Now, we write

$$\begin{aligned}\|x_j[t]-x_\ell[t]\| &\le \|x_\ell[t]-\overline{x}_\ell[t]\|+\|x_j[t]-\overline{x}_\ell[t]\| \\ &\le \|x[t]\|+\|x_j[t]-\overline{x}_\ell[t]\|. \end{aligned}\tag{22}$$

Besides

$$\begin{aligned}\|x_j[t]-\overline{x}_\ell[t]\| &\le \|x_j[0]-\overline{x}_\ell[0]\| \\ &\quad+\sum_{\tau=0}^{t-1}\|(x_j[\tau+1]-\overline{x}_\ell[\tau+1])-(x_j[\tau]-\overline{x}_\ell[\tau])\| \\ &=\|x_j[0]-\overline{x}_\ell[0]\|+h\sum_{\tau=0}^{t-1}\|v_j[\tau]-\overline{v}_\ell[\tau]\|.\end{aligned}$$

By the definition of $\overline{v}_\ell[t]$ and induction hypothesis (18) we have

$$\|v_j[\tau]-\overline{v}_\ell[\tau]\| \le \frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\|v_j[t]-v_i[t]\| \le 2N_{\ell-1}e^{-\nu_{\ell-1}t}$$

We then define ξ by

$$\begin{aligned}\|x_j[t]-\overline{x}_\ell[t]\| &\le \|x_j[0]-\overline{x}_\ell[0]\|+h\sum_{\tau=0}^{t-1}2N_{\ell-1}e^{-\nu_{\ell-1}\tau} \\ &\le \max\{\|x_j[0]-\overline{x}_\ell[0]\|\}+\frac{2hN_{\ell-1}}{1-e^{-\nu_{\ell-1}}}=:\xi.\end{aligned}\tag{23}$$

Hence, substituting (23) into (22) and the result into (21), for $a[t]\neq 0$, we have

$$a[t]\ge\frac{1}{(1+\xi+\|x[t]\|)^{\alpha}}$$

and $a[t]$ verifies (P2) with ξ defined in (23).

To check (P3) we observe that $\|v_k[t]\|$ and $\|\overline{v}_k[t]\|$ are bounded by $\|v[0]\|$ by (6), so by (19) we obtain $\|v[t]\|\le 2\|v[0]\|=:v_0$.

Let us finally verify (P4), the bound on $b[t]$. We have

$$\begin{aligned}\|b[t]\| &= \Bigg\|\sum_{j\in\mathcal{L}(\ell)}ha_{\ell j}\,(v_j[t]-\overline{v}_\ell[t]) \\ &\qquad -\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\sum_{j\in\mathcal{L}(i)}ha_{ij}\,(v_j[t]-v_i[t])\Bigg\| \\ &\le\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\Bigg\|\sum_{j\in\mathcal{L}(i)}h(a_{\ell i}-a_{ij})\,(v_i[t]-v_j[t]) \\ &\qquad +\sum_{j\in\{\mathcal{L}(\ell)\setminus\mathcal{L}(i)\}}ha_{\ell i}\,(v_i[t]-v_j[t])\Bigg\| \\ &\le 2N_{\ell-1}e^{-\nu_{\ell-1}t}\end{aligned}$$

for all t, in view of (18), as $|a_{\ell i}-a_{ij}|\le 1$.

Hence, conditions (P1) to (P4) are fulfilled. By Proposition 1 there exist random variables $N',\nu'>0$ independent of t such that $\|v[t]\|\le N'e^{-\nu' t}$ for all t. Therefore

$$\|v_\ell[t]\|\le\|v[t]\|+\|\overline{v}_\ell[t]\|\le N'e^{-\nu' t}+N_{\ell-1}e^{-\nu_{\ell-1}t}.$$

This inequality, toghether with (18), shows that $\max_{1\le i\le \ell}\|v_i[t]\|\le Ne^{-\nu t}$ with $N=N'+N_{\ell-1}$ and $\nu=\max\{\nu',\nu_{\ell-1}\}$. Theorem 1 follows. ■

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Steady-State Analysis and Regulation of Discrete-Time Nonlinear Systems

Alexey Pavlov and Nathan van de Wouw

***Abstract*—This paper presents results on steady-state analysis and regulation for nonlinear discrete-time systems subject to time-varying excitations. In the analysis part of the paper, for convergent nonlinear systems (which have uniquely defined steady-state responses to excitations) we provide a complete characterization of the steady-state responses to excitations generated by an exosystem. As a corollary, we obtain a nonlinear frequency response function which extends the well-known FRF defined for linear systems to the class of nonlinear convergent systems. In the control part of the paper, we present a characterization of all controllers solving the global output regulation problem. All these results are obtained using the machinery of convergent systems, extended to the discrete-time setting. For piecewise affine systems, general results are supplied with a constructive design procedure.**

***Index Terms*—Convergent systems, discrete-time systems, frequency response functions, incremental stability, nonlinear systems, output regulation, PWA systems.**

I. Introduction

In many theoretical and practical problems where a dynamical control system (either continuous- or discrete-time) is excited by time-

Manuscript received April 29, 2009; revised September 12, 2011; accepted September 15, 2011. Date of publication December 13, 2011; date of current version June 22, 2012. Recommended by Associate Editor D. Angeli.

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Digital Object Identifier 10.1109/TAC.2011.2178790

varying inputs, the dynamics of the system can be split into transient and steady-state parts. For linear systems, a wide range of tools exists for analysis and controller design for both of these parts (for example based on such a powerful tool as the frequency response function). For nonlinear systems, on the other hand, due to their inherent complexity, available results focus mostly on transient dynamics, in particular, on stability analysis and stabilization by means of feedback. There are relatively few results on analysis and controller design focusing on steady-state dynamics for nonlinear systems with inputs. At the same time, analysis of steady-state responses to excitations and shaping their properties by means of feedback are very important in practical applications. For example, tracking properties of a closed-loop system for reference signals in the low frequency range or its sensitivity to measurement noise in the high frequency range fall into the category of the steady-state analysis. For linear systems such powerful tools as sensitivity and complementary sensitivity functions allow one to quantify these properties. These sensitivity functions are based on the notion of the frequency response function, which does not have a proper counterpart in the domain of nonlinear systems. This partly explains the lack of tools for steady-state analysis and control for nonlinear systems.

Another explanation stems from the fact that for general nonlinear systems, the notion of steady-state response corresponding to a time-varying input is not well defined. Even if a nonlinear system has a globally asymptotically stable equilibrium for a zero input, solutions of the system corresponding to a non-zero input can diverge to infinity or converge to one or several bounded solutions depending on the initial condition. This makes any steady-state analysis for nonlinear systems very non-trivial. Still, some quantitative steady-state characteristics of solutions of a nonlinear system excited by inputs can be obtained in the framework of input-to-state stability (ISS) or L_2-gain analysis, see, e.g. [1], [2]. These frameworks provide quantitative estimates for bounds on system responses (in terms of the ISS- or L_2-gain) to general inputs. The estimated gains are, however, generally rather conservative especially when applied to particular types of excitations such as, for example, periodic excitations or excitations with a particular frequency content, which are often encountered in applications.

One possible approach to analyze steady-state behavior of a system is to consider it locally, in a small neighborhood of an equilibrium and with small inputs. It is natural to expect that in this case, provided the equilibrium is exponentially stable, a number of the system's steady-state properties can be captured from its linearization at the equilibrium. However, in many cases one needs to analyze or control a nonlinear system outside of such a small neighborhood. Moreover, linearization may not be defined, as, for example, for piecewise affine (PWA) systems.

From the control side, the problem of shaping steady-state responses can be efficiently addressed in the framework of nonlinear output regulation [3]–[5]. In this framework, the objective is to stabilize and shape, by means of control, steady-state responses of a nonlinear system to excitations generated by an exosystem. Shaping of steady-state responses is done to achieve zeroing of a regulated output of the system. Although there are plenty of results on output regulation for continuous-time systems, see, e.g., [3]–[6] and references therein, only a few results are available for discrete-time systems. The only result on discrete-time output regulation for nonlinear systems known to the authors [4] corresponds to the case of local output regulation (i.e. in a sufficiently small neighborhood of the origin and for sufficiently small inputs).

In this paper we present results on steady-state analysis and controller design for discrete-time nonlinear systems in a *non-local* and *non-equilibrium* setting. In particular, for the class of convergent nonlinear discrete-time systems (which have uniquely defined steady-state responses to external excitations), we provide a complete characterization of the steady-state responses to excitations generated by an exosystem. The result is non-local and provides this characterization in terms of an invariant manifold. As a corollary, for convergent systems we obtain a complete characterization of steady-state responses to harmonic excitations (in the discrete-time setting) of all amplitudes, frequencies and phases. This characterization is given in terms of a finite-dimensional function, which we call a nonlinear frequency response function. This function serves as an extension of the well-known FRF from the linear systems theory. Based on the obtained analysis results, we also provide a complete characterization of all controllers solving the global uniform output regulation problem. The presented results extend results obtained for continuous-time systems in [5], [7] to the case of discrete-time nonlinear systems. Moreover, we extend results on local nonlinear output regulation for discrete-time systems from [4] to the global setting.

These extensions have been made possible thanks to the notion of convergent systems, [5], [8], [9]. In this paper, we extend this notion from the continuous-time case to the case of discrete-time systems. Moreover, we provide constructive results to verify the convergence property and to achieve it by means of feedback for the class of piecewise affine (PWA) systems. Apart from serving the needs of this paper, these results are highly instrumental in solving tracking, synchronization and observer design problems for discrete-time PWA systems, as it has been illustrated for the continuous-time case in [10].

The paper is organized as follows. In Section II, we provide definitions of convergent discrete-time systems and sufficient conditions to verify the convergence property for general nonlinear systems. For PWA systems, these general conditions are translated to an LMI-based result and extended to a constructive controller design procedure to induce the convergence property by means of feedback. Section III contains results on steady-state analysis of convergent nonlinear systems. Here we also introduce the notion of nonlinear frequency response function for convergent systems. In Section IV, we formulate the global uniform output regulation problem for nonlinear discrete-time systems and provide a characterization of all controllers solving this problem. Finally, Section V summarizes the paper with conclusions.

We will use the following notations. $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of integer and real numbers, respectively. Given a matrix $P = P^T \succ 0$ and a vector x, $|x|_P := \sqrt{x^T P x}$.

II. Convergent Discrete-Time Systems

In this section we consider general discrete-time nonlinear systems described by equations of the form

$$x[k+1] = f\left(x[k], k\right) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \times \mathbb{Z} \to \mathbb{R}^n$ and $k \in \mathbb{Z}$ reflects the discrete time variable.

Definition 1: System (1) is called (uniformly, exponentially) convergent if

- there exists a unique solution $\bar{x}[k]$ that is defined and bounded on $\mathbb{Z}$ (from $-\infty$ to $+\infty$),
- $\bar{x}[k]$ is globally (uniformly, exponentially) asymptotically stable.[1]

The solution $\bar{x}[k]$ is called a steady-state solution.

[1]A solution $\bar{x}[k]$ of system (1) is called:
- *globally asymptotically stable (GAS)* if a) for any $\epsilon > 0$ and $k_0 \in \mathbb{Z}$ there exists $\delta = \delta(\epsilon, k_0) > 0$ such that if $|x[k_0] - \bar{x}[k_0]| < \delta$, then $|x[k] - \bar{x}[k]| < \epsilon$ for all $k \geq k_0$; b) any solution $x[k]$ satisfies $|x[k] - \bar{x}[k]| \to 0$, as $k \to +\infty$.
- *globally uniformly asymptotically stable (GUAS)* if a) for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $|x[k_0] - \bar{x}[k_0]| < \delta$, then $|x[k] - \bar{x}[k]| < \epsilon$ for all $k \geq k_0$; b) for any $R > 0$ and any $\tilde{\epsilon} > 0$ there exists $T = T(R, \tilde{\epsilon}) > 0$ such that if $|x[k_0] - \bar{x}[k_0]| < R$, then $|x[k] - \bar{x}[k]| < \tilde{\epsilon}$ for all $k \geq k_0 + T$.
- *globally exponentially stable (GES)* if there exist $c > 0$ and $0 < \lambda < 1$ such that $|x[k] - \bar{x}[k]| \leq c\lambda^{(k-k_0)}|x[k_0] - \bar{x}[k_0]|$ for all $k \geq k_0$.

As follows from this definition, any solution of a convergent system "forgets" its initial condition and converges to the steady-state solution, which is independent of the initial condition. The convergence property is an extension of stability properties of asymptotically stable linear systems excited by external inputs. Similar to the case of linear systems, the steady-state solution of a convergent system with a periodic right-hand side is also periodic with the same period. If the right-hand side of a convergent system is time-invariant, then the steady-state solution is an equilibrium. These properties can be shown to hold in the same way as for the continuous time case [5], [8]. Moreover, since all solutions of a convergent system converge to a bounded steady-state solution, the following two properties hold: all solutions are bounded in forward time and any two solutions converge to each other.

In the scope of control, time dependency of the right-hand side of system (1) is usually due to some input. This input may represent, for example, a disturbance or a feedforward control signal. In this case the system takes the form

$$x[k+1] = f\left(x[k], w[k]\right) \tag{2}$$

with state $x \in \mathbb{R}^n$ and input $w \in \mathbb{R}^m$. Below we define the convergence property for systems with inputs from a certain class $\mathcal{I}$.

Definition 2: System (2) is said to be (uniformly, exponentially) convergent for a class of inputs $\mathcal{I}$ that are defined on $\mathbb{Z}$ if it is (uniformly, exponentially) convergent for every input $w \in \mathcal{I}$. In order to emphasize the dependency on $w[k]$, the steady-state solution is denoted by $\bar{x}_w[k]$.

Next we give a definition of an additional property of a convergent system linking bounds on the inputs to the bounds on the corresponding steady-state solutions.

Definition 3: System (2) that is convergent for some class of inputs $\mathcal{I}$ is said to have the Uniformly Bounded Steady-State (UBSS) property if for any $\rho > 0$ there exists $\mathcal{R} > 0$ such that for any input $w \in \mathcal{I}$ the following implication holds:

$$|w[k]| \le \rho \;\; \forall k \in \mathbb{Z} \Rightarrow |\bar{x}_w[k]| \le \mathcal{R} \;\; \forall k \in \mathbb{Z}.$$

It can be shown that if convergent system (2) is Input-to-State Stable (ISS) [1], then it has the UBSS property.

The next theorem provides sufficient conditions under which system (1) is exponentially convergent. It is a discrete-time counterpart of the result on convergent continuous-time systems from [8] (see also [9]).

Theorem 1: Consider system (1) with a Lipschitz continuous right-hand side satisfying

$$|f(x_1,k) - f(x_2,k)|_P \le \lambda |x_1 - x_2|_P, \; \forall x_1, x_2 \in \mathbb{R}^n, \; k \in \mathbb{Z} \tag{3}$$

$$\sup_{k\in\mathbb{Z}} |f(0,k)|_P =: C < +\infty \tag{4}$$

for some matrix $P = P^T \succ 0$ and number λ such that $0 < \lambda < 1$. Then system (1) is exponentially convergent with the steady-state solution $\bar{x}[k]$ satisfying $\sup_{k\in\mathbb{Z}} |\bar{x}[k]|_P \le (C/1-\lambda)$. Moreover, any two solutions of system (1) satisfy

$$|x_1[k] - x_2[k]|_P \le \lambda^{(k-k_0)} \left|x_1[k_0] - x_2[k_0]\right|_P, \quad \forall k \ge k_0. \tag{5}$$

Proof: The proof can be found in [14], [11]. ■

Remark 1: It follows from Theorem 1, that for systems with inputs of the form (2) with $f(x, w)$ being continuous in both x and w, the condition $|f(x_1, w) - f(x_2, w)|_P \le \lambda |x_1 - x_2|_P$, for all $x_1, x_2 \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, satisfied for some matrix $P = P^T \succ 0$ and $0 < \lambda < 1$, implies that system (2) is exponentially convergent with the UBSS property for the class of inputs defined and bounded on $\mathbb{Z}$, and any two solutions corresponding to the same input $w[k]$ satisfy (5).

Remark 2: Note that inequality (5), which follows from (1) and (3), implies that every solution of system (1) is globally exponentially stable. Systems with such a property are referred to as incrementally stable or contracting, see, e.g. [12], [13]. The essential difference between Theorem 1 and the results from the above mentioned references is that Theorem 1 proves the existence and uniqueness of a steady-state solution as defined in Definition 1. In [12], [13] the existence and uniqueness of the steady-state solution has been proven only for the case of systems with periodic or time-invariant right-hand sides. Here we prove it for systems with arbitrary time-varying right-hand sides. The uniquely defined steady-state solution is highly instrumental in both the analysis of the steady-state dynamics and output regulation. It allows one to split system analysis into steady-state analysis and analysis of the transient dynamics.

A. Convergent PWA Systems

The general conditions of Theorem 1 can be difficult to check. Below we provide constructive sufficient conditions for exponential convergence of piecewise affine (PWA) systems characterized by continuous PWA maps. Consider the state space $\mathbb{R}^n$ that is divided into polyhedral cells Λ_i, $i = 1, \ldots, l$, by hyperplanes given by equations of the form $H_{ij}^T x + h_{ij} = 0$, such that $\Lambda_i \subset \{x \in \mathbb{R}^n : H_{ij}^T x + h_{ij} \ge 0\}$ and $\Lambda_j \subset \{x \in \mathbb{R}^n : H_{ij}^T x + h_{ij} < 0\}$, with $H_{ij} \in \mathbb{R}^n$ and $h_{ij} \in \mathbb{R}$ for $\{i, j\} = 1, \ldots, l$ and $i \ne j$. We will consider piecewise-affine systems of the form

$$x[k+1] = A_i x[k] + b_i + Bu[k] + Dw[k], \tag{6}$$

for $x[k] \in \Lambda_i$, $i = 1, \ldots, l$. The vectors $x[k] \in \mathbb{R}^n$, $u[k] \in \mathbb{R}^p$ and $w[k] \in \mathbb{R}^m$ are the state, control and input vectors at time k, respectively. $A_i, b_i, i = 1, \ldots, l$, B and D are constant matrices and vectors of appropriate dimensions. The hyperplanes $H_{ij}^T x + h_{ij} = 0$ are the switching surfaces. We assume that the inputs $w[k]$ are defined on the whole time axis $\mathbb{Z}$.

We will deal with piecewise affine systems which have continuous right-hand sides in x, u and w. This continuity requirement on the right-hand side of system (6) is equivalent to the condition that for any two cells Λ_i and Λ_j having a common boundary $H_{ij}^T x + h_{ij} = 0$ the corresponding matrices A_i and A_j and the vectors b_i and b_j satisfy the equalities

$$G_{ij} H_{ij}^T = A_i - A_j, \quad G_{ij} h_{ij} = b_i - b_j \tag{7}$$

for some vector $G_{ij} \in \mathbb{R}^n$ (see, e.g., [5], [10]). The following theorem establishes sufficient conditions for exponential convergence of the open-loop system (6) (i.e. with $u[k] \equiv 0$) for the class of bounded inputs $w[k]$.

Theorem 2: Suppose the right-hand side of (6) is continuous and there exist a matrix P and a number $\bar{\alpha}$ such that $0 < \bar{\alpha} < 1$ and

$$P = P^T \succ 0, \qquad A_i^T P A_i \preceq \bar{\alpha} P, \quad i = 1, \ldots, l \tag{8}$$

Then system (6) is exponentially convergent with the UBSS property for the class of inputs $w[k]$ that are defined and bounded on $\mathbb{Z}$. Moreover, any two solutions corresponding to the same input $w[k]$ satisfy $|x_1[k] - x_2[k]|_P \le \lambda^{(k-k_0)} |x_1[k_0] - x_2[k_0]|_P, \forall k \ge k_0$. for the given P and $\lambda = \sqrt{\bar{\alpha}}$.

Proof: The proof can be found in [14], [11]. ■

It may seem that the result of Theorem 2 can be directly obtained from the contraction analysis as presented in [13]. However, this is not the case for two reasons. Firstly, contraction analysis does not provide existence and uniqueness of the steady-state solution, see Remark 2. Secondly, the results on contraction analysis from [13] do not apply here, since they require the right-hand side to be smooth, which is not the case for PWA systems. Still it may seem that condition (8), which is the analog of the contraction condition from [13], and which guarantees the existence of a common quadratic Lyapunov function for every linear mode, is such a strong requirement that it alone guarantees the convergence property of system (6) for arbitrary bounded inputs $w[k]$. In general, this is not the case, as has been demonstrated in [14].

The following theorem provides conditions under which PWA system (6) can be made convergent by means of a piecewise affine feedback of the form

$$u[k] = K_i x[k] + d_i + v[k], \text{ for } x[k] \in \Lambda_i, \ i = 1, \dots, l \quad (9)$$

with $K_i \in \mathbb{R}^{m\times n}$ and $d_i \in \mathbb{R}^m$, $i = 1, \dots, l$, being feedback parameters to be designed. The vector $v \in \mathbb{R}^p$ is an auxiliary control input that can be used for shaping the steady-state solution.

Theorem 3: Consider PWA system (6) with a continuous right-hand side. Suppose there exist $\mathcal{P} \in \mathbb{R}^{n\times n}$, $\mathcal{Z}_i \in \mathbb{R}^{m\times n}$, $d_i \in \mathbb{R}^m$, $i = 1, \dots, l$, and a number $0 < \bar{\alpha} < 1$ such that

$$\mathcal{P} = \mathcal{P}^T \succ 0, \quad \begin{bmatrix} \bar{\alpha}\mathcal{P} & \mathcal{P}A_i^T + \mathcal{Z}_i^T B^T \\ A_i\mathcal{P} + B\mathcal{Z}_i & \mathcal{P} \end{bmatrix} \succ 0 \quad (10)$$

and for any pair of cells Λ_i and Λ_j having a common boundary given by $H_{ij}^T x + h_{ij} = 0$ there exists a vector $M_{ij} \in \mathbb{R}^m$ such that

$$\mathcal{Z}_i - \mathcal{Z}_j = M_{ij} H_{ij}^T \mathcal{P}, \quad d_i - d_j = M_{ij} h_{ij}. \quad (11)$$

Then system (6) in closed loop with controller (9) having feedback gains $K_i = \mathcal{Z}_i \mathcal{P}^{-1}$, and d_i, $i = 1, \dots, l$, satisfying (11) is exponentially convergent with the UBSS property with respect to inputs $w[k], v[k]$.

Proof: The dynamics of the closed-loop system (6), (9) is given by:

$$x[k+1] = \bar{A}_i x[k] + \bar{b}_i + Bv[k] + Dw[k] \quad (12)$$

for $x \in \Lambda_i$, $i = 1, \dots, l$, with $\bar{A}_i := A_i + BK_i$, $\bar{b}_i = b_i + Bd_i$, $i = 1, \dots, l$. Pre-and post-multiplication of the second inequality in (10) with $\begin{bmatrix} \mathcal{P}^{-1} & 0 \\ 0 & \mathcal{P}^{-1} \end{bmatrix}$, using the definition $\bar{A}_i := A_i + BK_i$ and the substitution of $P = \mathcal{P}^{-1}$ yields $\begin{bmatrix} \bar{\alpha}P & \bar{A}_i^T P \\ P\bar{A}_i & P \end{bmatrix} \succ 0$. Exploiting the Schur complement yields the inequality $\bar{A}_i^T P \bar{A}_i \prec \bar{\alpha}P$, with $P = P^T \succ 0$ and $0 < \bar{\alpha} < 1$. Moreover, the right-hand side of (12) is continuous with respect to x, since it can be represented as the sum $f_x(x) + Bf_u(x) + Bv[k] + Dw[k]$, where $f_x(x) := A_i x + b_i$ and $f_u(x) := K_i x + d_i$, for $x \in \Lambda_i$, $i = 1, \dots, l$. The continuity of $f_x(x)$ is explicitly required in the theorem. The continuity of $f_u(x)$ follows from (11) after post multiplying the first condition in (11) with $\mathcal{P}^{-1}$ and applying conditions in (7). Thus by Theorem 2 we conclude that the closed-loop system (12) is exponentially convergent with the UBSS property for the class of inputs $v[k]$ and $w[k]$ bounded on $\mathbb{Z}$. ■

Remark 3: Theorem 3 can be directly employed to design controllers solving the state tracking problem for PWA systems [14], [11].

III. Steady-State Analysis and Frequency Response Functions

In this section, we analyze the steady-state dynamics of convergent system (2) with inputs $w[k]$ being solutions of the system

$$w[k+1] = s(w[k]), \quad w \in \mathbb{R}^m \quad (13)$$

where $s : \mathbb{R}^m \to \mathbb{R}^m$. We will show that in this case the steady-state dynamics can be fully characterized by a finite-dimensional function $\alpha(w)$ that maps the input $w[k]$ to the corresponding steady-state solution $\bar{x}_w[k] = \alpha(w[k])$.

Firstly, let us introduce some assumptions on system (13). We assume that $s(w)$ is a mapping with a well-defined inverse $s^{-1}(w)$. Under this assumption, for any $w_0 \in \mathbb{R}^m$ there is a unique solution $w[k]$ defined on $\mathbb{Z}$ and satisfying $w[0] = w_0$. We denote the class of solutions of system (13) by $\mathcal{I}_s$. Where it is necessary, we will emphasize the dependency of $w[k]$ on the initial condition by writing $w[k, w_0]$. Moreover, we introduce the following boundedness assumption on solutions of system (13).

Assumption $\mathbf{BA}$*:* For every $r > |0$ there exists $\rho > 0$ such that $|w_0| < r \Rightarrow |w[k, w_0]| < \rho \, \forall k \in \mathbb{Z}$.

The next result provides a characterization, in terms of an invariant manifold, of all steady-state solutions of system (2) corresponding to inputs $w[k]$ generated by (13).

Theorem 4: Suppose system (2) is convergent for the class of inputs $\mathcal{I}_s$. Then there exists a function $\alpha : \mathbb{R}^m \to \mathbb{R}^n$ such that for any solution $w[k]$ of system (13) the corresponding steady-state solution of (2) equals $\bar{x}_w[k] = \alpha(w[k])$. If every solution $w[k]$ of system (13) is bounded on $\mathbb{Z}$, then $\alpha(w)$ is the unique solution of the equation

$$\alpha(s(w)) = f(\alpha(w), w), \quad w \in \mathbb{R}^m \quad (14)$$

in the class of functions mapping bounded sets to bounded sets. If, in addition, the functions $f(x, w)$, $s(w)$ and $s^{-1}(w)$ are continuous, system (2) is *uniformly* convergent with the UBSS property for the class of inputs $\mathcal{I}_s$ and system (13) satisfies the boundedness assumption $\mathbf{BA}$, then $\alpha(w)$ is continuous.

Proof: The proof of this theorem follows a similar line of reasoning as the proof of its continuous-time counterpart [5], [7]. For sake of brevity, we refer to [11] for the full proof for the discrete-time case.■

This invariant manifold theorem is essential in the analysis of the output regulation problem studied in Section IV. Moreover, it allows us to define frequency response functions for nonlinear convergent systems. For linear systems, frequency response functions—usually presented in the form of Bode plots—allow one to analyze system's sensitivity to external disturbances, see, e.g., [15], [16]. While being defined in the frequency domain, in the time domain these functions completely characterize steady-state responses of a linear system to harmonic excitations. Clearly, for general nonlinear systems the concept of a frequency response function does not exist. However, below we will show that for the class of convergent nonlinear systems, one can define a finite-dimensional function that completely characterizes the steady-state responses to harmonic excitations (in the discrete-time setting) for all amplitudes, frequencies and phases. We will call this function a nonlinear frequency response function. As an application example, it has been shown in [17] that knowledge of the steady-state responses to harmonic excitations can be essential in nonlinear performance-based control design for e.g. optical storage drives.

Theorem 5: Consider the system

$$x[k+1] = f(x[k], p[k]) \quad (15)$$

where $f(x, p)$ is a continuous function, $x \in \mathbb{R}^n$ is the state and $p \in \mathbb{R}$ is the input. Suppose it is uniformly convergent with the UBSS property for the class of bounded inputs. Then there exists a continuous function $\chi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^n$ such that for any a, ω and ϕ the steady-state solution of system (15) corresponding to the input $p[k] = a\sin(2\pi\omega k + \phi)$ is given by

$$\bar{x}_{a\omega\phi}[k] = \chi\left([a\cos(2\pi\omega k + \phi), a\sin(2\pi\omega k + \phi)], \omega\right). \quad (16)$$

The function $\chi(v, \omega)$ with $v = (v_1, v_2)^T$ is the unique continuous solution of the equation

$$\chi(S(\omega)v, \omega) = f(\chi(v, \omega), v_2), \ S(\omega) = \begin{bmatrix} \cos 2\pi\omega & -\sin 2\pi\omega \\ \sin 2\pi\omega & \cos 2\pi\omega \end{bmatrix}. \quad (17)$$

Proof: The proof of this theorem follows from the fact that signals of the form $p[k] = a\sin(2\pi\omega k + \phi)$ for various a, ω and ϕ are generated by the system

$$\begin{cases} v_{12}[k+1] = S\left(v_3[k]\right) v_{12}[k], \quad p[k] = v_2[k] \\ v_3[k+1] = v_3[k] \end{cases} \tag{18}$$

where $v_{12} = [v_1, v_2]^T \in \mathbb{R}^2$, $v_3 \in \mathbb{R}$, with the initial conditions $v_{12}[0] = [a\cos\phi, a\sin\phi]^T$, $v_3[0] = \omega$ and $S(\omega)$ as given in (17). Consequently, we can treat system (15) excited by the input $p[k] = a\sin(2\pi\omega k + \phi)$ as the system

$$x[k+1] = f\left(x[k], v_2[k]\right) \tag{19}$$

excited by a solution of the system (18). According to the conditions of the theorem, system (19) is uniformly convergent with the UBSS property for the class of bounded inputs. Since all solutions of (18) are bounded on $\mathbb{Z}$, system (19) is uniformly convergent with the UBSS property for the class of all solutions of (18). Moreover, one can easily check that system (18) with the state $w := [v_1, v_2, v_3]^T$ satisfies the boundedness assumption $\mathbf{BA}$. Therefore, by Theorem 4 there exists a unique continuous function $\chi : \mathbb{R}^3 \to \mathbb{R}^n$ satisfying (17), see (14) in Theorem 4, such that for any solution $w[k] = [v_1[k], v_2[k], \omega]^T$ of system (18) the corresponding steady-state solution of system (19) equals $\bar{x}_w[k] = \chi(v_1[k], v_2[k], \omega)$. In particular, for the solution of system (18) $[a\cos(2\pi\omega k + \phi), a\sin(2\pi\omega k + \phi), \omega]^T$, which corresponds to the input $p[k] = a\sin(2\pi\omega k + \phi)$, the steady-state solution equals $\bar{x}_{a\omega\phi}[k]$ given in (16). ∎

In the case of linear systems, the information on responses to harmonic excitations is provided by the frequency response function. Due to linearity, only frequency-dependent characteristics contain the essential information on the response. In the nonlinear case, the steady-state responses to harmonic excitations will depend also on the amplitude and phase. The function χ takes into account amplitude and phase information (in addition to the frequency) and provides a complete characterization of responses to such excitations. Still, following the linear systems tradition, we will call function χ a Frequency Response Function (FRF), or a nonlinear FRF, omitting the words amplitude and phase in this term. Knowing this nonlinear FRF allows one to compute any quantitative characteristics of the steady-state responses to harmonic excitations.

In theory, the nonlinear FRF can be found from (17). Analytical or numerical methods for solving this equation lie outside the scope of this paper. One numerical method can be found in [11], together with an example of computing the nonlinear FRF for a PWA system.

IV. The Global Output Regulation Problem

In this section, we apply results from the previous section to obtain a characterization of controllers solving the global output regulation problem. In the output regulation problem, we consider systems modeled by equations of the form

$$x[k+1] = f\left(x[k], u[k], w[k]\right) \tag{20}$$
$$e[k] = h_r\left(x[k], w[k]\right) \tag{21}$$
$$y[k] = h_m\left(x[k], w[k]\right) \tag{22}$$

with state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^p$, regulated output $e \in \mathbb{R}^{l_r}$, and measured output $y \in \mathbb{R}^{l_m}$. The exogenous signal $w[k] \in \mathbb{R}^m$, which can be viewed as a disturbance in (20), as a reference signal in (21), or as measurement noise in (22), is generated by exosystem

$$w[k+1] = s\left(w[k]\right), \quad w \in \mathbb{R}^m. \tag{23}$$

We assume that function $s(w)$ has a well-defined inverse and hence all solutions of (23) are defined on $\mathbb{Z}$. Moreover, we assume that the exosystem satisfies the boundedness assumption $\mathbf{BA}$ (see Section III). The set of solutions of the exosystem (23) is denoted by $\mathcal{I}_s$. The functions $f(x,u,w)$, $h_r(x,w)$, $h_m(x,w)$, $s(w)$ and $s^{-1}(w)$ are assumed to be continuous. The problem that we are going to analyze is formulated below.

The Global Uniform Output Regulation Problem: Find, if possible, an output feedback controller of the form

$$\xi[k+1] = \eta\left(\xi[k], y[k]\right), \quad \xi \in \mathbb{R}^q \tag{24}$$
$$u[k] = \theta\left(\xi[k], y[k]\right) \tag{25}$$

for some $q \geq 0$ with continuous functions $\eta(\xi, y)$ and $\theta(\xi, y)$ such that the closed-loop system

$$x[k+1] = f\left(x[k], \theta\left(\xi[k], h_m\left(x[k], w[k]\right)\right), w[k]\right) \tag{26}$$
$$\xi[k+1] = \eta\left(\xi[k], h_m\left(x[k], w[k]\right)\right) \tag{27}$$

satisfies the following conditions:

a) it is uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s$;
b) for all solutions of the closed-loop system and the exosystem starting in $(x[0], \xi[0]) \in \mathbb{R}^{n+q}$ and $w[0] \in \mathbb{R}^m$ it holds that $e[k] = h_r(x[k], w[k]) \to 0$ as $k \to +\infty$.

The uniform convergence requirement guarantees that the closed-loop system (26), (27), has a uniquely defined steady-state solution, which is bounded and UGAS. It is determined only by the corresponding input $w[k]$. The UBSS property (see Definition 3) guarantees that the steady-state solution has a bound which can be calculated from a bound on $w[k]$. Condition b reflects the main control goal: asymptotic zeroing of the regulated output.

Before formulating the main result of this section, we introduce some notations for ω-limit sets; an ω-limit set can be viewed as a set containing the steady-state dynamics of system (23). For a bounded trajectory $w[k, w_0]$ starting in $w_0 \in \mathbb{R}^m$, by $\Omega(w_0)$ we denote the set of points w^* such that $\lim_{i\to+\infty} w[k_i, w_0] = w^*$ for some sequence $\{k_i\}$ with $\lim_{i\to+\infty} k_i = +\infty$. By $\Omega(\mathbb{R}^m)$ we denote the union $\Omega(\mathbb{R}^m) := \bigcup_{w_0 \in \mathbb{R}^m} \Omega(w_0)$. For the case of an exosystem with bounded on $\mathbb{Z}$ trajectories and continuous $s(w)$ and $s^{-1}(w)$, the following statements can be proved in the same way as for the continuous-time case, see, e.g., [18]. For a bounded trajectory $w[k, w_0]$, the ω-limit set $\Omega(w_0)$ is a bounded nonempty invariant set. Moreover, $\Omega(\mathbb{R}^m)$ is an invariant set that attracts all trajectories of the exosystem. The latter fact justifies the comment that $\Omega(\mathbb{R}^m)$ can be considered as a set of the steady-state dynamics of system (23).

Now we can formulate necessary and sufficient conditions for a controller (24), (25), to solve the global uniform output regulation problem.

Theorem 6: Consider exosystem (23) and system (20)–(22) in closed loop with controller (24), (25). Under the condition that the closed-loop system is uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s$, controller (24),(25) solves the global uniform output regulation problem if and only if there exist continuous mappings $\pi(w)$, $c(w)$ and $\sigma(w)$ defined in some neighborhood of $\Omega(\mathbb{R}^m)$ such that relations

$$\pi\left(s(w)\right) = f\left(\pi(w), c(w), w\right) \tag{28}$$
$$0 = h_r\left(\pi(w), w\right) \tag{29}$$
$$\sigma\left(s(w)\right) = \eta\left(\sigma(w), h_m\left(\pi(w), w\right)\right) \tag{30}$$
$$c(w) = \theta\left(\sigma(w), h_m\left(\pi(w), w\right)\right) \tag{31}$$

hold for all $w \in \Omega(\mathbb{R}^m)$.

Proof: For the sake of brevity, we refer to [11] for a complete proof of this theorem, which is essentially based on Theorem 4 and follows a similar line of reasoning as the proof of its continuous-time counterpart in [5]. ■

Condition (28), (29) is a discrete-time counterpart of the well-known regulator equations [4]–[6]. Solvability of (28), (29) implies the existence of an input $u[k] = c(w[k])$ with a corresponding solution $x[k] = \pi(w[k])$ of system (20) along which the regulated output $e[k]$ is identically zero. Since this condition is independent of the particular controller, the existence of continuous $\pi(w)$ and $c(w)$ satisfying (28), (29) serves as a general necessary condition for the solvability of the problem.

Condition (30), (31) is the so-called immersion property, which means that controller (24), (25) is capable of generating the output-zeroing input $u[k] = c(w[k])$ from the measured output $y[k] = h_m(\pi(w[k]), w[k])$ corresponding to the solution $x[k] = \pi(w[k])$.

Both the solvability of the regulator equations and the immersion property guarantee the existence of bounded solution $(x[k], \xi[k]) := (\pi(w[k]), \sigma(w[k]))$ of the closed-loop system along which the regulated output equals zero. The uniform convergence property, on the other hand, guarantees that this solution is a globally uniformly asymptotically stable solution of the closed-loop system, which yields asymptotic zeroing of the regulated output. The UBSS property is not needed for the design purposes, but it is needed for the analysis purposes to conclude on the necessity of the conditions (28)–(31). Since all solutions of the exosystem eventually converge to the ω-limit set $\Omega(\mathbb{R}^m)$, we only need to check conditions (28)–(31) on this set.

On the one hand, Theorem 6 is a discrete-time counterpart of the solvability results for the non-local setting of the continuous-time nonlinear output regulation problem studied in [5], [6]. On the other hand, it is an extension of the local solvability results from [4] to the global setting. The results presented in [4] are based on the center manifold theorem [19], which is an essentially local result with limitations on the exosystem dynamics, whereas Theorem 6 is based on Theorem 4, a global invariant manifold theorem without local limitations on the exosystem dynamics.

From the design point of view, in addition to conditions (28)–(31), a controller solving the output regulation problem also has to make the closed-loop system uniformly convergent with the UBSS property. To develop such a controller, one can use the results of Section II and, for PWA systems, Theorem 3. An example illustrating the application of Theorem 6 and Theorem 3 to piecewise affine systems can be found in [11]. This example is not included here due to space limitations of the technical note format.

V. Conclusions

In this paper we have presented results on steady-state analysis and output regulation of nonlinear discrete-time systems. The first result provides a characterization of all steady-state solutions of a nonlinear convergent system excited by an exosystem. This characterization is obtained in terms of a finite-dimensional function. As a particular case of this result, we have proven the existence of a nonlinear frequency response function, which characterizes steady-state responses of a convergent nonlinear system to harmonic excitations of all frequencies, amplitudes and phases. This function is an extension of the frequency response function from the linear systems theory to the case of nonlinear systems. Secondly, for the global output regulation problem, we have presented a characterization of all controllers solving this problem. This result extends existing non-local solvability results for continuous-time systems to the case of discrete-time systems on the one hand, and local solvability results for discrete-time systems to the global setting on the other hand. All these results are based on the notion of convergent systems, which proves to be very instrumental in non-equilibrium and non-local analysis and control problems for nonlinear systems. To facilitate our studies, we have defined this notion for the case of discrete-time systems and presented the corresponding sufficient conditions. The obtained general results are supported by constructive results for piecewise affine (PWA) systems. The results presented in this paper may serve as a basis for further developments in non-equilibrium and non-local analysis and control design for nonlinear systems, such as those encountered in synchronization, observer design, steady-state performance analysis and nonlinear filtering.

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