Model Reduction for a Class of Convergent Nonlinear Systems

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***Abstract*—In this technical note, a model reduction procedure is presented for nonlinear systems that can be decomposed into a feedback interconnection of a linear and nonlinear subsystem. Conditions for stability of the reduced-order model and an error bound are given. Herein, the input-to-state convergence property is exploited, which proves to be useful in the definition and derivation of the error bound. The results are illustrated by application to a nonlinear mechanical system.**

***Index Terms*—Model reduction, nonlinear systems.**

I. Introduction

Practical engineering problems typically lead to complex, high-order models. Model reduction can be used to obtain a low-order approximation of these models, which facilitates controller design and implementation or allows for efficient analysis by fast simulation. Herein, it is desirable to preserve properties of the high-order model, of which stability is amongst the most crucial. Additionally, a bound on the reduction error is highly instrumental in determining the quality of the reduced-order model. For asymptotically stable linear systems, balanced truncation [7], [14], [17] and optimal Hankel norm approximation [9] provide these properties.

Besides giving rise to models of high order, complex high-tech systems often exhibit nonlinear behavior. Thus, nonlinearities have to be taken into account in the model reduction procedure. An approach exploiting linear model reduction techniques in the scope of model reduction for nonlinear systems is trajectory piecewise-linear approximation [18]. Stability of the reduced-order model can in general not be guaranteed, even though results on finite-gain input-output stability are available for systems with specific structure [6]. An alternative approach for model reduction of stable nonlinear systems is given by the extension of balanced truncation to nonlinear systems [8], [20], which guarantees local stability of the reduced-order model. However, input-output stability properties are not considered and the procedure is computationally challenging. The same properties hold for moment matching for nonlinear systems [3].

Hence, model reduction procedures for stable nonlinear systems do not generally guarantee stability of the reduced-order model, nor guarantee a bound on the reduction error. In the current technical note, a model reduction procedure for a class of nonlinear systems will be presented, including conditions guaranteeing stability of the reduced-order model as well as an error bound. Nonlinear systems will be considered that can be decomposed into a linear and nonlinear subsystem, which are bidirectionally coupled. In this configuration, it is assumed that the nonlinear subsystem is of relatively low order, whereas the linear subsystem is of high order. This is motivated by the observation that nonlinearities act only locally in many engineering (control) applications. Examples include mechanical systems with friction, hysteresis or nonlinear actuator dynamics. This assumption allows for an approach in which model reduction is applied to the linear subsystem only, making the approach computationally efficient.

In this configuration, the linear and nonlinear subsystems are assumed to be *input-to-state convergent* [16]. The convergence property implies, for every bounded input, the existence of a unique, globally asymptotically stable, bounded steady-state solution. Additionally, the input-to-state convergence property yields a bounded difference in state trajectories for a bounded difference in inputs. Loosely speaking, this amounts to the property that, for two input functions that are "close", the corresponding steady-state solutions are "close" as well. Since the total nonlinear system essentially consists of two bidirectionally coupled input-to-state convergent systems, a small-gain theorem is presented under which the coupled system is itself input-to-state convergent.

In this setting, conditions are given under which the reduced-order system is input-to-state convergent, thus preserving certain stability properties of the high-order model. Furthermore, an error bound on the steady-state solutions is derived. Herein, it is noted that convergence of the total nonlinear system implies, for every bounded input, the existence of a unique steady-state solution, allowing for a clear definition of an error bound. Next, the input-to-state convergence property of the subsystems is highly instrumental in the derivation of this error bound, since it quantifies the amplification of errors introduced by reduction of the linear subsystem.

This technical note is organized as follows. Preliminaries regarding convergent systems are reviewed in Section II. The system configuration and model reduction approach are given in Sections III and IV, whereas the main results on input-to-state convergence of the reduced-order system as well as an error bound are presented in Section V. The procedure is illustrated with an example in Section VI before stating conclusions in Section VII.

Notation: For a vector x, the Euclidian norm is denoted by $|x|$. For a signal x, defined on $\mathbb{R}$, the $\mathcal{L}_\infty$ signal norm is denoted by $\|x\|_\infty$ and defined as $\|x\|_\infty := \sup_{\tau\in\mathbb{R}} |x(\tau)|$. Functions of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ are defined according to [12]. The identity function id satisfies $\mathrm{id}(r) = r, \forall r \in \mathbb{R}$.

II. Convergent Systems

A model reduction procedure for a class of (input-to-state) convergent systems will be presented in this technical note. Thereto, convergent systems are reviewed below. Consider the system

$$\dot{x} = f(x, u) \tag{1}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. In (1), $f(x,u)$ with $f(0,0) = 0$ is assumed to be locally Lipschitz in x and continuous in u. Throughout this technical note, the input functions are assumed to be in the class of piecewise continuous vector functions, defined on $\mathbb{R}$ and bounded as $\|u\|_\infty < \infty$. This class is denoted by $\mathcal{L}_\infty^m$. The convergence property can be defined as follows [16]:

Definition 1: A system (1) is globally (uniformly, exponentially) convergent for a class of inputs $\mathcal{U} \subset \mathcal{L}_\infty^m$ if, for each input $u \in \mathcal{U}$, there exists a solution $\bar{x}_u$ such that: 1) $\bar{x}_u$ is defined and bounded on $\mathbb{R}$; 2) $\bar{x}_u$ is globally (uniformly asymptotically, exponentially) stable.

The solution $\bar{x}_u$ is known as the *steady-state* solution corresponding to the input u. In the case of uniform and exponential convergence, the steady-state solution is unique (see e.g. [16]). This allows for the definition of a steady-state operator $\mathcal{F} : \mathcal{L}_\infty^m \to \mathcal{L}_\infty^n$, defined as $\mathcal{F}u := \bar{x}_u$ [16]. The convergence property thus defines a stability property for

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nonlinear systems with inputs, in the sense that all solutions corresponding to a particular input converge to a unique steady-state solution. An even stronger stability property is given as follows.

Definition 2 ([16]): A system (1) is said to be input-to-state convergent if it is globally uniformly convergent for the class of inputs $\mathcal{L}_\infty^m$ and, for every input $u \in \mathcal{L}_\infty^m$, (1) is input-to-state stable (ISS) with respect to the steady-state solution $\bar{x}_u$, i.e. there exist functions $\beta \in \mathcal{KL}$, $\gamma_{xu} \in \mathcal{K}_\infty$ such that any solution $\tilde{x}$ of (1) corresponding to some input $\tilde{u} \in \mathcal{L}_\infty^m$ satisfies

$$|\tilde{x}(t) - \bar{x}_u(t)| \le \beta\left(|\tilde{x}(t_0) - \bar{x}_u(t_0)|, t - t_0\right) + \gamma_{xu}\left(\sup_{t_0 \le \tau \le t} |\tilde{u}(\tau) - u(\tau)|\right), \quad \forall t \ge t_0. \quad (2)$$

Here, β and γ_{xu} may depend on the particular input u.

In (2), γ_{xu} denotes the gain from u to x. Throughout the technical note, all gains are labeled similarly. The notion of input-to-state convergence differs from the similar notion of incremental input-to-state stability (δISS) [1] since the first only requires input-to-state stability (see e.g. [21]) with respect to the steady-state solution, rather than with respect to all solutions. A sufficient condition for the input-to-state convergence of nonlinear systems is given by the Demidovich condition, see e.g. [15], [16]. For linear time-invariant systems, input-to-state convergence is implied by asymptotic stability of the system without input. Here, it is stressed that such implication does not hold for nonlinear systems. Furthermore, the input-to-state convergence property implies an incremental bound on the steady-state operator:

Lemma 1: Let system (1) satisfy the input-to-state convergence property (2). Then, the steady-state operator $\mathcal{F}$ is incrementally bounded as $\|\mathcal{F}u_2 - \mathcal{F}u_1\|_\infty \le \gamma_{xu}(\|u_2 - u_1\|_\infty)$, where γ_{xu} is the class $\mathcal{K}_\infty$-function in Definition 2.

Proof: The result follows from application of (2) to compare two steady-state solutions. Since the steady-state solutions are bounded for all $t \in \mathbb{R}$, the influence of initial condition vanishes for $t_0 \to -\infty$. Then, taking $t \to \infty$ in (2) leads to the desired result. ■

In the approach for model reduction presented in Section III, a decomposition of the nonlinear system is considered, which basically consists of a feedback interconnection of two input-to-state convergent systems. Therefore, the bidirectionally coupled systems $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_z$ as given by

$$\mathbf{\Sigma}_x : \dot{x} = f(x, z, u_1), \quad x \in \mathbb{R}^{n_x}, u_1 \in \mathbb{R}^{m_1} \quad (3)$$

$$\mathbf{\Sigma}_z : \dot{z} = g(z, x, u_2), \quad z \in \mathbb{R}^{n_z}, u_2 \in \mathbb{R}^{m_2} \quad (4)$$

are considered. When $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_z$ are input-to-state convergent, input-to-state convergence of the coupled system (3), (4) can be guaranteed by the following small-gain theorem, which is related to the small-gain theorem for ISS systems [11].

Theorem 2: Consider two input-to-state convergent systems $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_z$ with gain functions $\gamma_{xz}, \gamma_{xu_1}$ and $\gamma_{zx}, \gamma_{zu_2}$, respectively. Then, the coupled configuration (3), (4) is input-to-state convergent if there exist functions $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that

$$(\mathrm{id} + \rho_1) \circ \gamma_{xz} \circ (\mathrm{id} + \rho_2) \circ \gamma_{zx}(s) \le s, \quad \forall s \ge 0. \quad (5)$$

Proof: The proof of this theorem can be found in [5]. ■

III. Problem Setting

Nonlinear systems that can be decomposed as in the configuration in Fig. 1 are considered. Here, the total coupled system $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ consists of a high-order linear subsystem $\mathbf{\Sigma}_{lin}$ and a convergent nonlinear subsystem $\mathbf{\Sigma}_{nl}$, where the linear dynamics are given by

$$\mathbf{\Sigma}_{lin} : \dot{x} = Ax + B_u u + B_v v, \quad y = C_y x, \quad w = C_w x \quad (6)$$

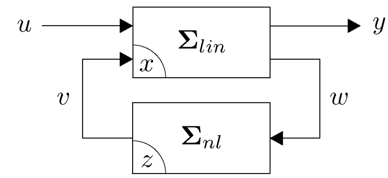

Fig. 1. Coupled system consisting of a high-order linear part and convergent nonlinear part.

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Here, $\mathbf{\Sigma}_{lin}$ is assumed to be asymptotically stable and input signals u are chosen from the class $\mathcal{L}_\infty^m$. The signals $v \in \mathbb{R}^s$ and $w \in \mathbb{R}^q$ connect $\mathbf{\Sigma}_{lin}$ to the nonlinear subsystem $\mathbf{\Sigma}_{nl}$, which is given as

$$\mathbf{\Sigma}_{nl} : \dot{z} = g(z, w), \quad v = h(z) \quad (7)$$

with $z \in \mathbb{R}^r$ and $g(0,0) = 0$. It is assumed that $h(0) = 0$ and that the incremental bound

$$|h(z_2) - h(z_1)| \le \chi_{vz}\left(|z_2 - z_1|\right) \quad (8)$$

holds for all $z_1, z_2 \in \mathbb{R}^r$ and with χ_{vz} of class $\mathcal{K}_\infty$. Furthermore, $\mathbf{\Sigma}_{nl}$ is assumed to be input-to-state convergent with respect to the input w. This allows for the definition of the steady-state operator $\mathcal{G}$ as $\mathcal{G}w := \bar{z}_w$, which satisfies $\|\mathcal{G}w_2 - \mathcal{G}w_1\|_\infty \le \gamma_{zw}(\|w_2 - w_1\|_\infty)$ with $\gamma_{zw} \in \mathcal{K}_\infty$ by Lemma 1. The steady-state output operator, defined as $\mathcal{G}_v w := h(\bar{z}_w)$, satisfies for all $w_1, w_2 \in \mathcal{L}_\infty^q$ [by (8)]

$$\|\mathcal{G}_v w_2 - \mathcal{G}_v w_1\|_\infty \le \chi_{vz} \circ \gamma_{zw}\left(\|w_2 - w_1\|_\infty\right). \quad (9)$$

Since the linear subsystem $\mathbf{\Sigma}_{lin}$ is asymptotically stable, it is input-to-state convergent with respect to the inputs u and v. Thus, steady-state operators can be defined as $\mathcal{F}(u, v) := \bar{x}_{u,v}$ and $\mathcal{F}_i(u, v) := C_i \bar{x}_{u,v}$, $i \in \{y, w\}$, where the latter defines the steady-state output operators for outputs y and w. These steady-state output operators are incrementally bounded as

$$\|\mathcal{F}_i(u_2, v_2) - \mathcal{F}_i(u_1, v_1)\|_\infty \le \chi_{ix}\left(\gamma_{xu}\left(\|u_2 - u_1\|_\infty\right) + \gamma_{xv}\left(\|v_2 - v_1\|_\infty\right)\right) \quad (10)$$

for $i \in \{y, w\}$ and for all $u_1, u_2 \in \mathcal{L}_\infty^m$, $v_1, v_2 \in \mathcal{L}_\infty^s$. In (10), $\gamma_{xu}, \gamma_{xv} \in \mathcal{K}_\infty$ denote the gain functions of the steady-state operator $\mathcal{F}(u, v)$, whereas $\chi_{yx}, \chi_{wx} \in \mathcal{K}_\infty$ represent incremental bounds on the output equations. Clearly, these functions are linear. The bounds (10) will be exploited in the scope of model reduction, where outputs for a given input u are of interest, thus considering the case $u_1 = u_2 = u$.

Besides the assumption of input-to-state convergence of the subsystems $\mathbf{\Sigma}_{lin}$ and $\mathbf{\Sigma}_{nl}$, it is assumed that there exist functions $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that the small-gain condition

$$(\mathrm{id} + \rho_1) \circ \gamma_{xv} \circ \chi_{vz} \circ (\mathrm{id} + \rho_2) \circ \gamma_{zw} \circ \chi_{wx}(s) \le s \quad (11)$$

holds for all $s \ge 0$. By Theorem 2, this implies that $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ is input-to-state convergent. Summarizing, the following assumptions are adopted.

Assumption 1: The subsystems $\mathbf{\Sigma}_{lin}$ as in (6) and $\mathbf{\Sigma}_{nl}$ as in (7), coupled as in Fig. 1, satisfy the following conditions: 1) A is Hurwitz; 2) $\mathbf{\Sigma}_{nl}$ is input-to-state convergent; 3) h in (7) satisfies (8); 4) the small-gain condition (11) holds.

The relevant subclass of Lur'e-type systems is obtained when the nonlinear subsystem $\mathbf{\Sigma}_{nl}$ in $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ is replaced by a static nonlinearity $\varphi : \mathbb{R}^q \to \mathbb{R}^s$. When φ is incrementally bounded (i.e. globally Lipschitz) as

$$|\varphi(w_2) - \varphi(w_1)| \leq \mu |w_2 - w_1|, \quad \forall w_1, w_2 \in \mathbb{R}^q \tag{12}$$

with $\mu > 0$, this bound plays the same role as the incremental bound (9) on the steady-state output operator $\mathcal{G}_v w$. Specifically, $\chi_{vz} \circ \gamma_{vw}(r) = \mu r$ holds for all $r \geq 0$.

IV. Model Reduction

For the nonlinear system $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$, a model reduction procedure is proposed in which only the high-order linear subsystem $\mathbf{\Sigma}_{lin}$ is reduced, hereby exploiting existing reduction techniques. This leads to the reduced-order system $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{\Sigma}}_{lin}, \mathbf{\Sigma}_{nl})$, with the reduced-order linear subsystem as

$$\hat{\mathbf{\Sigma}}_{lin} : \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}_u u + \hat{B}_v \hat{v}, \quad \hat{y} = \hat{C}_y \hat{x}, \quad \hat{w} = \hat{C}_w \hat{x} \tag{13}$$

with $\hat{x} \in \mathbb{R}^k$, $k < n$, $\hat{y} \in \mathbb{R}^p$, $\hat{v} \in \mathbb{R}^s$ and $\hat{w} \in \mathbb{R}^q$. It is assumed that the linear model reduction procedure preserves stability, allowing for the introduction of the steady-state output operators $\hat{\mathcal{F}}_y(u, \hat{v})$ and $\hat{\mathcal{F}}_w(u, \hat{v})$, which represent the steady-state outputs $\bar{\hat{y}}_{u,\hat{v}}$ and $\bar{\hat{w}}_{u,\hat{v}}$ for given inputs u and $\hat{v}$, respectively. As for the high-order linear subsystem [see (10)], these operators can be incrementally bounded (for fixed u) as

$$\|\hat{\mathcal{F}}_i(u, \hat{v}_2) - \hat{\mathcal{F}}_i(u, \hat{v}_1)\|_\infty \leq \hat{\chi}_{ix} \circ \hat{\gamma}_{xv} \left(\|\hat{v}_2 - \hat{v}_1\|_\infty\right) \tag{14}$$

for all $\hat{v}_1, \hat{v}_2 \in \mathcal{L}_\infty^s$, $i \in \{y, w\}$. Here, $\hat{\gamma}_{xv}$ and $\hat{\chi}_{ix}$ are linear gain functions.

Besides stability of $\hat{\mathbf{\Sigma}}_{lin}$, it is assumed that the linear model reduction procedure provides a bound on the error introduced by model reduction, stated in the $\mathcal{L}_\infty$ signal norm. This specifies an $\mathcal{L}_\infty$-induced norm on the error system $\mathbf{\Delta}_{lin} = \mathbf{\Sigma}_{lin} - \hat{\mathbf{\Sigma}}_{lin}$, where $v = \hat{v}$. As the parallel interconnection of two input-to-state convergent systems, $\mathbf{\Delta}_{lin}$ is input-to-state convergent with respect to inputs u and v [16]. Thus, steady-state error operators can be defined as $\mathcal{E}_i(u, v) = \mathcal{F}_i(u, v) - \hat{\mathcal{F}}_i(u, v)$, $i \in \{y, w\}$. Then, it is assumed that the error bound on reduction of the linear subsystem is given as

$$\begin{aligned} \|\mathcal{E}_i(u_2, v_2) - \mathcal{E}_i(u_1, v_1)\|_\infty \leq \varepsilon_{iu} \left(\|u_2 - u_1\|_\infty\right) \\ + \varepsilon_{iv} \left(\|v_2 - v_1\|_\infty\right) \end{aligned} \tag{15}$$

for all $u_1, u_2 \in \mathcal{L}_\infty^m$, $v_1, v_2 \in \mathcal{L}_\infty^s$ and where ε_{ij} ($i \in \{y, w\}$, $j \in \{u, v\}$) are functions of class $\mathcal{K}_\infty$. Even though the availability of error bounds in the form (15) appears to be a restrictive assumption, it is noted that (15) represents an error bound for the linear subsystem only. Due to linearity, the functions ε_{ij} will be linear, whereas the incremental form follows directly from an ordinary (i.e. non-incremental) bound on the output errors. In fact, an *a priori* error bound exists when $\hat{\mathbf{\Sigma}}_{lin}$ is obtained by balanced truncation. Namely, an error bound on the $\mathcal{L}_1$ norm on the impulse response as in [10], [13] provides a bound on the $\mathcal{L}_\infty$-induced system norm [22]. Alternatively, an error bound can be computed *a posteriori* using results from [19], typically leading to a tighter bound. This approach might be applied to any asymptotically stable reduced-order model. Here, it is noted that these error bounds typically do not make a distinction between the error bounds for different input-output combinations as in (15). In this case, the error functions in (15) can typically be bounded as $\varepsilon_{ij}(r) \leq \varepsilon_{lin} r$ for all $r \geq 0$ with $i \in \{y, w\}$, $j \in \{u, v\}$, with ε_{lin} a single linear error bound.

V. Stability and Error Bound

For the model reduction procedure discussed in Section IV, conditions for stability of the reduced-order model and an error bound are given in the following theorem.

Theorem 3: Let $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \mathbf{\Sigma}_{nl})$ satisfy Assumption 1. Furthermore, let $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{\Sigma}}_{lin}, \mathbf{\Sigma}_{nl})$ be a reduced-order approximation, where $\hat{\mathbf{\Sigma}}_{lin}$ as in (13) is asymptotically stable and the error bound (15) on the linear subsystem holds. Then, $\hat{\mathbf{\Sigma}}$ is input-to-state convergent if there exist functions $\hat{\rho}_1, \hat{\rho}_2 \in \mathcal{K}_\infty$ such that the following condition holds for all $s \geq 0$:

$$(\mathrm{id} + \hat{\rho}_1) \circ \chi_{vz} \circ (\mathrm{id} + \hat{\rho}_2) \circ \gamma_{zw} \circ (\varepsilon_{wv} + \chi_{wx} \circ \gamma_{xv})(s) \leq s. \tag{16}$$

When (16) holds, the steady-state error $\delta \bar{y}_u = \bar{y}_u - \bar{\hat{y}}_u$ is bounded as $\|\delta \bar{y}_u\|_\infty \leq \varepsilon(\|u\|_\infty)$ with

$$\begin{aligned} \varepsilon(r) = \big(\varepsilon_{yu} + \varepsilon_{yv} \circ \left(\mathrm{id} + \rho_5^{-1}\right) \circ \eta_{vu} \\ + \left(\chi_{yx} \circ \gamma_{xv} + \varepsilon_{yv} \circ (\mathrm{id} + \rho_5)\right) \circ \delta_{vu}\big)(r) \end{aligned} \tag{17}$$

for all $r \geq 0$ and where $\eta_{vu}, \delta_{vu} \in \mathcal{K}_\infty$ are defined in the proof [see (28) and (36)] and $\rho_5 \in \mathcal{K}_\infty$ is an arbitrary function.

Proof: The proof can be found in the Appendix. ■

In Theorem 3, (16) guarantees the fulfilment of the small-gain condition for the reduced-order nonlinear system. Furthermore, (17) gives an *a priori* error bound.

The input-to-state convergence property plays an important role in the definition and derivation of the error bound. First, uniform convergence of both the high-order and reduced-order nonlinear system (as implied by input-to-state convergence) implies uniform convergence of the error system $\mathbf{\Delta} = \mathbf{\Sigma} - \hat{\mathbf{\Sigma}}$ and thus the existence of a unique steady-state output error, bounded for all bounded inputs. It is stressed that this property does not generally hold for nonlinear systems.

Second, input-to-state convergence of the subsystems is crucial in the derivation of the error bound. In the error analysis in the proof of Theorem 3, the (steady-state) solutions of the high-order and reduced-order system are compared. In doing so, it is of interest how the (steady-state) error between these solutions is amplified when passing through the nonlinear subsystem $\mathbf{\Sigma}_{nl}$. The input-to-state convergence property specifies a bound on this amplification, whereas the small-gain theorem guarantees boundedness of the steady-state error for the coupled configuration.

Remark 1: It is noted that the assumption that $\mathbf{\Sigma}_{lin}$ is linear is not required in the definition and proof of Theorem 3. The results of Theorem 3 are therefore also applicable to coupled input-to-state convergent nonlinear systems, where one of the subsystems is reduced such that the error bound (15) holds. In the current setting, with a linear subsystem $\mathbf{\Sigma}_{lin}$, it is recalled that the functions χ_{ix}, γ_{xj} and ε_{ij} with $i \in \{y, w\}$, $j \in \{u, v\}$ are linear, where the latter can be constructed using linear model reduction techniques such as e.g. balanced truncation.

Remark 2: The results of Theorem 3 on input-to-state convergence of $\hat{\mathbf{\Sigma}}$ and the error bound can be evaluated a priori. However, when the gains $\hat{\chi}_{yx}, \hat{\chi}_{yx}$ and $\hat{\gamma}_{xu}, \hat{\gamma}_{xv}$ of $\hat{\mathbf{\Sigma}}_{lin}$ are computed after model reduction has been performed (see e.g. [19]), a less conservative condition for input-to-state convergence of $\hat{\mathbf{\Sigma}}$ and a tighter error bound can be obtained.

Remark 3: The explicit expression (17) for the error bound allows for a reduction procedure in which the error is minimized. Namely, in the reduction of the linear subsystem, emphasis can be placed on the input-output combination that has the largest effect on the overall error ε in (17).

For the important subclass of Lur'e-type systems, the following corollary of Theorem 3 is obtained, in which the linearity of the subsystem $\mathbf{\Sigma}_{lin}$ is explicitly used by denoting the gains as linear functions, i.e. $\chi_{ix}(r) = \tilde{\chi}_{ix} r$, $\gamma_{xj}(r) = \tilde{\gamma}_{xj} r$ for all $r \geq 0$ with $i \in \{y, w\}$, $j \in \{u, v\}$. Furthermore, a single error bound ε_{lin} for the reduced linear system is used.

Corollary 4: Let $\mathbf{\Sigma} = (\mathbf{\Sigma}_{lin}, \varphi)$ be a Lur'e-type system, where $\mathbf{\Sigma}_{lin}$ is asymptotically stable, φ is a static nonlinearity satisfying (12) and the small-gain condition (11) holds. Let $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{\Sigma}}_{lin}, \varphi)$ be a reduced-order Lur'e-type system, where $\hat{\mathbf{\Sigma}}_{lin}$ as in (13) is asymptotically stable and satisfies the error bound (15) with $\varepsilon_{ij}(r) \leq \varepsilon_{lin} r$ for all $r \geq 0$ with $i \in \{y, w\}$, $j \in \{u, v\}$. If $\mu(\tilde{\chi}_{wx}\tilde{\gamma}_{xv} + \varepsilon_{lin}) < 1$, then $\hat{\mathbf{\Sigma}}$ is input-to-state convergent and the error is bounded as $\|\delta\bar{y}_u\|_\infty \leq \varepsilon \|u\|_\infty$ with

$$\varepsilon = \varepsilon_{lin}\left(1 + \frac{\mu\tilde{\chi}_{wx}\tilde{\gamma}_{xu}}{1-\mu\tilde{\chi}_{wx}\tilde{\gamma}_{xv}}\right)\left(1 + \frac{\mu(\tilde{\chi}_{yx}\tilde{\gamma}_{xv} + \varepsilon_{lin})}{1-\mu(\tilde{\chi}_{wx}\tilde{\gamma}_{xv} + \varepsilon_{lin})}\right). \quad (18)$$

Remark 4: The results on Lur'e-type systems are obtained by exploiting the input-to-state convergence property of the linear dynamics and the incremental bound on the nonlinearity φ, which are both stated in terms of the $\mathcal{L}_\infty$ signal norm. An alternative approach is taken in [4], where error analysis for model reduction of Lur'e-type systems is performed using the $\mathcal{L}_2$ signal norm, leading to an error bound with a structure similar to (18). In [4], convergence properties rather than input-to-state convergence properties are used in the analysis.

Theorem 3 only provides a bound on the *steady-state* output error, which is defined for all $t \in \mathbb{R}$. On the other hand, existing error bounds in model reduction (see e.g. [2]) are defined typically for signals on $t \in [0, \infty)$ and zero initial condition. The bound on the steady-state output error used in the current technical note includes this case, as stated next.

Proposition 1: Let the system (1), with $f(0, 0) = 0$, be globally uniformly convergent and let an input u satisfy $u(t) = 0$ for all $t \in (-\infty, t_0)$. Then, the corresponding steady-state solution $\bar{x}_u$ satisfies $\bar{x}_u(t) = 0$ for all $t \in (-\infty, t_0]$.

Proof: The uniform convergence property guarantees that the steady-state solution $\bar{x}_u$ exists and is unique, in the sense that it is the only solution that is bounded for all $t \in \mathbb{R}$. It thus has to be shown that $\bar{x}_u(t) = 0$ is the only bounded solution for $t \in (-\infty, 0)$, forming a part of the total steady-state solution. Thereto, it is noted that, for $t \in (-\infty, 0)$, the steady-state solution satisfies $\dot{x} = f(x, 0)$. By the uniform convergence property and $f(0, 0) = 0$, the origin of $\dot{x} = f(x, 0)$ is globally asymptotically stable. Then, a converse Lyapunov theorem (see e.g. [12]) guarantees the existence of a smooth positive definite radially unbounded function V and a positive definite function α such that $(\partial V/\partial x) f(x, 0) \leq -\alpha(x)$ for all $x \in \mathbb{R}^n$. Let $\bar{x}_u^*$ be a solution that satisfies $\bar{x}_u^*(t^*) \neq 0$ for some $t^* \in (-\infty, t_0)$. Then, by tracing this solution in backward time it is easily seen that $V(x(t)) \to \infty$ for $t \to -\infty$. Since V is radially unbounded and smooth, this implies that the state grows unbounded. Hence, $\bar{x}_u(t) = 0$ is the only solution that remains bounded on the time interval $t \in (-\infty, t_0)$. By continuity, $\bar{x}_u(t_0) = 0$ as well. ∎

VI. Illustrative Example

To illustrate the model reduction procedure outlined in Section IV, the clamped flexible beam as in Fig. 2 is considered. Here, $u \in \mathbb{R}$ is an external force acting on the beam, whereas $y \in \mathbb{R}$ denotes the beam deflection. The beam is modeled using Euler beam elements, leading to a linear subsystem $\mathbf{\Sigma}_{lin}$ of order $n = 60$. At one end, the beam is supported by a mount which exhibits nonlinear viscoelastic behavior, leading to a nonlinear subsystem $\mathbf{\Sigma}_{nl}$ with internal dynamics $\dot{z} = -z - z^3 + w$, with $w \in \mathbb{R}$ the transversal velocity of the beam tip. The output $v = -\kappa z$, with $v \in \mathbb{R}$ and $\kappa = 6$, represents the force the mount exerts on the beam. It can be shown that $\mathbf{\Sigma}_{nl}$ is input-to-state convergent with γ_{zw} given by its inverse as $\gamma_{zw}^{-1}(r) = r + (1/4)r^3$, implying $\gamma_{zw}(r) < r$ for all $r > 0$. Clearly, the output satisfies an incremental bound with gain function $\chi_{vz}(r) = \kappa r$. After computation of the gains for all input-output combinations of the linear system, the small-gain condition is evaluated as $\gamma_{xv} \circ \chi_{vz} \circ \gamma_{zw} \circ \chi_{wx}(s) \leq 0.8724s < s, \forall s > 0$. This condition is equivalent to (11), since the

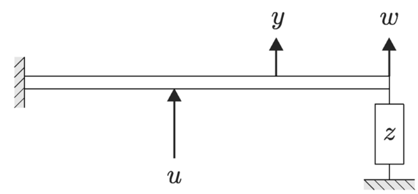

Fig. 2. Flexible beam with nonlinear support.

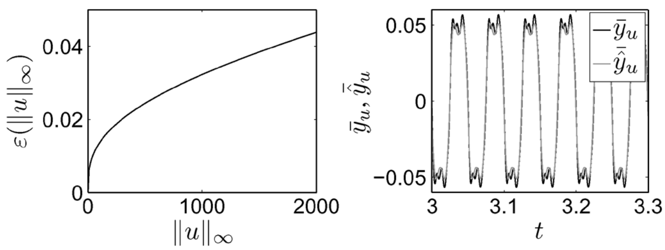

Fig. 3. Error bound $\varepsilon(\|u\|_\infty)$ (left) and steady-state output for $u(t) = 10^4\ \mathrm{sign}(\sin(2\pi 20t))$ of the high-order and reduced-order models (right) for $k = 4$.

strict inequality and the linearity of the gain functions implies the existence of the functions ρ_i, $i \in \{1, 2\}$. Hence, the system satisfies Assumption 1 and is input-to-state convergent.

Balanced truncation is applied to the linear beam model to obtain a reduced-order model of order $k = 4$. An a posteriori computation of the error bound leads to the linear error functions ε_{ji} for $j \in \{y, w\}$, $i \in \{u, v\}$, where $\varepsilon_{wv}(r) = (8.69 \cdot 10^{-3})r$. Now, application of Theorem 3 indicates that input-to-state convergence of the reduced-order nonlinear system is guaranteed. Furthermore, the error bound ε as in (17) is obtained and depicted in the left graph of Fig. 3. Here, it is recalled that the error bound holds for all bounded inputs. A time simulation is shown in the right graph of Fig. 3, showing a good approximation.

VII. Conclusion

In this technical note, a model reduction procedure is presented for nonlinear systems that can be decomposed into a feedback interconnection of a linear and nonlinear subsystem, both satisfying the input-to-state convergence property. Model reduction is applied to the linear subsystem only, allowing for the application of well-developed existing model reduction techniques. In this approach, conditions for stability of the reduced-order nonlinear system are given, hereby exploiting a small-gain theorem for input-to-state convergent systems. Furthermore, the input-to-state convergence property is shown to be instrumental in the derivation of an error bound.

Appendix
Proof of Theorem 3

The two statements are proven separately. Herein, the following property is extensively used.

Property 1 (Weak Triangular Inequality [11]): For any $\gamma \in \mathcal{K}$, $\rho \in \mathcal{K}_\infty$, the following inequality holds for all $a, b \geq 0$: $\gamma(a + b) \leq \gamma((\mathrm{id} + \rho)(a)) + \gamma((\mathrm{id} + \rho^{-1})(b))$.

1) Input-to-State Convergence of the Reduced-Order System: Theorem 2 guarantees stability of the reduced-order system $\hat{\mathbf{\Sigma}}$ if there exists functions $\hat{\rho}_1, \hat{\rho}_2 \in \mathcal{K}_\infty$ such that

$$(\mathrm{id} + \hat{\rho}_1) \circ \hat{\gamma}_{xv} \circ \chi_{vz} \circ (\mathrm{id} + \hat{\rho}_2) \circ \gamma_{zw} \circ \hat{\chi}_{wx}(s) \leq s, \ \forall s \geq 0. \quad (19)$$

Here, $\hat{\gamma}_{xv}$ and $\hat{\chi}_{wx}$ are incremental bounds on the input-to-state gain and output operator of the reduced-order linear subsystem $\hat{\mathbf{\Sigma}}_{lin}$. These exist since $\hat{\mathbf{\Sigma}}_{lin}$ is asymptotically stable, but are not known a priori.

Therefore, an (*a priori*) upper bound on $\hat{\gamma}_{xv}$ and $\hat{\chi}_{wx}$ is derived by considering

$$\|\hat{\mathcal{F}}_w(u,v_2) - \hat{\mathcal{F}}_w(u,v_1)\|_\infty = \|\mathcal{E}_w(u,v_2) - \mathcal{E}_w(u,v_1) + \mathcal{F}_w(u,v_2) - \mathcal{F}_w(u,v_1)\|_\infty \quad (20)$$

$$\leq \|\mathcal{E}_w(u,v_2) - \mathcal{E}_w(u,v_1)\|_\infty + \|\mathcal{F}_w(u,v_2) - \mathcal{F}_w(u,v_1)\|_\infty. \quad (21)$$

Here, the first and second term in (21) are bounded by (15) and (10), respectively, leading to

$$\|\hat{\mathcal{F}}_w(u,v_2) - \hat{\mathcal{F}}_w(u,v_1)\|_\infty \leq (\varepsilon_{wv} + \chi_{wx} \circ \gamma_{xv})(\|v_2 - v_1\|_\infty). \quad (22)$$

Now, (22) provides an upper bound on the incremental gain of the reduced-order steady-state output operator $\hat{\mathcal{F}}_w(u,v)$ with respect to v. Hence, $\hat{\chi}_{wx} \circ \hat{\gamma}_{xv}(s) \leq (\varepsilon_{wv} + \chi_{wx} \circ \gamma_{xv})(s)$ holds for all $s \geq 0$. By introduction of the function $\hat{\rho}_1 = \hat{\gamma}_{xv}^{-1} \circ \check{\rho}_1 \circ \hat{\gamma}_{xv}$, which satisfies

$$(\mathrm{id} + \check{\rho}_1) \circ \hat{\gamma}_{xv}(s) = \hat{\gamma}_{xv} \circ (\mathrm{id} + \hat{\rho}_1)(s), \quad \forall s \geq 0 \quad (23)$$

the terms $\hat{\chi}_{wx}$ and $\hat{\gamma}_{xv}$ in (19) can be grouped. Using $\hat{\chi}_{wx} \circ \hat{\gamma}_{xv}(s) \leq (\varepsilon_{wv} + \chi_{wx} \circ \gamma_{xv})(s)\ \forall s \geq 0$, (16) implies (19), such that Theorem 2 proves input-to-state convergence of $\hat{\mathbf{\Sigma}}$.

2) Error Bound: As a first step, bounds on the signals related to the high-order system $\mathbf{\Sigma}$ are derived. Here, the incremental bound on the steady-state operator $\mathcal{F}_w$ [see (10)] leads to

$$\|\bar{w}_u - 0\|_\infty \leq \chi_{wx}\Big(\gamma_{xu}(\|u - 0\|_\infty) + \gamma_{xv}(\|\bar{v}_u - 0\|_\infty)\Big) \quad (24)$$

where it is noted (by using $g(0,0) = 0, h(0) = 0$) that $\bar{x}_u = \bar{z}_u = 0$ is the unique steady-state solution for zero input. In (24), $\bar{v}_u$ and $\bar{w}_u$ denote the steady-state solutions of the signals v and w for input u, respectively. Then, (9) gives

$$\|\bar{w}_u\|_\infty \leq \chi_{wx}\Big(\gamma_{xu}(\|u\|_\infty) + \gamma_{xv} \circ \chi_{vz} \circ \gamma_{zw}\left(\|\bar{w}_u\|_\infty\right)\Big) \quad (25)$$

$$\leq \chi_{wx} \circ (\mathrm{id} + \rho_3) \circ \gamma_{xv} \circ \chi_{vz} \circ \gamma_{zw}\left(\|\bar{w}_u\|_\infty\right) + \chi_{wx} \circ \left(\mathrm{id} + \rho_3^{-1}\right) \circ \gamma_{xu}(\|u\|_\infty) \quad (26)$$

where the weak triangular inequality (Property 1) is applied to split the terms related to u and $\bar{w}_u$ and where ρ_3 is an arbitrary function of class $\mathcal{K}_\infty$. Rewriting (26) leads to

$$\|\bar{w}_u\|_\infty \leq \Big(\mathrm{id} - \chi_{wx} \circ (\mathrm{id} + \rho_3) \circ \gamma_{xv} \circ \chi_{vz} \circ \gamma_{zw}\Big)^{-1} \circ \chi_{wx} \circ \left(\mathrm{id} + \rho_3^{-1}\right) \circ \gamma_{xu}(\|u\|_\infty) =: \eta_{wu}(\|u\|_\infty) \quad (27)$$

with $\eta_{wu} \in \mathcal{K}_\infty$. Here, the small-gain condition (11) guarantees the existence of a function ρ_3 (e.g. $\rho_3 = \rho_1$) such that the inverse in (27) exists. Now, substitution of (27) in (9) gives

$$\|\bar{v}_u\|_\infty \leq \chi_{vz} \circ \gamma_{zw}(\|\bar{w}_u\|_\infty) \leq \chi_{vz} \circ \gamma_{zw} \circ \eta_{wu}(\|u\|_\infty) =: \eta_{vu}(\|u\|_\infty), \quad \eta_{vu} \in \mathcal{K}_\infty. \quad (28)$$

As a second step, steady-state errors are considered, with the error on w given as

$$\|\delta\bar{w}_u\|_\infty = \|\bar{w}_u - \bar{\hat{w}}_u\|_\infty = \left\|\mathcal{F}_w(u,\bar{v}_u) - \hat{\mathcal{F}}_w(u,\bar{\hat{v}}_u)\right\|_\infty \quad (29)$$

$$\leq \left\|\mathcal{F}_w(u,\bar{v}_u) - \mathcal{F}_w(u,\bar{\hat{v}}_u)\right\|_\infty + \left\|\mathcal{F}_w(u,\bar{\hat{v}}_u) - \hat{\mathcal{F}}_w(u,\bar{\hat{v}}_u)\right\|_\infty. \quad (30)$$

Here, the first term is related to the steady-state operator of the high-order linear subsystem, which satisfies the incremental bound (10). The second term equals the steady-state error operator $\mathcal{E}_w(u,\bar{\hat{v}}_u)$, which is, by assumption, bounded as in (15). Using these bounds, (30) is bounded as

$$\|\delta\bar{w}_u\|_\infty \leq \chi_{wx} \circ \gamma_{xv}\left(\|\bar{v}_u - \bar{\hat{v}}_u\|_\infty\right) + \varepsilon_{wu}(\|u\|_\infty) + \varepsilon_{wv}\left(\|\bar{\hat{v}}_u\|_\infty\right) \quad (31)$$

where the property $\mathcal{E}_w(0,0) = 0$ is used. Furthermore, the first term in (31) is now related to the steady-state error in v, i.e. $\delta\bar{v}_u = \bar{v}_u - \bar{\hat{v}}_u$. Thus, application of (9) gives

$$\|\delta\bar{w}_u\|_\infty \leq \chi_{wx} \circ \gamma_{xv} \circ \chi_{vz} \circ \gamma_{zw}\left(\|\delta\bar{w}_u\|_\infty\right) + \varepsilon_{wu}(\|u\|_\infty) + \varepsilon_{wv}\left(\|\bar{v}_u\|_\infty + \|\delta\bar{v}_u\|_\infty\right). \quad (32)$$

The triangle inequality and the assumption that ε_{wv} is of class $\mathcal{K}_\infty$ is used to obtain the last term. Subsequently, the last term in (32) can be split by using Property 1 to obtain

$$\|\delta\bar{w}_u\|_\infty \leq \chi_{wx} \circ \gamma_{xv} \circ \chi_{vz} \circ \gamma_{zw}(\|\delta\bar{w}_u\|_\infty) + \varepsilon_{wu}(\|u\|_\infty) + \varepsilon_{wv} \circ (\mathrm{id} + \rho_4)(\|\delta\bar{v}_u\|_\infty) + \varepsilon_{wv} \circ (\mathrm{id} + \rho_4^{-1})(\|\bar{v}_u\|_\infty) \quad (33)$$

with ρ_4 an arbitrary class $\mathcal{K}_\infty$-function. In (33), (9) can be used to bound $\|\delta\bar{v}_u\|_\infty$ as before, whereas $\|\bar{v}_u\|_\infty$ can be bounded by application of the earlier result (28), which yields

$$\|\delta\bar{w}_u\|_\infty \leq \Big(\chi_{wx} \circ \gamma_{xv} + \varepsilon_{wv} \circ (\mathrm{id} + \rho_4)\Big) \circ \chi_{vz} \circ \gamma_{zw}\left(\|\delta\bar{w}_u\|_\infty\right) + \left(\varepsilon_{wu} + \varepsilon_{wv} \circ (\mathrm{id} + \rho_4^{-1}) \circ \eta_{vu}\right)(\|u\|_\infty). \quad (34)$$

Rewriting (34) gives (with δ_{wu} of class $\mathcal{K}_\infty$)

$$\|\delta\bar{w}_u\|_\infty \leq \Big(\mathrm{id} - (\chi_{wx} \circ \gamma_{xv} + \varepsilon_{wv} \circ (\mathrm{id} + \rho_4)) \circ \chi_{vz} \circ \gamma_{zw}\Big)^{-1} \circ \left(\varepsilon_{wu} + \varepsilon_{wv} \circ \left(\mathrm{id} + \rho_4^{-1}\right) \circ \eta_{vu}\right)(\|u\|_\infty), =: \delta_{wu}(\|u\|_\infty). \quad (35)$$

The small-gain condition (16) guarantees the existence of a function ρ_4 such that the inverse indeed exists (e.g. $\rho_4 = \hat{\rho}_1$). Applying (9) gives a bound on $\delta\bar{v}_u$ as

$$\|\delta\bar{v}_u\|_\infty \leq \chi_{vz} \circ \gamma_{zw}(\|\delta\bar{w}_u\|_\infty) \leq \chi_{vz} \circ \gamma_{zw} \circ \delta_{wu}(\|u\|_\infty), =: \delta_{vu}(\|u\|_\infty), \quad \delta_{vu} \in \mathcal{K}_\infty. \quad (36)$$

To find the bound on the output error $\delta\bar{y}_u = \bar{y}_u - \bar{\hat{y}}_u$, the first steps of the analysis of the error $\delta\bar{w}_u$ are repeated. Repeating the procedure in (29) and (30) gives

$$\|\delta\bar{y}_u\|_\infty \leq \chi_{yz} \circ \gamma_{xv}\left(\|\delta\bar{v}_u\|_\infty\right) + \varepsilon_{yu}(\|u\|_\infty) + \varepsilon_{yv}\left(\|\bar{\hat{v}}_u\|_\infty\right) \quad (37)$$

which is similar to the result (31). By using the triangle inequality on the norm of $\bar{\hat{v}}_u = \bar{v}_u - \delta\bar{v}_u$ and the weak triangular inequality (Property 1), (37) leads to

$$\|\delta\bar{y}_u\|_\infty \leq \chi_{yz} \circ \gamma_{xv}(\|\delta\bar{v}_u\|_\infty) + \varepsilon_{yu}(\|u\|_\infty) + \varepsilon_{yv} \circ (\mathrm{id} + \rho_5)(\|\delta\bar{v}_u\|_\infty) + \varepsilon_{yv} \circ (\mathrm{id} + \rho_5^{-1})(\|\bar{v}_u\|_\infty) \quad (38)$$

with ρ_5 of class $\mathcal{K}_\infty$. In (38), the terms related to $\delta\bar{v}_u$ and $\bar{v}_u$ can be bounded using (36) and (28), respectively. This leads to the final result (17).

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Exact Differentiation of Signals With Unbounded Higher Derivatives

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***Abstract*—Arbitrary-order homogeneous differentiators based on high-order sliding modes are generalized to ensure exact robust kth-order differentiation of signals with a given functional bound of the $(k+1)$th derivative. The asymptotic accuracies in the presence of noises and discrete sampling are estimated. The results are applicable for the global observation of system states with unbounded dynamics. Computer simulation demonstrates the applicability of the modified differentiators.**

***Index Terms*—High-order sliding mode, homogeneity, nonlinear observers, robustness.**

I. INTRODUCTION

Signal differentiation is a well-known problem mostly related to various observation problems. The main differentiation difficulty is its sensitivity to small high-frequency input noises. Since one cannot reliably distinguish between the noise and the basic signal, practical differentiation is a trade-off between exact differentiation and robustness with respect to noises.

The usual assumption is that the noise corresponds to the high-frequency signal component to be filtered out (e.g., [8], [9]). Respectively, the traditional sliding-mode (SM) differentiators [5], [14], [15], as well as high-gain differentiators [1], do not provide for exact differentiation due to filtration involved. The differentiator from [2] is based on a second-order SM (2-SM) controller using the derivative sign, whose evaluation requires the possibly-lacking knowledge of the noise magnitude.

Exact derivatives of arbitrary kth order can be obtained by the robust exact finite-time-convergent differentiator [10], provided the $(k+1)$th-order derivative is bounded by a known constant. The differentiator is based on 2-SMs, and features the best possible asymptotics in the presence of infinitesimal Lebesgue-measurable sampling noises. It has already found numerous practical and theoretical applications (e.g., [3], [4], [7], [12], [13]). While it solves main differentiation problems of local output-feedback implementation, its global implementation requires the global boundedness of the $(k+1)$th-order output derivative, which is quite restrictive. Though a global constant bound could be always chosen for the whole practical operation region, the constant would be excessively large and would increase differentiator errors. Thus, the satisfactory performance of the differentiator at the operation region boundary inevitably causes performance degradation somewhere inside the region.

On the other hand, main system features are often determined by a few variables available or observable in real time. In that case upper bounds of the highest output derivatives and sampling noises can also be often estimated as functions of these variables. For example, aerodynamic features of an aircraft are mostly determined by the dynamic pressure and the Mach number.

It is assumed in this note that the $(k+1)$th-order derivative has a variable upper bound available in real time. It is proved that the kth-order

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