inputs.

For systems of Lur'e-type form, sufficient conditions for exponential convergence were obtained by V.A. Yakubovich [56]. Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Hw(t) \\ y &= Cx + Dw(t) \\ u &= -\varphi(y, w(t)), \end{aligned} \tag{3.5}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is the output, $w(t) \in \mathbb{R}^m$ is a piecewise continuous input and $\varphi(y, w)$ is a static nonlinearity.

[3]See [48] for a definition of the input-to-state stability property.

Theorem 3.2 ([35, 56]) *Consider system (3.5). Suppose the matrix A is Hurwitz, the nonlinearity $\varphi(y, w)$ satisfies*

$$|\varphi(y_2, w) - \varphi(y_1, w)| \leq K\,|y_2 - y_1|\,, \tag{3.6}$$

for all $y_1, y_2 \in \mathbb{R}$ and $w \in \mathbb{R}^m$, and the frequency response function $G_{yu}(j\omega) = C(j\omega I - A)^{-1}B$ from u to y satisfies

$$\sup_{\omega\in\mathbb{R}} |G_{yu}(j\omega)| =: \gamma_{yu} < \frac{1}{K}. \tag{3.7}$$

Then, system (3.5) is exponentially convergent with the UBSS property and input-to-state convergent for the class of piecewise continuous bounded inputs.[4]

Below follows an alternative result, not based on quadratic Lyapunov functions.

Theorem 3.3 ([35]) *Consider system (3.2). Suppose there exist C^1 functions $V_2(x)$ and $V_1(x_1, x_2)$, $\mathscr{K}$-functions $\alpha_2(s)$, $\alpha_3(s)$, $\alpha_5(s)$, $\gamma(s)$, and $\mathscr{K}_\infty$-functions $\alpha_1(s)$, $\alpha_4(s)$ satisfying the conditions*

$$\alpha_1(|x_1 - x_2|) \leq V_1(x_1, x_2) \leq \alpha_2(|x_1 - x_2|), \tag{3.8}$$

$$\frac{\partial V_1}{\partial x_1}(x_1, x_2)F(x_1, w) + \frac{\partial V_1}{\partial x_2}(x_1, x_2)F(x_2, w) \leq -\alpha_3(|x_1 - x_2|), \tag{3.9}$$

$$\alpha_4(|x|) \leq V_2(x) \leq \alpha_5(|x|), \tag{3.10}$$

$$\frac{\partial V_2}{\partial x}(x)F(x, w) \leq 0 \text{ for } |x| \geq \gamma(|w|) \tag{3.11}$$

for all $x_1, x_2, x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$. Then, system (3.2) is globally uniformly convergent and has the UBSS property for the class of bounded piecewise continuous inputs.

One can show that conditions of Theorems 3.1–3.3 imply incremental stability [2]. In fact, the proof of convergence in these results is based on two basic components:

(1) incremental stability: it guarantees global asymptotic stability of any solution,
(2) existence of a compact positively invariant set: it guarantees existence of a solution $\bar{x}(t)$ that is bounded on $\mathbb{R}$ [10, 56]. By virtue of (1), $\bar{x}(t)$ is globally asymptotically stable, which proves convergence.

Although here incremental stability is used in the sufficient conditions for convergence given above, in general these two properties are not equivalent. However, as

[4]This result is a particular case of a more general condition on $G_{yu}(j\omega)$ in the form of Circle criterion [56].

shown in [46], uniform convergence and incremental stability are equivalent on compact positively invariant sets. In the latter reference, also a necessary and sufficient condition for uniform convergence is formulated, which reads as follows.

Theorem 3.4 ([46]) *Assume that system (3.1) is globally uniformly convergent, with associated steady-state solution $\bar{x}(t)$. Assume that the function F is continuous in (x,t) and C^1 with respect to the x variable. Assume also that the Jacobian $\frac{\partial}{\partial x} f(x,t)$ is bounded, uniformly in t. Then there exist a C^1 function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$, functions α_1, α_2 and $\alpha_3 \in \mathcal{K}_\infty$, and a constant $c \geq 0$ such that*

$$\alpha_1(|x - \bar{x}(t)|) \leq V(t,x) \leq \alpha_2(|x - \bar{x}(t)|), \tag{3.12}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} F(x,t) \leq -\alpha_3(|x - \bar{x}(t)|) \tag{3.13}$$

and

$$V(t,0) \leq c, \ \ t \in \mathbb{R}. \tag{3.14}$$

Conversely, if a differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ and functions α_1, α_2 and $\alpha_3 \in \mathcal{K}_\infty$, and a constant $c \geq 0$ are given such that for some trajectory $\bar{x} : \mathbb{R} \to \mathbb{R}^n$ estimates (3.12)–(3.14) hold, then system (3.1) must be globally uniformly convergent and the solution $\bar{x}(t)$ is the unique bounded solution as in Definition 3.1.

For interconnections of convergent systems, one can obtain similar results as for interconnections of systems with a stable equilibrium. In particular, a series connection of input-to-state convergent (ISC) systems is again an ISC system [35]. Feedback interconnection of two ISC systems is again an ISC system under a small-gain condition on the gain functions $\gamma(r)$ for each subsystem, see details in [6]. Theorems 3.1–3.3 in combination with these interconnection properties provide a practical toolbox of sufficient conditions for the convergence property.

Sufficient conditions for convergence for other classes of systems, such as discrete-time nonlinear systems, piecewise affine systems, linear complementarity systems, switched systems, measure differential inclusions, delay differential equations, have been pursued in [8, 9, 25, 26, 35, 37, 39, 43, 54].

3.2.2 Basic Properties of Convergent Systems

The convergence property is an extension of stability properties of asymptotically stable linear systems with inputs:

$$\dot{x} = Ax + Bw, \tag{3.15}$$

where A is Hurwitz. For any piecewise continuous input $w(t)$ that is bounded on $\mathbb{R}$, this system has a unique solution $\bar{x}_w(t)$ which is defined and bounded on $\mathbb{R}$:

$$\bar{x}_w(t) := \int_{-\infty}^{t} \exp(A(t-s))Bw(s)ds. \tag{3.16}$$

This solution is globally exponentially stable since A is Hurwitz. Therefore, system (3.15) is exponentially convergent for the class of bounded piecewise continuous inputs. This example also illustrates the selection of the steady-state solution in the definition of convergent systems. The steady-state solution is not only a solution that attracts all other solutions in forward time–all solutions of system (3.15) have this property. It is key to realize that, among all these solutions, only one remains bounded both in forward and backward time. The selection of this bounded on $\mathbb{R}$ solution defines the steady state in a unique way for uniformly convergent systems [35]. The natural choice for the definition of the steady-state solution is further illustrated by the following property.

Property 3.1 ([10]) Suppose system (3.2) with a given input $w(t)$ is uniformly convergent. If the input $w(t)$ is constant, the corresponding steady-state solution $\bar{x}_w(t)$ is also constant; if $w(t)$ is periodic with period T, then the corresponding steady-state solution $\bar{x}_w(t)$ is also periodic with the same period T.

As it will be demonstrated in subsequent sections, the following two basic properties of convergent systems will be very instrumental in design and analysis problems within systems and control:

(i) a convergent system defines a steady-state operator $\mathscr{F}w(t) := \bar{x}_w(t)$ that maps bounded on $\mathbb{R}$ inputs to bounded on $\mathbb{R}$ steady-state solutions and periodic inputs to periodic steady-state solutions with the same period.
(ii) any solution of a uniformly convergent system starting in a compact positively invariant set $\mathscr{X}$ is uniformly asymptotically stable in $\mathscr{X}$.

Property (i) is highly instrumental in problems focused on steady-state dynamics, while property (ii) significantly simplifies stability proofs for particular solutions. The latter property follows from [46], where it is shown that for a compact positively invariant set uniform convergence and incremental stability are equivalent. In subsequent sections, we will demonstrate how these two basic properties can be used in various design and analysis problems.

3.3 Controller and Observer Design

In controller and observer design problems for nonstationary systems (e.g., systems with time-varying inputs), the common objective is to ensure, by means of controller design, that a certain solution with desired properties is asymptotically stable with a given domain of attraction (e.g., global). For example, in the observer design problem, the desired solution is the solution of the observed system. In the output tracking and disturbance rejection problem, the desired solution is the one that matches the desired output of the system regardless of the disturbance.

The conventional approach to prove whether a controller/observer solves these problems consists of the following steps:

(a) find a solution of the closed-loop system/observer $\bar{x}(t)$ with desired properties,
(b) translate that solution to the origin through the transformation $z(t) = x(t) - \bar{x}(t)$,
(c) prove asymptotic stability of $z(t) \equiv 0$ with a desired domain of attraction.

Although stability analysis of an equilibrium should be simpler, in many cases this simplicity is essentially reduced by the coordinate transformation: the right-hand side of the system in the new coordinates z typically depends on $\bar{x}(t)$. This makes the analysis challenging and in some cases even prohibitively complex as, for example, for piecewise affine systems [53]. On the other hand, the same design problems can be approached using the property of convergence:

(1) design a feedback controller that ensures uniform convergence of the closed-loop system: as a result, any solution starting in a compact positively invariant set is uniformly asymptotically stable in this set.
(2) design a feedforward controller that ensures that the system with the feedback and feedforward controllers has a solution $\bar{x}(t)$ with the desired properties.

Thus for any compact positively invariant set of initial conditions, the solution $\bar{x}(t)$ will be uniformly asymptotically stable in this set. This approach allows one to avoid the coordinate transformation $z = x - \bar{x}(t)$ and subsequent cumbersome stability analysis of the transformed system.[5]

Let us illustrate the benefit of the above convergence-based approach in the scope of the observer design problem. Consider the system

$$\begin{cases} \dot{x} = F(x, w), \\ y = h(x, w) \end{cases} \tag{3.17}$$

with input w and output y. The objective is to design an observer that asymptotically reconstructs from the measured input and output the state $x(t)$ starting at an unknown initial condition $x(t_0) = x_0$. A conventional way to design an observer is to construct it as a copy of the system dynamics with an output injection term:

$$\begin{cases} \dot{\hat{x}} = F(\hat{x}, w) + L(y, \hat{y}, w), \\ \hat{y} = h(\hat{x}, w), \end{cases} \tag{3.18}$$

where the injection function $L(y, \hat{y}, w)$ satisfies $L(y, y, w) \equiv 0$. The latter condition guarantees that the observer, if initiated in the same initial condition as system (3.17), $\hat{x}(t_0) = x_0$, has a solution $\hat{x}(t) \equiv x(t)$, i.e., condition (2) above is satisfied. If the injection term $L(y, \hat{y}, w)$ is designed in such a way that the observer (3.18) is uniformly convergent, then the desired solution $\hat{x}(t) \equiv x(t)$ is uniformly asymptotically stable in any compact positively invariant set of initial conditions. Hence, one

[5] This benefit has recently also been explicitly recognized in [13].

can think of the observer design as aiming to ensure that the observer is a convergent system rather than aiming at rendering the observer error dynamics asymptotically stable.

The problem of controlled synchronization (e.g., master–slave synchronization) has a lot in common with the observer design problem (see, e.g., [31]). Therefore, it can be treated in the same way, as the observer design problem. The same holds for the output tracking and disturbance rejection problems.

For a class of piecewise affine (PWA) systems, this convergence-based approach allows one to solve the output tracking, synchronization, and observer design problems in a relatively simple manner [53]. For PWA systems these nonstationary problems become very hard to solve by conventional methods if the number of cells with affine dynamics is larger than two.

For the tracking and disturbance rejection problems, in the approach mentioned above one needs to answer the following questions:

- How to find a feedback that makes the closed-loop system uniformly convergent?
- How to find a bounded feedforward input that shapes the output of the steady-state solution to a desired value, and whether such a feedforward exists at all?

For an answer to the first question the reader is referred, for example, to [35], where controller design tools based on quadratic stability, backstepping, separation principle and passivity were developed. The second question will be addressed in the next section.

3.4 Stable Inversion of Nonlinear Systems

The problem of finding a bounded input that ensures the existence of a bounded solution with a desired output is called the stable inversion problem. Conventionally, it is studied after transforming the system into a normal form:

$$\dot{\xi} = p(\xi, \bar{y}, u), \tag{3.19}$$
$$y^{(r)} = q(\bar{y}, \xi) + s(\bar{y}, \xi)u, \tag{3.20}$$

where $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the input; $\xi \in \mathbb{R}^n$ and $\bar{y} := (y, \dot{y}, \ldots, y^{(r-1)})^T$ constitute the state of the system. The functions p, q and s are locally Lipschitz and $s(\bar{y}, \xi)$ is nonzero for all $\bar{y}$ and ξ. For simplicity of the presentation, we assume that the normal form (3.19), (3.20) is defined globally. The reference output trajectory is given by $y_d(t)$, which is bounded together with its r derivatives. From (3.20), we compute an input u corresponding to the reference output trajectory $y_d(t)$:

$$u = s(\bar{y}_d, \xi)^{-1}(y_d^{(r)} - q(\bar{y}_d, \xi)) =: U(\xi, \bar{y}_d, y_d^{(r)}), \tag{3.21}$$

where $\bar{y}_d := (y_d, \dot{y}_d, \ldots, y_d^{(r-1)})^T$. Substituting this control into (3.19), we obtain the tracking dynamics

$$\dot{\xi} = p(\xi, \bar{y}_d(t), U(\xi, \bar{y}_d(t), y_d^{(r)}(t))) =: \bar{p}(\xi, t). \tag{3.22}$$

If we can find a bounded solution $\bar{\xi}(t)$ of (3.22), then the corresponding bounded input $u_d(t)$ can be computed from (3.21) by substituting this $\bar{\xi}(t)$ for ξ. The desired bounded solution of (3.19), (3.20) equals $(\bar{\xi}^T(t), \bar{y}_d^T(t))^T$.

For minimum phase systems, bounded $\bar{y}_d(t)$, $y_d^{(r)}(t)$ ensure boundedness of any solution of the tracking dynamics in (3.22) in forward time. For non-minimum phase systems, this is not the case, since the tracking dynamics are unstable. However, there may still exist a bounded solution, as it has been shown in [11, 12] for the local case.

To extend that result to the nonlocal case, we can, first, observe that the unstable tracking dynamics can be convergent in backward time. In this case, all solutions except for one diverge to infinity as $t \to +\infty$. The only bounded on $\mathbb{R}$ solution is the steady-state solution from the definition of convergence.

We can apply similar reasoning to the case if (3.22) can be decomposed (after, possibly, a coordinate transformation) into a series connection of two systems:

$$\dot{\eta} = F(\eta, t), \quad \eta \in \mathbb{R}^{n_s}, \tag{3.23}$$
$$\dot{\zeta} = G(\zeta, \eta, t), \quad \zeta \in \mathbb{R}^{n_u}. \tag{3.24}$$

If system (3.23) is convergent and (3.24) with η as input is convergent in backward time for the class of bounded continuous inputs, one can easily verify that the bounded on $\mathbb{R}$ solution of (3.23), (3.24) is unique and it equals $(\bar{\eta}^T(t), \bar{\zeta}_{\bar{\eta}}^T(t))^T$, where $\bar{\eta}(t)$ is the steady-state solution of (3.23) and $\bar{\zeta}_{\bar{\eta}}(t)$ is the steady-state solution of (3.24) corresponding to $\bar{\eta}(t)$.

If the tracking dynamics can be represented as a feedback interconnection of a convergent system in forward time and a convergent system in backward time,

$$\dot{\eta} = F(\eta, \zeta, t), \quad \eta \in \mathbb{R}^{n_s}, \tag{3.25}$$
$$\dot{\zeta} = G(\zeta, \eta, t), \quad \zeta \in \mathbb{R}^{n_u}, \tag{3.26}$$

then one can still ensure the existence of a unique bounded on $\mathbb{R}$ solution if a certain small-gain condition is satisfied, as formalized in the result below.

Theorem 3.5 *Consider system (3.25). Suppose that*

1. *system (3.25) with ζ as input is convergent for the class of continuous bounded inputs with the corresponding steady-state operator $\mathscr{F}$ being Lipschitz continuous with a Lipschitz constant γ_F, i.e., $\|\mathscr{F}\zeta_1 - \mathscr{F}\zeta_1\|_\infty \le \gamma_F \|\zeta_1 - \zeta_2\|_\infty$,*
2. *system (3.26) with η as input is convergent in backward time for the class of continuous bounded inputs with the corresponding steady-state operator $\mathscr{G}$ being Lipschitz continuous with a Lipschitz constant γ_G, i.e., $\|\mathscr{G}\eta_1 - \mathscr{G}\eta_1\|_\infty \le \gamma_G \|\eta_1 - \eta_2\|_\infty$.*

If the small-gain condition

$$\gamma_F \gamma_G < 1, \tag{3.27}$$

is satisfied, then system (3.25) has a unique bounded on $\mathbb{R}$ *solution.*

Finding the Lipschitz constant for the steady-state operator as well as a numerical method for the calculation of the bounded solution are described in [33].

The simple convergence-based considerations presented above extend the local results from [11, 12] on stable inversion of non-minimum phase systems to the nonlocal nonlinear case.

3.5 The Output Regulation Problem

In [35], the notion of convergent systems was successfully applied to solve the output regulation problem for nonlinear systems in a nonlocal setting. Before that, the output regulation problem was solved for linear systems, see, e.g., [14], resulting in the well-known internal model principle, and for nonlinear systems in a local setting [7, 20]. The application of the convergent systems property allowed us to extend these results to nonlocal problem settings for nonlinear systems. In particular, necessary and sufficient conditions for the solvability of the global nonlinear output regulation problem were found [35].[6] These conditions included, as their particular case, the solvability conditions for the linear and the local nonlinear cases.

The output regulation problem can be treated as a special case of the tracking and disturbance rejection problem, where the reference signal for the output and/or disturbance are generated by an external autonomous system. Consider systems modeled by equations of the form

$$\dot{x} = f(x, u, w), \tag{3.28}$$

$$e = h_r(x, w), \tag{3.29}$$

$$y = h_m(x, w), \tag{3.30}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^k$, regulated output $e \in \mathbb{R}^{l_r}$, and measured output $y \in \mathbb{R}^{l_m}$. The exogenous signal $w(t) \in \mathbb{R}^m$, which can be viewed as a disturbance in (3.28) or as a reference signal in (3.29), is generated by an external autonomous system

$$\dot{w} = s(w), \tag{3.31}$$

starting in a compact positively invariant set of initial conditions $\mathscr{W}_+ \subset \mathbb{R}^m$. System (3.31) is called an exosystem.

[6]These results were obtained in parallel with [21], where an alternative approach to nonlocal nonlinear output regulation problem was pursued.

The global uniform output regulation problem is to find, if possible, a controller of the form

$$\dot{\xi} = \eta(\xi, y), \quad \xi \in \mathbb{R}^q, \tag{3.32}$$
$$u = \theta(\xi, y), \tag{3.33}$$

for some $q \geq 0$ such that the closed-loop system

$$\dot{x} = f(x, \theta(\xi, h_m(x, w)), w), \tag{3.34}$$
$$\dot{\xi} = \eta(\xi, h_m(x, w)) \tag{3.35}$$

satisfies three conditions:

- *regularity:* the right-hand side of the closed-loop system is locally Lipschitz with respect to (x, ξ) and continuous with respect to w;
- *uniform convergence:* the closed-loop system is uniformly convergent with the UBSS property for the class of bounded continuous inputs;
- *asymptotic output zeroing:* for all solutions of the closed-loop system and the exo-system starting in $(x(0), \xi(0)) \in \mathbb{R}^{n+q}$ and $w(0) \in \mathcal{W}_+$ it holds that $e(t) = h_r(x(t), w(t)) \to 0$ as $t \to +\infty$.[7]

In conventional formulations of the output regulation problem, some other stability requirement on the closed-loop system is used instead of the requirement of uniform convergence, e.g., (global) asymptotic stability of the origin for zero input or boundedness of solutions. For linear and local nonlinear cases, it can be shown that these requirements are equivalent to the requirement of the uniform convergence. For nonlocal nonlinear problem settings, boundedness of solutions of the uniformly convergent system follows from the definition of uniform convergence and boundedness of $w(t)$. Thus the choice of uniform convergence as a “stability requirement” is natural in this problem. It leads to a necessary and sufficient solvability condition for the output regulation problem that includes, as its particular case, the solvability conditions for the linear and local nonlinear output regulation problems, see e.g., [20]. This fact indicates that the right problem formulation for the nonlocal output regulation problem is captured in this way.

From the controller design point of view, this problem can be addressed using the same approach as described in Sect. 3.3: design a controller such that the closed-loop system is (a) uniformly convergent with the UBSS property and (b) has a solution along which the asymptotic output zeroing condition holds, see Sect. 3.3. The questions to be addressed in this approach are, first, whether the structure of the controlled system and the exo-system allows for a solution of the problem; second, how to design a controller that makes the closed-loop system uniformly convergent; and, third, how to check that this controller ensures existence of a solution with the asymptotic output zeroing property.

[7] Other variants of the uniform output regulation problem can be found in [35].

While the second question is addressed in a number of papers, see, e.g., [35], the first and the third questions are answered by the following result.

Theorem 3.6 ([35]) *Consider system (3.28)–(3.30) and exo-system (3.31) with a compact positively invariant set of initial conditions $\mathcal{W}_+ \subset \mathbb{R}^m$.*

(i) The global uniform output regulation problem is solvable only if there exist continuous mappings $\pi(w)$ and $c(w)$ defined in some neighborhood of $\Omega(\mathcal{W}_+)$– the ω-limit set for solutions of exo-system (3.31) starting in $\mathcal{W}_+$– and satisfying the regulator equations

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \tag{3.36}$$

$$0 = h_r(\pi(w(t)), w(t)), \tag{3.37}$$

for all solutions of exo-system (3.31) satisfying $w(t) \in \Omega(\mathcal{W}_+)$ for $t \in \mathbb{R}$.

(ii) If controller (3.32), (3.33) makes the closed-loop system uniformly convergent with UBSS property for the class of bounded continuous inputs, then it solves the global uniform output regulation problem if and only if for any $w(t) \in \Omega(\mathcal{W}_+)$ the controller has a bounded solution with input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ and output $\bar{u}_w(t) = c(w(t))$.

Notice that solvability of the regulator equations implies that for any $w(t)$ from the omega-limit set $\Omega(\mathcal{W}_+)$, system (3.28) has a stable inversion $\bar{u}_w(t)$ for the desired output $e(t) \equiv 0$. The second condition implies that the controller, being driven by the output $\bar{y}_w(t)$, can generate the control signal $\bar{u}_w(t)$. Since all solutions of the exo-system starting in $\mathcal{W}_+$ converge to the omega-limit set $\Omega(\mathcal{W}_+)$, it is enough to verify conditions (i) and (ii) only on $\Omega(\mathcal{W}_+)$.

Here we see that the notion of uniform convergence allows us to extend the solvability conditions and controller design methods for the output regulation problem from the linear and local nonlinear cases to nonlocal nonlinear case.

3.6 Frequency Response Functions for Nonlinear Convergent Systems

Frequency response functions (FRF) for linear time invariant systems form a foundation for a large number of analysis and design tools. One can define FRF for linear systems through the Laplace transform. For nonlinear systems, however, the Laplace transform is not defined. If we notice that, for linear systems, FRF can also be viewed as a function that fully characterizes steady-state responses to harmonic excitations, we can extend the notion of FRF to nonlinear convergent systems of the form

$$\dot{z} = F(x, w), \tag{3.38}$$

$$y = h(x) \tag{3.39}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ and scalar input w. Recall that convergent systems have a uniquely defined periodic response to a periodic excitation (with the same period time). We can define the FRF as a mapping that maps input $w(t) = a \sin \omega t$ to the corresponding periodic steady-state solution $\bar{x}_{a,\omega}(t)$. As follows from the next result, this mapping has quite a simple structure.

Theorem 3.7 ([38]) *Suppose system (3.38) is uniformly convergent with UBSS property for the class of continuous bounded inputs $w(t)$. Then, there exists a continuous function $\alpha : \mathbb{R}^3 \to \mathbb{R}^n$ such that for any harmonic excitation of the form $w(t) = a \sin \omega t$, the corresponding (asymptotically stable) steady-state solution equals*

$$\bar{x}_{a\omega}(t) := \alpha(a \sin(\omega t), a \cos(\omega t), \omega). \tag{3.40}$$

As follows from Theorem 3.7, the function $\alpha(v_1, v_2, \omega)$ contains *all* information on the steady-state solutions of system (3.38) corresponding to harmonic excitations. For this reason, we give the following definition.

Definition 3.5 The function $\alpha(v_1, v_2, \omega)$ defined in Theorem 3.7 is called the *state frequency response function*. The function $h(\alpha(v_1, v_2, \omega))$ is called the *output frequency response function*.

In general, it is not easy to find such frequency response functions analytically. In some cases they can be found based on the following lemma.

Lemma 3.1 ([38]) *Under the conditions of Theorem 3.7, if there exists a continuous function $\alpha(v_1, v_2, \omega)$ differentiable in $v = [v_1, v_2]^T$ and satisfying*

$$\frac{\partial \alpha}{\partial v}(v, \omega) S(\omega) v = F(\alpha(v, \omega), v_1), \quad \forall\, v, \omega \in \mathbb{R}^2 \times \mathbb{R}, \tag{3.41}$$

then this $\alpha(v_1, v_2, \omega)$ is the state frequency response function. Conversely, if the state frequency response function $\alpha(v_1, v_2, \omega)$ is differentiable in v, then it is a unique solution of (3.41).

With this definition of the frequency response function, we can further define Bode magnitude plots for convergent systems that would map frequency ω and amplitude a of the harmonic input to a measure of the steady-state output (e.g., L_2 norm) normalized with the input amplitude a. This extension of the Bode plot enables graphical representation of convergent system steady-state responses at various frequencies and amplitudes of the excitation (due to nonlinearity it will depend on both). In this sense, such frequency response functions are instrumental in supporting frequency domain analysis of nonlinear convergent systems, similar to that employed for linear systems.

3.7 Steady-State Performance Optimization

For linear systems, Bode plots are commonly used to evaluate steady-state sensitivities of the closed-loop system to measurement noise, disturbances, and reference signals. If the performance of the closed-loop system, evaluated through the (frequency domain) sensitivity functions, is not satisfactory, controller parameters can be tuned to achieve desired or optimal steady-state performance. For nonlinear systems, such performance-based controller tuning is much more challenging. Even in the simple case of a convergent closed-loop system with a linear plant being controlled by a linear controller with a variable gain element [19, 55], this problem is far from straightforward. First, one needs to evaluate/calculate steady-state responses to the noise, disturbance and/or reference signals. In practice, this may be challenging already for excitations with only one harmonic (see previous section). In reality, the excitations will consist of multiple harmonics, and calculation of the steady-state solution to these excitations can be a challenge in itself. Second, after the steady-state solution is evaluated, one needs to find how to tune controller parameters to improve/optimize certain performance characteristics of the steady state responses.

For a subclass of nonlinear convergent systems, both of these problems can be solved numerically in a computationally efficient way. Let us consider Lur'e-type systems of the form

$$\dot{x} = Ax + Bu + Hw(t) \tag{3.42}$$

$$y = Cx + Dw(t) \tag{3.43}$$

$$u = -\varphi(y, w(t), \theta) \tag{3.44}$$

$$e = C_e x + D_e w(t), \tag{3.45}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is the output, $w(t) \in \mathbb{R}^m$ is a piecewise continuous input, and $e \in \mathbb{R}$ is a performance output. We assume that the nonlinearity $\varphi : \mathbb{R} \times \mathbb{R}^m \times \Theta \to \mathbb{R}$ is memoryless and may depend on n_θ parameters collected in the vector $\theta = [\theta_1, \ldots, \theta_{n_\theta}]^T \in \Theta \subset \mathbb{R}^{n_\theta}$. We also assume that $\varphi(0, w, \theta) = 0\ \forall w \in \mathbb{R}^m$ and $\theta \in \Theta$. For simplicity, we only consider the case in which the parameters θ appear in the nonlinearity φ and none of the system matrices. An extension to the latter situation is relatively straightforward. The functions $G_{yu}(s)$, $G_{yw}(s)$, $G_{eu}(s)$ and $G_{ew}(s)$ are the corresponding transfer functions from inputs u and w to outputs y and e of the linear part of system (3.42)–(3.45).

In this section, we consider the case of periodic disturbances $w(t)$. Recall that if system (3.42)–(3.45) satisfies the conditions of Theorem 3.2 for all $\theta \in \Theta$, then it is exponentially convergent and thus for each periodic $w(t)$ it has a unique periodic steady-state solution $\bar{x}_w(t, \theta)$.

Once the steady-state solution is uniquely defined, we can define a performance measure to quantify the steady-state performance of the system for a particular T-periodic input $w(t)$ and particular parameter θ. For example, it can be defined as

$$J(\theta) = \frac{1}{T}\int_0^T \bar{e}_w(t,\theta)^2 dt, \tag{3.46}$$

where $\bar{e}_w(t,\theta)$ is the performance output response corresponding to the steady-state solution. If we are interested in quantifying simultaneously the steady-state performance corresponding to a family of disturbances, $w_1(t), w_2(t), \dots, w_N(t)$, with periods $T_1, \dots, T_N$, we, e.g., can choose a weighted sum of the functionals of the form (3.46). The choice of the performance objective strongly depends on the needs of the particular application.

System (3.42)–(3.45) may represent a closed-loop nonlinear control system with θ being a vector of controller parameters. Ultimately, we aim to optimize the steady-state performance of this system by tuning $\theta \in \Theta$. To this end, we propose to use gradient-like optimization algorithms, which provide a direction for decrease of $J(\theta)$ based on the gradient of $\partial J/\partial\theta(\theta)$. This approach requires computation of the gradient of $J(\theta)$. For the performance objective as in (3.46), the gradient equals

$$\frac{\partial J}{\partial \theta}(\theta) = \frac{2}{T}\int_0^T \bar{e}_w(t,\theta)\frac{\partial \bar{e}_w}{\partial\theta}(t,\theta)dt, \tag{3.47}$$

under the condition that $\bar{e}_w(t,\theta)$ is C^1 with respect to θ. Here we see that in order to compute the gradient of $J(\theta)$ we need to know both $\bar{e}_w(t,\theta)$ and $\partial\bar{e}_w/\partial\theta(t,\theta)$. The following theorem provides, firstly, conditions under which $\bar{x}_w(t,\theta)$ (and therefore $\bar{e}_w(t,\theta)$) is C^1 with respect to θ, and, secondly, gives us an equation for the computation of $\partial\bar{e}_w/\partial\theta(t,\theta)$.

Theorem 3.8 ([40]) *If system (3.42)–(3.45) satisfies the conditions of Theorem 3.2 for all $\theta \in \Theta$, and the nonlinearity $\varphi(y,w,\theta)$ is C^1 for all $y \in \mathbb{R}$, $w \in \mathbb{R}^m$ and θ in the interior of Θ, then the steady-state solution $\bar{x}_w(t,\theta)$ is C^1 in θ. The corresponding partial derivatives $\partial\bar{x}_w/\partial\theta_i(t,\theta)$ and $\partial\bar{e}_w/\partial\theta_i(t,\theta)$ are, respectively, the unique T-periodic solution $\bar{\Psi}(t)$ and the corresponding periodic output $\bar{\mu}(t)$ of the system*

$$\dot{\Psi} = A\Psi + BU + BW_i(t) \tag{3.48}$$

$$\lambda = C\Psi \tag{3.49}$$

$$U = -\frac{\partial\varphi}{\partial y}(\bar{y}(t,\theta), w(t), \theta)\lambda \tag{3.50}$$

$$\mu = C_e\Psi, \tag{3.51}$$

where $W_i(t) = -\partial\varphi/\partial\theta_i(\bar{y}_w(t,\theta), w(t), \theta)$.

To calculate the steady-state output $\bar{e}(t,\theta)$, notice that it is a solution of the following equation:

$$\bar{y} = \mathscr{G}_{yu} \circ \mathscr{F}\bar{y} + \mathscr{G}_{yw}w, \tag{3.52}$$

$$\bar{e} = \mathscr{G}_{eu} \circ \mathscr{F}\bar{y} + \mathscr{G}_{ew}w, \tag{3.53}$$

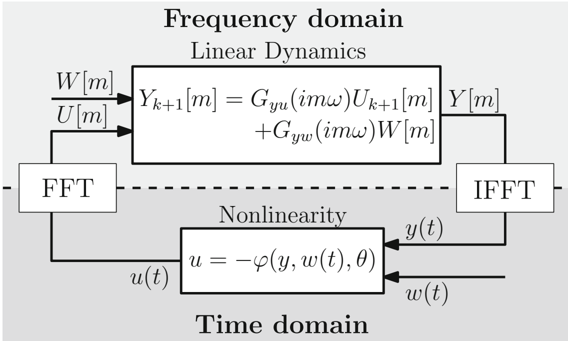

Fig. 3.1 Mixed-Time-Frequency algorithm to compute the steady-state solutions

where $\mathscr{G}_{yu}$, $\mathscr{G}_{yw}$, $\mathscr{G}_{eu}$ and $\mathscr{G}_{ew}$ are the linear operators mapping periodic inputs $u(t)$ and $w(t)$ to periodic steady-state outputs $y(t)$ and $e(t)$ of the linear part of system (3.42)–(3.45); and $\mathscr{F}$ is the operator defined by $\mathscr{F}y(t) := -\varphi(y(t), w(t), \theta)$. The conditions of Theorem 3.2 imply that the superposition operator $\mathscr{G}_{yu} \circ \mathscr{F}$ is a contraction operator acting from $L_2(T)$ to $L_2(T)$. Therefore, $\bar{y}$ (and then $\bar{e}$) can be calculated from the iterative process

$$u_{k+1} = \mathscr{F} y_k \tag{3.54}$$

$$y_{k+1} = \mathscr{G}_{yu} u_{k+1} + \mathscr{G}_{yw} w, \tag{3.55}$$

starting from an arbitrary initial guess y_0. To speed up the calculation, this iterative process can be implemented both in frequency domain (to compute $\mathscr{G}_{yu} u_{k+1}$ and $\mathscr{G}_{yw} w$) and in time domain (to compute the output of the nonlinearity $\mathscr{F} y_k$). This is schematically shown in Fig. 3.1, where Y, W and U denote the vectors of the Fourier coefficients (indexed by m) of the signals $y(t)$, $w(t)$ and $u(t)$, respectively. and (I)FFT denotes the (inverse) Fast Fourier Transform.

If at every iteration we truncate the Fourier coefficients for $u_k(t)$ and $w(t)$ to keep only the N first harmonics (which is inevitable in any numerical implementation of the algorithm), the algorithm will still converge from an arbitrary initial guess $y_0(t)$ to a unique solution $\bar{y}^N$. The error caused by the truncation can be estimated by [40]:

$$\begin{aligned} \|\bar{y} - \bar{y}^N\|_{L_2} \leq \Big\{ & \sup_{|m|>N} |G_{yu}(im\omega)| \gamma_{yw} \tfrac{K\|w\|_{L_2}}{1-\gamma_{yu}K} \\ & + \gamma_{yw} \|w - w^N\|_{L_2} \Big\} \tfrac{1}{1-\gamma_{yu}K}, \end{aligned} \tag{3.56}$$

where $\gamma_{yw} := \sup_{m\in\mathbb{Z}} |G_{yw}(im\omega)|$ and $\|w - w^N\|_{L_2}$ is the error of truncation of harmonics in $w(t)$ higher than N. From this estimate, we can conclude that by choosing N high enough, one can reach any desired accuracy of approximation $\bar{y}(t)$ by the solution $\bar{y}^N$ of the algorithm with the truncation. Notice that this algorithm with truncation can be considered as a multiharmonic variant of the describing function method, described in [30] for *autonomous* systems.

After computing $\bar{y}_w$ and $\bar{e}_w$, one can then compute in the same way the partial derivatives $\partial \bar{e}_w / \partial \theta(t, \theta)$, since system (3.48)–(3.51) satisfies the same conditions as the original system (3.42)–(3.45) in Theorem 3.2. Then one can compute $\partial J / \partial \theta$ and proceed to gradient-like optimization of θ.

Details on implementation of these numerical algorithms can be found in [40], where they, in combination with a gradient optimization method, were applied for tuning parameters of a variable gain controller for wafer stage control. The results presented in [40] demonstrated very fast convergence of the algorithms for calculation of the steady-state solution and its gradients, as well as efficient performance of the gradient based tuning algorithm. This turns the algorithm into a powerful numerical method for optimizing steady-state performance of nonlinear closed-loop systems of Lur'e-type form.

In this section, we have shown how the convergence property can be instrumental in supporting the model-based performance optimization of nonlinear control systems. In the next section, we also consider the problem of performance optimization of nonlinear systems, where again performance is characterized by periodically time-varying steady-state solutions. However, now it is assumed that no model or disturbance information is available to support performance optimization and therefore a model-free optimization approach called extremum seeking is adopted.

3.8 Extremum Seeking Control

Extremum seeking control is a model-free, online approach for performance optimization for dynamical systems. The large majority of the works in extremum seeking is considering the case in which the performance of the system is quantified in terms of a (unknown) performance objective function depending on the *equilibrium state* of the system [23, 51, 52]. However, in many cases the performance of a system is characterized by *time-varying behaviors*; as an example, one can think of tracking control problems for high-tech positioning systems, such as industrial robots, wafer scanners or pick-and-place machines in which the machine's functioning relies on the accurate realization of time-varying (or periodic for repetitive tasks) reference trajectories.

In this section, we will show how the concept of convergence can be a key underlying property of the dynamical system subjected to an extremum seeker when the performance objective depends on periodic steady-state trajectories of the plant. For more details on extremum seeking for nonlinear plants with periodic steady-state solutions, we refer to [17].

Let us consider a nonlinear dynamical system of the form

$$\dot{x} = f(x, u, \theta, w(t)), \tag{3.57}$$

$$y = h(x, w(t)), \tag{3.58}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and the input, $w \in \mathbb{R}^l$ is an input disturbance and $\theta \in \mathbb{R}$ is a system parameter. The disturbances $w(t)$ are generated by an exo-system of the following form:

$$\dot{w} = \varphi(w). \tag{3.59}$$

We assume that for any initial condition $w(0) \in \mathbb{R}^l$, the solution of the exo-system (3.59) is uniformly bounded (in backward and forward time) and periodic with a known constant period $T_w > 0$.

Consider a state-feedback controller of the following form[8]

$$u = \alpha(x, \theta). \tag{3.60}$$

Now we assume that the closed-loop plant (3.57), (3.60) is uniformly convergent for any fixed $\theta \in \mathbb{R}$. The convergence property implies that for any fixed $\theta \in \mathbb{R}$, there exists a unique, bounded for all $t \in \mathbb{R}$, uniformly globally asymptotically stable steady-state solution $\bar{x}_{\theta,w}(t)$ of the closed-loop plant (3.57), (3.60).[9] As explained in Sect. 3.2, the convergence property implies that the steady-state response $\bar{x}_{\theta,w}(t)$ is periodic with period T_w, given the nature of the exo-system, which produces periodic disturbance inputs with period T_w.

We aim to find the fixed value of $\theta \in \mathbb{R}$ that optimizes the steady-state performance of the closed-loop plant (3.57), (3.60). To this end, we design a cost function that defines performance in terms of the system output y. As a stepping stone, we introduce various signal-norm-based performance measures of the following form:

$$L_p(y_d(t)) := \left(\frac{1}{T_w} \int_{t-T_w}^{t} |y(\tau)|^p d\tau \right)^{\frac{1}{p}}, \tag{3.61}$$

$$L_\infty(y_d(t)) := \max_{\tau \in [t-T_w, t]} |y(\tau)| \tag{3.62}$$

with $p \in [1, \infty)$. The argument $y_d(t)$ of the performance measures in (3.61), (3.62) represents a (past) function segment of the output, characterizing the performance, and is defined by $y_d(t) := y(t + \tau)$ for all $\tau \in [-t_d, 0]$, for some $t_d > T_w$, see [17] for details. We use one of the performance measures in (3.61), (3.62) in the design of the cost function, which is given by

$$q = Q_i(y_d(t)) := g \circ L_i(y_d(t)), \; i \in [1, \infty], \tag{3.63}$$

where the function $g(\cdot)$ further characterizes the performance cost.

Note that, by the grace of convergence, the cost function Q_i is constant in steady state. Finally, it is assumed that the steady-state performance map, i.e., the map from constant θ to q in steady state, exhibits a unique maximum. It is this maximum that we

[8] Sufficient smoothness of the functions f, h and α is assumed.

[9] In fact, a particular Lyapunov-based stability certificate is required for the solution $\bar{x}_{\theta,w}(t)$ in the scope of this section, see [17].

aim to find using an extremum seeking controller, without employing knowledge on the plant dynamics, the performance map or the amplitude or phase of the disturbance.

Next, we introduce the extremum seeker that will optimize (maximize) the steady-state performance output q. The total extremum seeking scheme is depicted schematically in Fig. 3.2 and consists of a gradient estimator (estimating the gradient of the cost function with respect to θ) and an optimizer (devised to steer the parameter θ to the optimum). The optimizer is given by

$$\dot{\hat{\theta}} = Ke, \tag{3.64}$$

where K is the optimizer gain and e is the gradient estimate provided by the gradient estimator. The gradient estimator employed here is based on a moving average filter called the mean-over-perturbation-period (MOPP) filter:

$$e = \frac{\omega}{a\pi} \int_{t-\frac{2\pi}{\omega}}^{t} q(\tau) \sin(\omega(t - \phi)) d\tau, \tag{3.65}$$

where ω and a are the frequency and amplitude of the dither signal used to perturb the parameter input to the plant (therewith facilitating gradient estimation), see Fig. 3.2, and ϕ is a nonnegative constant. We note that both the performance measure in (3.61)–(3.63) and the MOPP estimation filter in (3.65) introduce delay in the closed-loop dynamics therewith challenging the analysis of stability properties of the resulting closed-loop system.

Still, it can be shown [17] that, under the assumptions posed above (in particular the convergence property), the total closed-loop system (3.57), (3.60), (3.63) including extremum seeking controller (3.64), (3.65) is semi-globally practically asymptotically stable in the sense that the parameter θ converges arbitrarily closely to its optimal value and the state solution of the plant converges arbitrarily closely to the optimal steady-state plant behavior, for arbitrarily large sets of initial conditions.

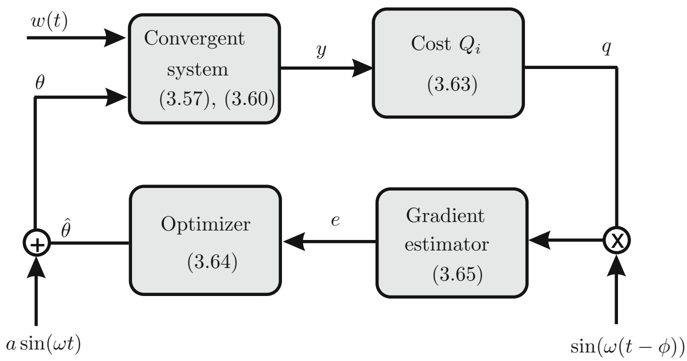

Fig. 3.2 Schematic representation of the extremum seeking scheme

n.v.d.wouw@tue.nl

The latter can be achieved by making the parameters a, ω, and K of the extremum seeker small enough.

Summarizing, the convergence property is instrumental in guaranteeing a unique and asymptotically stable periodic output response, which allows for a unique steady-state performance definition and, in turn, facilitates employing an extremum seeker to optimize the performance characterized by periodic steady-state solutions.

3.9 Model Reduction

In this section, we show how the convergence property can be instrumental in the scope of model reduction for a class of nonlinear systems.[10] The class of systems under consideration involves a feedback interconnection $\Sigma = (\Sigma_{lin}, \Sigma_{nl})$ between a linear system

$$\Sigma_{lin} : \begin{cases} \dot{x} &= Ax + B_u u + B_v v \\ y &= C_y x \\ w &= C_w x, \end{cases} \tag{3.66}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $v \in \mathbb{R}^s$ and $w \in \mathbb{R}^q$, and a nonlinear system

$$\Sigma_{nl} : \begin{cases} \dot{z} &= g(z, w) \\ v &= h(z), \end{cases} \tag{3.67}$$

where, $z \in \mathbb{R}^r$, see the left part of Fig. 3.3.

As will be made more precise below, we assume that the plant Σ is (input-to-state) convergent and we will preserve such property after the model reduction. At a conceptual level, the system being convergent (before and after reduction) helps to reason about the quality of a reduced-order model. To understand this, suppose that the plant and its reduction are not convergent. Then, these systems may have complex nonlinear dynamics characterized by multiple (stable and/or unstable)

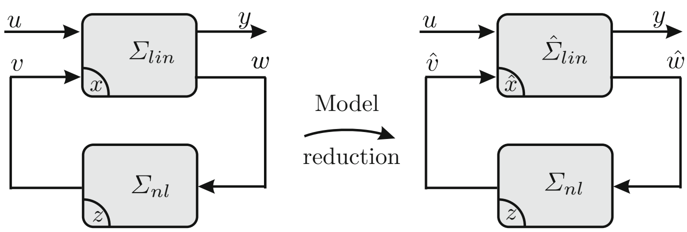

Fig. 3.3 Feedback interconnection plant dynamics and model reduction strategy

[10]The model reduction approach described here is based on [6].

attractors, such as e.g., equilibria and limit cycles, each associated with a potentially complex region of attraction. When reasoning about the quality of the reduction, one typically desires to compare solutions of the reduced-order and original system, especially when aiming to quantify a reduction error. It is hard to envision how such a comparison could be made if both systems have multiple attractors, even with regions of attraction defined on state spaces of different dimension (due to the reduction). The assumption of convergence facilitates the unique comparison of the output solutions of the reduced-order and original system, for a given identical input, as both systems now have a unique attractor (characterized by the unique steady-state solution); hence, the convergence property significantly simplifies establishing a clear definition of reduction error, as will be further explained below.

Figure 3.3 also expresses the fact that we pursue model reduction of the total system Σ by reducing the linear part Σ_{lin} of the dynamics and reconnecting the reduced-order linear dynamics $\hat{\Sigma}_{lin}$ to the nonlinear dynamics Σ_{nl}. This approach is inspired by practical applications in which, firstly, the high-dimensional nature of the dynamics is due to the linear dynamics and, second, the nonlinearities only act locally. Examples of such systems, e.g., can be found in mechanical systems in which the structural dynamics leads to high-dimensional models and local nonlinearities relate to friction, hysteresis, or nonlinear actuator dynamics. Applications in which such models arise can, e.g., be found in high-speed milling or drilling applications. A benefit of such an approach in which model reduction is applied to the linear subsystem only is the fact that a wide range of computationally efficient model reduction methods for linear systems exist.

Assumption 3.1 Now, we adopt the following assumptions on the system Σ:

- Σ_{lin} is asymptotically stable (i.e., A is Hurwitz), implying that Σ_{lin} is input-to-state convergent,
- Σ_{nl} is input-to-state convergent.

By the grace of the first bullet in Assumption 3.1, we have that there exist steady-state operators defined as $\mathscr{F}(u, v) := \bar{x}_{u,v}$, with $\bar{x}_{u,v}$ being the steady-state solutions of the convergent system Σ_{lin}, and $\mathscr{F}_i(u, v) = C_i \bar{x}_{u,v}$, $i \in \{y, w\}$, where the latter define the steady-state output operators of Σ_{lin} for outputs y and w. These steady-state output operators are (by linearity) incrementally bounded as

$$\|\mathscr{F}_i(u_2, v_2) - \mathscr{F}_i(u_1, v_1)\|_\infty \leq \chi_{ix}(\gamma_{xu}\|u_2 - u_1\|_\infty + \gamma_{xv}\|v_2 - v_1\|_\infty), \quad (3.68)$$

for $i \in \{y, w\}$. In (3.68), γ_{xu}, γ_{xv} denote the gain functions of the steady-state operator $\mathscr{F}(u, v)$, whereas χ_{ix} represent incremental bounds on the output equations of Σ_{lin}. The assumption in the second bullet of Assumption 3.1 implies that there exists a steady-state operator $\mathscr{G}w := \bar{z}_w$, which satisfies $\|\mathscr{G}w_2 - \mathscr{G}w_1\| \leq \gamma_{zw}\|w_2 - w_1\|_\infty$. If additionally, there exists an incremental bound for the output function h of Σ_{nl} such that $\|h(z_2) - h(z_1)\|_\infty \leq \chi_{vz}\|z_2 - z_1\|_\infty$, then we have that the steady-state output operator $\mathscr{G}_v w := h(\bar{z}_w)$ of Σ_{nl} satisfies

n.v.d.wouw@tue.nl

$$\|\mathscr{G}_v(w_2) - \mathscr{G}_v(w_1)\|_\infty \le \chi_{vz} \circ \gamma_{zw} \|w_2 - w_1\|_\infty. \tag{3.69}$$

All the gain functions above are class $\mathscr{K}_\infty$ functions.

Next, the total nonlinear system Σ is assumed to satisfy the following small-gain condition.

Assumption 3.2 There exist class $\mathscr{K}_\infty$ functions ρ_1 and ρ_2 such that Σ satisfies the small-gain condition

$$(id + \rho_1) \circ \gamma_{xv} \circ \chi_{vz} \circ (id + \rho_2) \circ \gamma_{zw} \circ \chi_{wx}(s) \le s, \tag{3.70}$$

for all $s \ge 0$.

Assumption 3.2 implies, see [4, 6], that the feedback interconnection Σ is input-to-state convergent.

As a next step, we assume that the reduced-order linear system $\hat{\Sigma}_{lin}$, see Fig. 3.3, given by

$$\hat{\Sigma}_{lin} : \begin{cases} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_u u + \hat{B}_v \hat{v} \\ \hat{y} &= \hat{C}_y \hat{x} \\ \hat{w} &= \hat{C}_w \hat{x}, \end{cases} \tag{3.71}$$

where $\hat{x} \in \mathbb{R}^k$, with $k < n$ establishing the order reduction, $\hat{y} \in \mathbb{R}^p$, $\hat{v} \in \mathbb{R}^s$ and $\hat{w} \in \mathbb{R}^q$, is asymptotically stable. This implies that there exist steady-state output operators for $\hat{\Sigma}_{lin}$: $\hat{\mathscr{F}}_i(u, \hat{v})$, $i \in \{y, w\}$, that are incrementally bounded, i.e.,

$$\|\hat{\mathscr{F}}_i(u_2, \hat{v}_2) - \hat{\mathscr{F}}_i(u_1, \hat{v}_1)\|_\infty \le \hat{\chi}_{ix} \circ \hat{\gamma}_{xv} \|\hat{v}_2 - \hat{v}_1\|_\infty, \tag{3.72}$$

for $i \in \{y, w\}$. Moreover, we assume that there exists an error bound for the reduction of the linear part of the system according to

$$\|\mathscr{E}_i(u_2, v_2) - \mathscr{E}_i(u_1, v_1)\|_\infty \le \varepsilon_{iu} \|u_2 - u_1\|_\infty + \varepsilon_{iv} \|v_2 - v_1\|_\infty, \tag{3.73}$$

where $\mathscr{E}_i(u, v) := \mathscr{F}_i(u, v) - \hat{\mathscr{F}}_i(u, v)$, for $i \in \{y, w\}$, and ε_{ij}, for $i \in \{y, w\}$ and $j \in \{u, v\}$, are positive constants.

In fact, the above assumption on the stability of the reduced-order system and the availability of an error bound for the linear reduced-order system can be directly satisfied since reduction techniques exist that guarantee the satisfaction of both assumptions. In fact, an a priori error bound exists when the reduced-order system $\hat{\Sigma}_{lin}$ is obtained by balanced truncation. Namely, an error bound on the norm on the impulse response as in [16, 24] provides a bound on the $\mathscr{L}_\infty$-induced system norm. Alternatively, an error bound can be computed a posteriori using results from [47], typically leading to a tighter bound.

Now, the following result can be formulated, which guarantees input-to-state convergence of the nonlinear reduced-order system $\hat{\Sigma} = (\hat{\Sigma}_{lin}, \Sigma_{nl})$ and provides and error bound for $\hat{\Sigma}$.

Theorem 3.9 *Let $\Sigma = (\Sigma_{lin}, \Sigma_{nl})$ satisfy Assumptions 3.1 and 3.2. Furthermore, let $\hat{\Sigma} = (\hat{\Sigma}_{lin}, \Sigma_{nl})$ be a reduced-order approximation, where $\hat{\Sigma}_{lin}$ is asymptotically stable and let there exist an error bound as in (3.73) on the linear subsystem. Then, the reduced-order nonlinear system $\hat{\Sigma}$ is input-to-state convergent if there exist class $\mathscr{K}_\infty$ functions $\hat{\rho}_1$ and $\hat{\rho}_2$ such that the following small-gain condition is satisfied:*

$$(id + \hat{\rho}_1) \circ \chi_{vz} \circ (id + \hat{\rho}_2) \circ (\varepsilon_{wv} + \chi_{wx} \circ \gamma_{xv})(s) \leq s, \tag{3.74}$$

for all $s \geq 0$.

When (3.74) holds, then the steady-state error $\|\bar{y}_u - \bar{\hat{y}}_u\|_\infty$ is bounded as $\|\bar{y}_u - \bar{\hat{y}}_u\|_\infty \leq \varepsilon\|u\|_\infty$, where $\varepsilon(r)$ is an error bound function.

For the proof and a detailed expression for the error bound function $\varepsilon(r)$ we refer to [6]. This error bound function $\varepsilon(r)$ depends on the properties (gain functions) of the original system Σ and the error bounds for the linear reduction (3.73). As the latter error bound can be obtained a priori (i.e., before the actual reduction is performed), the error bound in Theorem 3.9 also represents an a priori error bound. Note, moreover, that if the small-gain condition on the original system in Assumption 3.2 is satisfied with some margin, the small-gain condition in (3.74) can be satisfied by making the reduction of Σ_{lin} accurate enough, i.e., making ε_{wv} small enough.

Finally, we note that with this convergence-based approach to model reduction we obtain an error bound on the $\mathscr{L}_\infty$-norm of the reduction error. Alternative approaches exist, see [4, 5], exploiting incremental $\mathscr{L}_2$-gain or incremental passivity properties, instead of convergence properties, to obtain reduced-order systems (for a class of nonlinear systems of the same form as considered here) preserving such incremental system properties and complying with an $\mathscr{L}_2$ error bound.

3.10 Conclusions

In this chapter, we have reviewed the notion of convergent systems and its applications to a wide range of design and analysis problems for nonlinear (control) systems. It appears that nonlinear convergent systems inherit certain simplicity from asymptotically stable linear systems. This simplicity is not common to generic nonlinear systems. It allows one to solve a number of analysis and design problems for nonlinear systems in a nonlocal setting and extend previously known local results to nonlocal cases. For Lur'e-type systems it provides a powerful tool for optimization of steady-state performance. Open problems for further work relate to convergence properties for hybrid systems, to investigating how convergence and the existence of FRFs can be used to support system identification for certain classes of nonlinear convergent systems, and to applications of convergent systems to filtering.

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