provided to show that the proposed annular bound is less conservative than the existing results reported recently.

Appendix
Proof of Theorem 1

Suppose that $r \in Z[f(z)]$. Then, it is easy to see that $r \in Z[g(z)]$. This implies that

$$\begin{aligned} &r^n + a_{n-1}r^{n-1} + a_{n-2}r^{n-2} + \cdots + a_1 r + a_0 \\ &\quad = r^{2n} + b_{2n-2}r^{2n-2} + b_{2n-4}r^{2n-4} + \cdots + b_2 r^2 + b_0 \\ &\quad = 0. \end{aligned}$$

Thus, one has

$$|r|^n + |a_{n-1}||r|^{n-1} + |a_{n-2}||r|^{n-2} + \cdots + |a_1||r| \geq |a_0|$$
$$|r|^{2n} + |b_{2n-2}||r|^{2n-2} + |b_{2n-4}||r|^{2n-4} + \cdots + |b_2||r|^2 \geq |b_0|$$

$$|r|^n \leq |a_{n-1}||r|^{n-1} + |a_{n-2}||r|^{n-2} + \cdots + |a_1||r| + |a_0|$$
$$|r|^{2n} \leq |b_{2n-2}||r|^{2n-2} + |b_{2n-4}||r|^{2n-4} + \cdots + |b_2||r|^2 + |b_0|$$

which imply that

$$f_1(|r|) \geq 0,\ f_2(|r|) \geq 0,\ f_3(|r|) \leq 0,\ \text{and } f_4(|r|) \leq 0. \tag{2}$$

In the following, we separate two cases to discuss the annular bounds for the zeros of the polynomial of (1).

Case 1: $a_0 \neq 0$ (or, equivalently, $b_0 \neq 0$)

By Descartes' rule of signs, it can be readily obtained that each polynomial equation $f_i(x) = 0$, with $i \in \underline{4}$, has a unique positive root. Moreover, it is easy to see that

$$f_1(x) < 0, \quad \forall x \in [0, \alpha_1) \quad f_1(\alpha_1) = 0 \quad \text{and} \quad f_1(x) > 0 \quad \forall x \in (\alpha_1, \infty) \tag{3a}$$
$$f_2(x) < 0 \quad \forall x \in [0, \alpha_2), \quad f_2(\alpha_2) = 0 \quad \text{and} \quad f_2(x) > 0 \quad \forall x \in (\alpha_2, \infty) \tag{3b}$$
$$f_3(x) < 0 \quad \forall x \in [0, \alpha_3), \quad f_3(\alpha_3) = 0 \quad \text{and} \quad f_3(x) > 0 \quad \forall x \in (\alpha_3, \infty) \tag{3c}$$
$$f_4(x) < 0 \quad \forall x \in [0, \alpha_4), \quad f_4(\alpha_4) = 0 \quad \text{and} \quad f_4(x) > 0 \quad \forall x \in (\alpha_4, \infty) \tag{3d}$$

which imply that

$$|r| \geq \alpha_1, \quad |r| \geq \alpha_2 \quad |r| \leq \alpha_3 \quad \text{and} \quad |r| \leq \alpha_4 \tag{4}$$

in view of (2) and (3).

Case 2: $a_0 = 0$ (or, equivalently, $b_0 = 0$)

By Descartes' rule of signs, it is easy to see that $\alpha_1 = \alpha_2 = 0$ and

$$f_1(x) \geq 0 \quad \text{and} \quad f_2(x) \geq 0 \quad \forall x \in [\alpha_1, \infty) \tag{5a}$$
$$f_3(x) < 0 \quad \forall x \in [0, \alpha_3), \quad f_3(\alpha_3) = 0 \quad \text{and} \quad f_3(x) > 0 \quad \forall x \in (\alpha_3, \infty) \tag{5b}$$
$$f_4(x) < 0 \quad \forall x \in [0, \alpha_4), \quad f_4(\alpha_4) = 0 \quad \text{and} \quad f_4(x) > 0 \quad \forall x \in (\alpha_4, \infty) \tag{5c}$$

which imply that

$$|r| \geq \alpha_1 \quad |r| \geq \alpha_2 \quad |r| \leq \alpha_3 \quad \text{and} \quad |r| \leq \alpha_4 \tag{6}$$

in view of (2) and (5).

From (4) and (6), we conclude that $l := \max\{\alpha_1, \alpha_2\} \leq |r| \leq \min\{\alpha_3, \alpha_4\} := u$, with $r \in Z[f(z)]$. This completes the proof.

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The Local Output Regulation Problem: Convergence Region Estimates

Alexei Pavlov, Nathan van de Wouw, and Henk Nijmeijer

***Abstract*—In this note, the local output regulation problem is considered. The presented results answer the question: Given a controller solving the local output regulation problem, how do you estimate the set of admissible initial conditions for which this controller makes the regulated output converge to zero? The results are illustrated by a disturbance rejection problem for the transitional oscillator with a rotational actuator (TORA) system.**

***Index Terms*—Convergence region, disturbance rejection, nonlinear systems, output regulation.**

I. Introduction

In this note, we consider the problem of asymptotic regulation of the output of a dynamical system, which is subject to disturbances generated by an external system. This problem is known as the output regulation problem. For nonlinear systems, solutions to the *local* output regulation problem were given in [1] and [2]. In [1], necessary and sufficient conditions for the solvability of the problem in some neighborhood of an equilibrium were obtained and a procedure for designing a controller that solves the problem was presented. That paper was fol-

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lowed by publications regarding the local approximate output regulation problem [6], [7] and other aspects of the output regulation problem for nonlinear systems: regulation in the presence of uncertainties, adaptive, semiglobal and global output regulation (see [3]–[5], [9], and the references therein). At the same time, one problem regarding the *local* output regulation problem remained open: given a controller solving the problem in *some* neighborhood of the origin, how to determine (or estimate) a neighborhood of admissible initial conditions? Without answering this question, solutions to the local output regulation problem may not be satisfactory from an engineering point of view.

The first answers to that question were given in [10] and [11]. In those papers, procedures for estimating the set of admissible initial conditions were proposed. In this note, we extend the results obtained in [10] and [11] in order to obtain improved estimation results. The analysis is based on the results of [12], [13], and [20], which give sufficient conditions for every trajectory in a certain set to be exponentially stable. More information related to such properties of dynamical systems can be found in [14]–[16], and [10].

This note is organized as follows. In Section II, we recall the local output regulation problem and state the problem of estimating the set of admissible initial conditions. In Section III, some auxiliary technical results are presented. Section IV contains the main estimation results. In Section V, the obtained results are applied to a disturbance rejection problem in the transitional oscillator with a rotational actuator (TORA) system (see [17] and [18] for details about the TORA system). Conclusions are presented in Section VI. The proofs of all results are given in the Appendix.

The notations used in the note are the following. $\mathcal{A}^T$ is the transpose of matrix $\mathcal{A}$ and $\mathcal{A}^{-T} := (\mathcal{A}^{-1})^T$; the norm of a vector is denoted as $|z| = (z^T z)^{1/2}$; for a positive–definite matrix $P = P^T > 0$, we define the vector norm $|\cdot|_P$ as $|z|_P := \sqrt{z^T P z}$; $\|P\|$ is the operator norm of the matrix P induced by the vector norm $|\cdot|$; I is the identity matrix; the largest eigenvalue of a symmetric matrix $J = J^T$ is denoted $\Lambda(J)$ and $\mathcal{D}F_z(z)$ is the Jacobian matrix of $F(z)$.

II. Estimation Problem Statement

First, we recall the problem of local output regulation. Following [1], consider systems modeled by equations of the form

$$\dot{x} = f(x, u, w) \tag{1}$$

$$e = h(x, w) \quad y = h_m(x, w) \tag{2}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, regulated output $e \in \mathbb{R}^{l_r}$, measured output $y \in \mathbb{R}^{l_m}$, and exogenous input $w \in R^m$ generated by the linear exosystem

$$\dot{w} = Sw. \tag{3}$$

The functions f, h and h_m are at least continuously differentiable and $f(0,0,0) = 0, h(0,0) = 0, h_m(0,0) = 0$. It is assumed that exosystem (3) is *neutrally stable*, i.e., the equilibrium $w = 0$ is Lyapunov stable in forward and backward time [4]. The assumption of linearity of the exosystem is introduced in order to avoid unnecessary technical complications. All results presented later can be extended to the case of a general neutrally stable exosystem. Due to the neutral stability assumption, the spectrum of S consists of simple eigenvalues on the imaginary axis with, possibly, multiple eigenvalues at zero. Without loss of generality, we assume that S is skew-symmetric and, thus, any solution of (3) has the property $|w(t)| \equiv \mathrm{Const}$. Notice that if the right-hand side of (1) depends on a vector ψ of unknown constant parameters, w and ψ can be united and treated together as an external signal (w, ψ) generated by an extended exosystem given by (3) and $\dot{\psi} = 0$. This extended exosystem also satisfies the assumptions given before. Here, we assume that such extension has already been made and that (3) corresponds to an extended exosystem.

The local output regulation problem is to find, if possible, a feedback of the form

$$\dot{\xi} = \eta(\xi, y) \tag{4}$$

$$u = \theta(\xi, y) \tag{5}$$

with $\eta(0,0) = 0$ and $\theta(0,0) = 0$ such that **a)** $e(t) = h(x(t), w(t)) \to 0$ as $t \to \infty$ along every solution of the system

$$\dot{x} = f(x, \theta(\xi, h_m(x, w)), w) \tag{6}$$

$$\dot{\xi} = \eta(\xi, h_m(x, w)) \tag{7}$$

$$\dot{w} = Sw \tag{8}$$

starting close enough to the origin; and **b)** for $w(t) \equiv 0$, the equilibrium point $(x, \xi) = (0, 0)$ of the closed-loop system (6), (7) is locally exponentially stable.

A controller solving the local output regulation problem makes the output e tend to zero at least for small initial conditions $(x(0), \xi(0), w(0))$. Without specifying the region of admissible initial conditions for which output regulation occurs, such solution may not be satisfactory from an engineering point of view. Thus, we come to the following **estimation problem:** *Given the closed-loop system* (6), (7) *and the neutrally stable exosystem* (8), *estimate the region of admissible initial conditions for which the regulated output* $e(t) = h(x(t), w(t))$ *tends to zero.*

Denote $z := (x^T, \xi^T)^T \in \mathbb{R}^{n+k}$ (k is the dimension of ξ). Then, the closed-loop system (6), (7) can be written as

$$\begin{aligned} \dot{z} &= F(z, w) \\ e &= \bar{h}(z, w) := h(x, w) \end{aligned} \tag{9}$$

where $F(z, w)$ is the right-hand side of (6), (7). It is well known (see [1] and [3]) that a controller solves the local output regulation problem if and only if the corresponding closed-loop system (9) satisfies the following conditions.

$\mathcal{A}$) *The Jacobian matrix $\mathcal{D}F_z(0, 0)$ is Hurwitz.*

$\mathcal{B}$) *There exists a mapping $z = \pi(w)$ defined in a neighborhood $\mathcal{W}$ of the origin, with $\pi(0) = 0$, such that*

$$\begin{aligned} \frac{\partial \pi}{\partial w}(w) S w &= F(\pi(w), w) \\ 0 &= \bar{h}(\pi(w), w), \qquad \forall\, w \in \mathcal{W}. \end{aligned} \tag{10}$$

We will give a solution to the estimation problem based on the functions $F(z, w)$ and $\pi(w)$, which are found at the stage of controller design [1]–[5]. To simplify the subsequent analysis, it is assumed that the closed-loop system (9) and the mapping $\pi(w)$ are defined globally for all $z \in \mathbb{R}^{n+k}$ and $w \in \mathbb{R}^m$ (i.e., $\mathcal{W} = \mathbb{R}^m$). If this assumption does not hold, one should restrict the subsequent results to the sets $\mathcal{Z} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^m$ for which $F(z, w)$ and $\pi(w)$ are well defined.

Before proceeding with solving the estimation problem, we discuss the main idea of the solution. First, we find two sets $\mathcal{C} \subseteq \mathbb{R}^{n+k}$ and $\mathcal{W}_c \subseteq \mathbb{R}^m$ having the following property: If $w(t) \in \mathcal{W}_c$ for $t \geq 0$, then any two solutions $z_1(t)$ and $z_2(t)$ of system (9) lying in $\mathcal{C}$ for all $t \geq 0$ converge to each other: $|z_1(t) - z_2(t)| \to 0$ as $t \to \infty$. We call such set $\mathcal{C}$ a *convergence set* and the set $\mathcal{W}_c$ a *companion* to the set $\mathcal{C}$. Such sets exist, due to condition $\mathcal{A}$). This condition implies that near the origin, for small $w(t)$, the closed-loop system (9) behaves like a linear asymptotically stable system and, in particular, all its solutions are exponentially stable. Second, we find a set $\mathcal{Y} \subset \mathcal{C} \times \mathcal{W}_c$ of initial conditions $(z(0), w(0))$ such that any trajectory $(z(t), w(t))$ starting in this set satisfies the following conditions: $w(t) \in \mathcal{W}_c$, $\pi(w(t)) \in \mathcal{C}$

and $z(t) \in \mathcal{C}$ for all $t \geq 0$. As follows from condition $\mathcal{B}$), $\bar{z}(t) := \pi(w(t))$ is a solution of system (9) along which $e(t) \equiv 0$. Thus, by the properties of $\mathcal{C}$ and $\mathcal{W}_c$, it holds that $z(t) \to \bar{z}(t) := \pi(w(t))$ as $t \to +\infty$ and, hence, $e(t) = \bar{h}(z(t), w(t)) \to \bar{h}(\pi(w(t)), w(t)) \equiv 0$. So, $\mathcal{Y}$ is an estimate of the set of admissible initial conditions $(z(0), w(0))$ for which output regulation occurs.

III. Convergence Sets and the Demidovich Condition

In this section, we present and discuss a technical result about convergence sets for a system with input w given by

$$\dot{z} = F(z, w), \quad \text{where} \quad z \in \mathbb{R}^{n+k}, \qquad w \in \mathbb{R}^m \quad F(\cdot,\cdot) \in C^1. \tag{11}$$

The next lemma gives sufficient conditions for sets $\mathcal{C} \subseteq \mathbb{R}^{n+k}$ and $\mathcal{W}_c \subseteq \mathbb{R}^m$ to be a convergence set and its companion, respectively.

Lemma 1 [12] [20]: Suppose a convex set $\mathcal{C} \subseteq \mathbb{R}^{n+k}$ and a set $\mathcal{W}_c \subseteq \mathbb{R}^m$ satisfy the Demidovich condition

$$\sup_{z\in\mathcal{C}, w\in\mathcal{W}_c} \Lambda(P\mathcal{D}F_z(z,w) + \mathcal{D}F_z^T(z,w)P) =: -\alpha < 0 \tag{12}$$

for some positive–definite matrix $P = P^T > 0$. Then, for any continuous input $w(t)$ such that $w(t) \in \mathcal{W}_c$ for $t \geq 0$, any two solutions $z(t)$ and $\bar{z}(t)$ of (11) lying in $\mathcal{C}$ for all $t \geq 0$ satisfy

$$|z(t) - \bar{z}(t)| \leq Ce^{-\beta t}|z(0) - \bar{z}(0)| \tag{13}$$

for some $\beta > 0$ and $C > 0$ that are independent of the particular $z(t)$, $\bar{z}(t)$ and $w(t)$. □

The proof of this result is based on the Lyapunov-like function $V(z, \bar{z}) = |z - \bar{z}|_P^2$. Condition (12) guarantees that if $z(t)$ and $\bar{z}(t)$ lie in $\mathcal{C}$ and $w(t) \in \mathcal{W}_c$, then the function $V(z(t), \bar{z}(t))$ satisfies $\dot{V} \leq -\alpha/\|P\|V$. This, in particular, implies that if the ellipsoid $\mathcal{E}_P(\bar{z}(t), r) := \{z : V(z, \bar{z}(t)) < r\}$ is contained in $\mathcal{C}$ for all $t \geq 0$, then $\mathcal{E}_P(\bar{z}(t), r)$ is invariant. Uniting this observation with Lemma 1, we obtain the following corollary.

Corollary 1: Suppose $\mathcal{C}$ and $\mathcal{W}_c$ satisfy the conditions of Lemma 1. Let $w(t) \in \mathcal{W}_c$ for all $t \geq 0$ and $\bar{z}(t)$ be a solution of (11) such that $\bar{z}(t) \in \mathcal{C}$ for all $t \geq 0$. If the ellipsoid $\mathcal{E}_P(\bar{z}(t), r)$ is contained in $\mathcal{C}$ for all $t \geq 0$, then any solution of (11) starting in $z(0) \in \mathcal{E}_P(\bar{z}(0), r)$ satisfies (13). □

In order to solve the estimation problem stated in Section II, we need to find sets $\mathcal{C}$ and $\mathcal{W}_c$ satisfying the Demidovich condition. If $\mathcal{D}F_z(0,0)$ is Hurwitz (this is the case in the output regulation problem), one can choose a matrix $P = P^T > 0$ satisfying the matrix inequality $P\mathcal{D}F_z(0,0) + \mathcal{D}F_z^T(0,0)P < 0$. By continuity, $P\mathcal{D}F_z(z,w) + \mathcal{D}F_z^T(z,w)P$ is negative definite at least for small z and w. Hence, the Demidovich condition (12) is satisfied for $\mathcal{C}(\mathcal{R}) := \{z : |z| < \mathcal{R}\}$ and $\mathcal{W}(\rho) := \{w : |w| < \rho\}$ for some small $\mathcal{R}$ and ρ. If $P\mathcal{D}F_z(z,w) + \mathcal{D}F_z^T(z,w)P$ depends only on part of the coordinates z, then the Demidovich condition is satisfied for $\mathcal{C}_N(\mathcal{R}) := \{z : |Nz| < \mathcal{R}\}$ and $\mathcal{W}(\rho) := \{w : |w| < \rho\}$, where the matrix N is such that Nz consists of the coordinates that are present in $P\mathcal{D}F_z(z,w) + \mathcal{D}F_z^T(z,w)P$. Having chosen the matrix N, the numbers ρ and $\mathcal{R}$ can be found numerically (in some simple cases this can be done analytically).

IV. Main Results

We begin with answering the following question: under what conditions solves a controller, which solves the *local* output regulation problem, the *global* output regulation problem? Note, that due to condition $\mathcal{A}$), the closed-loop system (9) satisfies the Demidovich condition (12) locally, i.e., for $\mathcal{C}$ and $\mathcal{W}_c$ being some neighborhoods of the origin in $\mathbb{R}^{n+k}$ and $\mathbb{R}^m$, respectively. If the Demidovich condition is satisfied globally, then output regulation is attained globally, as stated by the following theorem.

Theorem 1: Let the local output regulation problem be solved. Suppose, the closed-loop system (9) satisfies the Demidovich condition (12) globally, i.e., for $\mathcal{C} = \mathbb{R}^{n+k}$ and $\mathcal{W}_c = \mathbb{R}^m$. Then, any trajectory $(z(t), w(t))$ of the closed-loop system (9) and the exosystem (8) satisfies

$$|z(t) - \pi(w(t))| \leq Ce^{-\beta t}|z(0) - \pi(w(0))| \tag{14}$$

for certain $\beta > 0$ and $C > 0$ independent of $(z(t), w(t))$, and $e(t) = \bar{h}(z(t), w(t)) \to 0$ as $t \to +\infty$. □

This theorem is a straightforward consequence of Lemma 1 and the fact that $\bar{z}(t) := \pi(w(t)$ is a solution of (9) along which $e(t) \equiv 0$.

If the Demidovich condition is satisfied only locally, we can find the sets $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\rho)$ for which this condition holds. This can be done numerically or, in some simple cases, analytically. Having found such sets, we can solve the estimation problem stated in Section II. Prior to formulating the solution, let us introduce the following function:

$$m_N(w_0) := \sup_{t\geq 0} |N\pi(w(t, w_0))| \tag{15}$$

where $w(t, w_0)$ is a solution of the exosystem (8) satisfying $w(0, w_0) = w_0$. The function $m_N(w_0)$ indicates whether $\pi(w(t, w_0))$ lies in the set $\mathcal{C}_N(\mathcal{R})$. Denote d to be the smallest number such that the inequality $|Nz| \leq d|z|_P$ is satisfied for all $z \in \mathbb{R}^{n+k}$. The number d can be found from the formula $d = \|NP^{-1/2}\|$. Indeed

$$\begin{aligned} d &= \sup_{|z|_P=1} |Nz| = \sup_{|P^{1/2}z|=1} |Nz| \\ &= \sup_{|\tilde{z}|=1} |NP^{-1/2}\tilde{z}| = \|NP^{-1/2}\|. \end{aligned}$$

The following theorem gives an estimate of the set of admissible initial conditions in the form of a neighborhood of the output-zeroing manifold $z = \pi(w)$.

Theorem 2: Let the local output regulation problem be solved. Suppose, the closed-loop system (9) satisfies the Demidovich condition (12) with $\mathcal{C}_N(\mathcal{R}) := \{z : |Nz| < \mathcal{R}\}$ and $\mathcal{W}_c(\rho) := \{w : |w| < \rho\}$ for some $\mathcal{R} > 0, \rho > 0$ and some matrix N. Then, any trajectory $(z(t), w(t))$ of the closed-loop system (9) and the exosystem (8) starting in the set

$$\begin{aligned} \mathcal{Y} := \Big\{(z_0, w_0) : |w_0| < \rho, m_N(w_0) \\ < \mathcal{R}, |z_0 - \pi(w_0)|_P < \frac{1}{d}(\mathcal{R} - m_N(w_0))\Big\} \end{aligned} \tag{16}$$

satisfies

$$|z(t) - \pi(w(t))| \leq Ce^{-\beta t}|z(0) - \pi(w(0))| \tag{17}$$

for some $\beta > 0$ and $C > 0$ independent of $(z(t), w(t))$, and $e(t) = \bar{h}(z(t), w(t)) \to 0$ as $t \to \infty$. □

The relation between the sets $\mathcal{Y}$, $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\rho)$ is schematically shown in Fig. 1.

If we want the closed-loop system (9) and the exosystem (3) to start in the set $\mathcal{Y}$, we need to guarantee that, first, the exosystem starts in a point w_0 in the set $\mathcal{M} := \{w_0 : |w_0| < \rho, m_N(w_0) < \mathcal{R}\}$ and, second, that the closed-loop system (9) starts in the set $\mathcal{A}(w_0) := \{z_0 : (z_0, w_0) \in \mathcal{Y})\}$. As can be seen from Fig. 2, the sets $\mathcal{A}(w_0)$ may be different for different values of w_0. Thus, the knowledge of w_0 is important. In practice, however, we may not know the exact value of w_0. For example, if the exosystem generates disturbances, then, knowing the level of disturbances, we can establish that $w_0 \in \mathcal{M}$, but still the

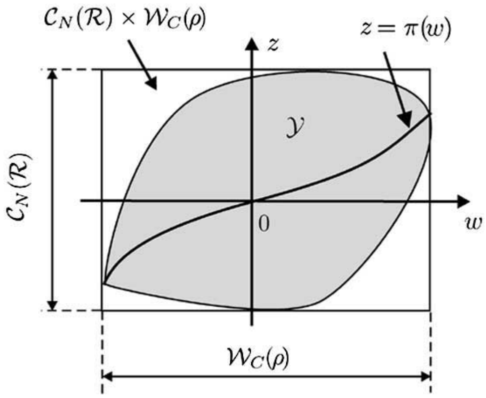

Fig. 1. Relation between the sets $\mathcal{Y}, \mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\rho)$: $\mathcal{Y}$ is an invariant set inside $\mathcal{C}_N(\mathcal{R}) \times \mathcal{W}_c(\rho)$.

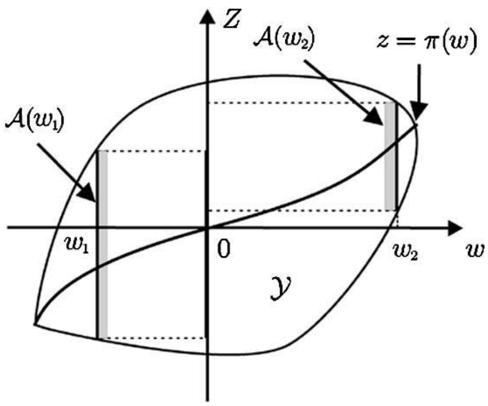

Fig. 2. Sets $\mathcal{Y}$ and $\mathcal{A}(w)$: for different w_1 and w_2, the sets $\mathcal{A}(w_1)$ and $\mathcal{A}(w_2)$ may be different.

exact value of w_0 is unknown. In order to cope with this difficulty, in the next result we find sets Z_0 and W_0 such that in whatever point $w_0 \in W_0$ the exosystem is initialized, output regulation will occur if the closed-loop system starts in $z_0 \in Z_0$. Prior to formulating the result, we define the functions

$$\delta(r) := \sup_{|w| \le r} \left(|N\pi(w)| + d|\pi(w)|_P\right), \; R(r) := (\mathcal{R} - \delta(r))/d. \quad (18)$$

The function $\delta(r)$ is nondecreasing and $\delta(0) = 0$. Let $r_* > 0$ be the largest number such that $r_* \le \rho$ and $\delta(r) < \mathcal{R}$ for all $r \in [0, r_*)$. Now, we can formulate the result.

Theorem 3: The conclusion of Theorem 2 holds for any trajectory $(z(t), w(t))$ starting in

$$z(0) \in E_P(R(r)) := \{z : |z|_P < R(r)\}$$
$$w(0) \in \mathcal{B}_w(r) := \{w : |w| < r\}$$

for some $r \in [0, r_*)$. □

The proof of this theorem is based on the fact that for every $r \in [0, r_*)$ the set $E_P(R(r)) \times B_w(r)$ is a subset of $\mathcal{Y}$, as shown in Fig. 3.

We can enlarge the obtained estimates by redefining $R(r)$ [see (18)] in the following way:

$$R(r) := (\mathcal{R}_m(r) - \delta(r))/d \quad (19)$$

where $\mathcal{R}_m(r)$ is defined as the largest number such that the Demidovich condition is satisfied for $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(r) = \{w : |w| < r\}$ for any $\mathcal{R} \in [0, \mathcal{R}_m(r))$. In this case, the convergence of solutions to the output-zeroing manifold will be exponential, but not uniform.

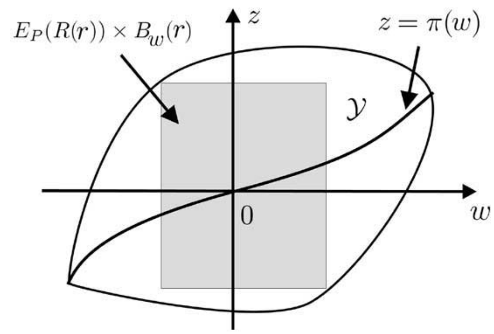

Fig. 3. Relation between the sets $\mathcal{Y}$, $E_P(R(r))$ and $B_w(r)$.

In practice, the function $\mathcal{R}_m(r)$ can be determined numerically and in some cases analytically.

A. Estimates for the Local Approximate Output Regulation Problem

Even though the local output regulation can be solvable, it can be extremely difficult to find a controller that solves it. For such controller, the closed-loop system would satisfy conditions $\mathcal{A}$) and $\mathcal{B}$). Condition $\mathcal{B}$) is the one that is difficult to satisfy. At the same time, in many cases it is easy to find a controller that satisfies (10) in condition $\mathcal{B}$) approximately (see [6]–[3]), i.e., as follows.

$\mathcal{B}^*$) *There exists a mapping $z = \tilde{\pi}(w)$ defined in a neighborhood $\mathcal{W}$ of the origin, with $\tilde{\pi}(0) = 0$, such that*

$$\frac{\partial \tilde{\pi}}{\partial w}(w) S w = F(\tilde{\pi}(w), w) + \varepsilon_1(w)$$
$$0 = \bar{h}(\tilde{\pi}(w), w) + \varepsilon_2(w) \quad (20)$$

for all $w \in \mathcal{W}$, where $\varepsilon_1(w)$ and $\varepsilon_2(w)$ are small (in some sense) continuous functions satisfying $\varepsilon_1(0) = 0$ and $\varepsilon_2(0) = 0$.

It is known (see [6]), that if the closed-loop system satisfies conditions $\mathcal{A}$) and $\mathcal{B}^*$), then for all sufficiently small initial conditions $z(0)$ and $w(0)$ the regulated output $e(t)$ converges to a function $\tilde{e}(w(t))$, where $\tilde{e}(w)$ is of the same order of magnitude as $\varepsilon_1(w)$ and $\varepsilon_2(w)$. This is called local approximate output regulation. Since it is required that the initial conditions must be sufficiently small, the problem of estimating this set of admissible initial conditions is also relevant in the case of approximate output regulation. This estimation problem can be solved using the same techniques as in the case of exact output regulation. The main idea is to find a set of initial conditions $\tilde{\mathcal{Y}} \subset \mathcal{C} \times \mathcal{W}_c$ (where $\mathcal{C}$ and $\mathcal{W}_c$ satisfy the Demidovich condition) such that if $(z(0), w(0)) \in \tilde{\mathcal{Y}}$, then $z(t) \in \mathcal{C}$, $\tilde{\pi}(w(t)) \in \mathcal{C}$ and $w(t) \in \mathcal{W}_c$, for all $t \ge 0$. As follows from (20), $\tilde{z}(t) := \tilde{\pi}(w(t))$ can be considered as a solution of the perturbed system $\dot{z} = F(z, w) + \varepsilon_1(w(t))$ and along this solution the regulated output equals $\varepsilon_2(w(t))$. Since $z(t)$ is exponentially stable (because of the Demidovich condition), a small perturbation $\varepsilon_1(w(t))$ implies, in the limit, a small difference between $z(t)$ and $\tilde{\pi}(w(t))$ (see [19, Ch. 5]). More precisely

$$\limsup_{t \to +\infty} |z(t) - \tilde{\pi}(w(t))| \le C \limsup_{t \to +\infty} |\varepsilon_1(w(t))| \quad (21)$$

for some constant C independent of $z(0)$ and $w(0)$. This, in turn, implies

$$\limsup_{t \to +\infty} |e(t)| \le \tilde{C} \limsup_{t \to +\infty} |\varepsilon_1(w(t))| + \limsup_{t \to +\infty} |\varepsilon_2(w(t))| \quad (22)$$

for some $\tilde{C}$. Hence, $\tilde{\mathcal{Y}}$ is an estimate of the set of admissible initial conditions. Estimates in the form of direct product $\tilde{Z}_0 \times \tilde{W}_0$ can be found in a similar way as in Theorem 3.

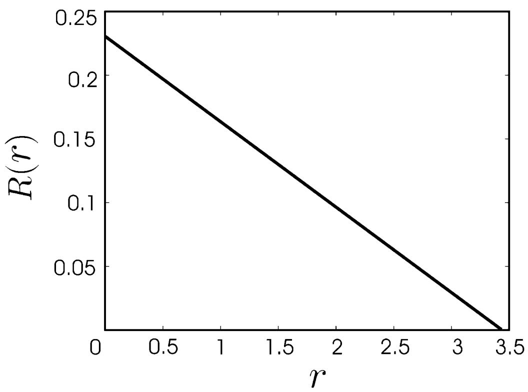

Fig. 4. $R(r)$ and r for the estimates $E_P(R(r)) \times B_w(r)$.

V. EXAMPLE

Let us illustrate the application of Theorem 3. Consider the so-called TORA-system described by equations of the form

$$\begin{aligned} \dot{x}_1 &= x_2, \quad \dot{x}_2 = -x_1 + \epsilon \sin x_3 + \mu D \\ \dot{x}_3 &= x_4 \quad \dot{x}_4 = v \\ e &= x_1 - \epsilon \sin x_3 \end{aligned} \tag{23}$$

where μ and $\epsilon < 1$ are some positive parameters, v is a control input and D is a disturbance force. For simplicity, we assume that both x and w are measured, i.e., $y = (x, w)$. This system is a nonlinear benchmark mechanical system that was introduced in [17] (see also [18]). The control problem is to find a controller such that e tends to zero in the presence of a harmonic disturbance D of known frequency, but unknown amplitude and phase. This is a particular case of the output regulation problem. The disturbance force D can be considered as an output of the linear harmonic oscillator

$$\dot{w}_1 = \Omega w_2 \quad \dot{w}_2 = -\Omega w_1 \quad D = w_1. \tag{24}$$

For small initial conditions $x(0)$ and $w(0)$, this output regulation problem is solved by a static controller of the form $v = c(w) + K(x - \pi(w))$, where the mappings $\pi(w) \in \mathbb{R}^4$ and $c(w) \in \mathbb{R}$ are defined by the formulas

$$\begin{aligned} \pi_1(w) &:= -\frac{\mu w_1}{\Omega^2} \quad \pi_2(w) := -\frac{\mu w_2}{\Omega} \\ \pi_3(w) &:= -\arcsin\left(\frac{\mu w_1}{\Omega^2 \epsilon}\right) \\ \pi_4(w) &:= -\frac{\mu \Omega w_2}{\sqrt{\Omega^4 \epsilon^2 - \mu^2 w_1^2}} \end{aligned} \tag{25}$$

$$ec(w) := \frac{\mu \Omega^2 w_1 \left(\Omega^4 \epsilon^2 - \mu^2 \left(w_1^2 + w_2^2\right)\right)}{\left(\sqrt{\Omega^4 \epsilon^2 - \mu^2 w_1^2}\right)^3} \tag{26}$$

and the matrix K is such that the closed-loop system has an asymptotically stable linearization at the origin. Indeed, it is easy to check that for such controller the closed-loop system satisfies conditions $\mathcal{A}$) and $\mathcal{B}$) with the specified $\pi(w)$ (see [3] and [5] for details on solving the local output regulation problem). Let us apply Theorem 3 to estimate the set of admissible $(x(0), w(0))$ (since the controller is static, then $z = x$) for the following values of the parameters: $\epsilon = 0.5, \mu = 0.04, \Omega = 1$, $K = (12, -4, -8, -5)$.

First, we must choose a matrix $P = P^T > 0$ such that $P\mathcal{D}F_x(0,0) + (\mathcal{D}F_x(0,0))^T P < 0$. We find such P from the Lyapunov equation $P\mathcal{D}F_x(0,0) + (\mathcal{D}F_x(0,0))^T P = -Q$, where Q is the diagonal matrix $\mathrm{diag}(2,8,1,1)$. For convenience, P is normalized such that $\|P\| = 1$. Since $\mathcal{D}F_x(x,w)$ depends only on x_3, the matrix N for the set $\mathcal{C}_N(\mathcal{R})$ is chosen equal to $N = (0,0,1,0)$, i.e., such that $Nx = x_3$. So, the convergence set $\mathcal{C}$ is sought in the form $\mathcal{C}_N(\mathcal{R}) := \{x : |x_3| < \mathcal{R}\}$ (see Section III for details). Since $\mathcal{D}F_x(x,w)$ does not depend on w, the companion set $\mathcal{W}_c$ can be taken equal to $\mathbb{R}^m$ and $\mathcal{R}_m(\rho) \equiv \mathrm{Const}$. Numerical computation gives $\mathcal{R}_m = 1.03$. Finally, computation of $R(r)$ given by (19) gives us the estimates of the admissible initial conditions sets: $E_P(R(r)) \times B_w(r)$. The function $R(r)$ is shown in Fig. 4. Note, that the mappings $\pi(w)$ and $c(w)$ and, thus, the closed-loop system are defined only for $|w_1| < \Omega^2 \epsilon / \mu$. For the given values of the system parameters this constraint is given by $|w_1| < 12.5$. The obtained estimates satisfy this condition. The estimates are fairly conservative. According to simulations, for a fixed level of disturbance r, output regulation still occurs for $x(0) \in E_P(\bar{R}(r))$ with $\bar{R}(r)$ about four-times larger than the obtained $R(r)$. One possible reason for such conservativeness is a bad choice of the matrix P. A different choice of P may result in better estimates. At the moment, it is an open question how to choose P in order to obtain the best (in some sense) estimates.

VI. CONCLUSION

In this note, we have considered the problem of estimating the sets of admissible initial conditions for a solution to the local output regulation problem. The presented solutions to this estimation problem are based on the so-called Demidovich condition. If a controller solves the local output regulation problem, then the closed-loop system satisfies the Demidovich condition locally. If the Demidovich condition is satisfied globally, then output regulation is attained globally (under the assumption that the output-zeroing manifold is defined globally). If this is not the case, results providing estimates of the sets of admissible initial conditions are given. The obtained estimates consist of initial conditions for which the trajectories of the forced closed-loop system converge to the output-zeroing manifold exponentially. The results are illustrated by application to a disturbance rejection problem in the TORA system. Since the exosystem is allowed to generate constant signals, the obtained results are also suitable for systems with parametric uncertainties. Although the analysis in the note was performed under the assumption of linearity of the exosystem, the results can be extended to the case of a general neutrally stable exosystem. Similar results can be obtained for the local *approximate* output regulation problem, for which estimating the set of admissible initial conditions is also relevant.

The obtained estimates are, in general, fairly conservative, since they are based on quadratic stability analysis and strongly depend on the choice of the matrices N and P. Despite this conservatism, the results can be rather useful in the following situations. First, one can directly use the estimates in practice (for certain simple systems they may be quite satisfactory). Second, if the estimates are too conservative, one can use them as a starting point for obtaining larger estimates by means of, for example, backward integration. The third way is to use the estimates as a criterion for choosing/tuning certain controller parameters. Since controller design admits some freedom in choosing certain controller parameters (like the matrix K in the TORA example), one can pick such parameters that guarantee larger estimates. For example, one can aim at finding controller parameters that guarantee satisfaction of the Demidovich condition globally. Such controller would solve the output regulation problem globally.

APPENDIX

Proof of Theorem 2: We need to show that (17) holds for any solution $(z(t), w(t))$ that starts in $(z(0), w(0))$ satisfying the relations: $|w(0)| < \rho, m_N(w(0)) < \mathcal{R}$ and $z(0) \in \mathcal{E}_P(\pi(w(0)), r)$, where $\mathcal{E}_P(\bar{z}, r) := \{z : |z - \bar{z}|_P < r\}$ and $r := (\mathcal{R} - m_N(w(0)))/d$. Due to the conditions on the initial conditions and the properties of the exosystem, $|w(t)| \equiv |w(0)| < \rho$ and the solution $\bar{z}(t) := \pi(w(t))$ satisfies

$$|N\bar{z}(t)| \le \sup_{t \ge 0} |N\pi(w(t))| = m_N(w(0)) < \mathcal{R}.$$

Hence, $\bar{z}(t) \in C_N(\mathcal{R})$ and $w(t) \in \mathcal{W}_c(\rho)$ for all $t \geq 0$. Let us show that $\mathcal{E}_P(\bar{z}(t), r) \subset \mathcal{C}_N(\mathcal{R})$ for all $t \geq 0$. Suppose $z \in \mathcal{E}_P(\bar{z}(t), r)$ for some $t \geq 0$. Then

$$\begin{aligned} |Nz| &\leq |N\bar{z}(t)| + |N(z - \bar{z}(t))| \leq m_N(w(0)) + d|z - \bar{z}(t)|_P \\ &< m_N(w(0)) + dr = \mathcal{R}. \end{aligned}$$

Consequently, $\mathcal{E}_P(\bar{z}(t), r) \subset \mathcal{C}_N(\mathcal{R})$ for all $t \geq 0$. The sets $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\rho)$ satisfy the conditions of Lemma 1. By Corollary 1, we obtain (17). Finally, $e(t) = \bar{h}(z(t), w(t)) \to \bar{h}(\pi(w((t)), w(t)) \equiv 0$ as $t \to +\infty$. □

Proof of Theorem 3: It is sufficient to show that $E_P(R(r)) \times B_w(r) \subset \mathcal{Y}$ for any $r \in [0, r_*)$. Then, the statement of Theorem 3 follows from Theorem 2. Suppose $z_0 \in E_P(R(r))$ and $w_0 \in B_w(r)$ for some fixed $r \in [0, r_*)$. According to the definition of $\mathcal{Y}$, we first need to show that $|w_0| < \rho$. This is true due to the fact that $|w_0| < r < r_* \leq \rho$. Next, we show that $m_N(w_0) < \mathcal{R}$. By the definition of $\delta(r)$, it holds that $|N\pi(w)| \leq \delta(r)$ for all $|w| < r$. The choice of $|w_0| < r$ implies $|w(t, w_0)| \equiv |w_0| < r$. Hence, by the definition of $m_N(w_0)$ we obtain

$$m_N(w_0) = \sup_{t \geq 0} |N\pi(w(t, w_0))| \leq \sup_{|w| < r} |N\pi(w)| \leq \delta(r).$$

The choice of $r < r_*$ implies that $\delta(r) < \mathcal{R}$ and consequently $m_N(w_0) < \mathcal{R}$.

Next, we need to show that $|z_0 - \pi(w_0)|_P < (\mathcal{R} - m_N(w_0))/d$. The triangle inequality implies

$$|z_0 - \pi(w_0)|_P \leq |z_0|_P + |\pi(w_0)|_P. \tag{27}$$

By the choice of z_0 and by the definition of $R(r)$

$$\begin{aligned} |z_0|_P < R(r) &= (\mathcal{R} - \delta(r))/d \\ &= (\mathcal{R} - \sup_{|w| \leq r} (|N\pi(w)| + d|\pi(w)|_P))/d \\ &\leq (\mathcal{R} - m_N(w_0))/d - |\pi(w_0)|_P. \end{aligned}$$

Substituting this inequality in (27), we obtain $|z_0 - \pi(w_0)|_P < (\mathcal{R} - m_N(w_0))/d$. This completes the proof. □

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Further Results on the Bounds of the Zeros of Quasi-Critical Polynomials

Zifang Zhang, Daoyi Xu, and Jianren Niu

***Abstract*—On the basis of the relationship of the mth power of a polynomial and its modular form (polynomial whose coefficients are the moduli of the coefficients of that polynomial), we derive a necessary and sufficient condition for the modulus of the mth power of a polynomial for contacting its modular form on the boundary of a disc. Combined with the result about distribution of zeros of analytic function, some new sufficient conditions are derived which give bounds of the absolute values of the roots of a quasi-critical polynomial. These results extend certain earlier similar tests for linear discrete-time systems. Finally, four examples are given to demonstrate the results, Example 2.1 gives a state feedback application, Examples 2.2 and 2.4 deal with r-stability, and Example 2.3 display that our theorems give better results when m increases but at the cost of increasing complexity.**

***Index Terms*—D-stability, linear discrete-time system, quasi-critical situation, Schur stability, state feedback control system.**

I. INTRODUCTION

A linear time-invariant discrete-time system

$$x(k) = a_1 x(k-1) + \cdots + a_n x(k-n) \; (k = 0, 1, \ldots) \tag{1.1}$$

is asymptotically stable if all the roots of its characteristic polynomial

$$f(z) = z^n - a_1 z^{n-1} - \cdots - a_n \tag{1.2}$$

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