

# Tracking Control for Hybrid Systems With State-Triggered Jumps

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**Abstract**—This paper addresses the tracking problem in which the controller should stabilize time-varying reference trajectories of hybrid systems. Despite the fact that discrete events (or jumps) in hybrid systems can often not be controlled directly, as, e.g., is the case in impacting mechanical systems, the controller should still stabilize the desired trajectory. A major complication in the analysis of this hybrid tracking problem is that, in general, the jump times of the plant do not coincide with those of the reference trajectory. Consequently, the conventional Euclidean tracking error does not converge to zero, even if trajectories converge to the reference trajectory in between jumps, and the jump times converge to those of the reference trajectory. Hence, standard control approaches can not be applied. We propose a novel definition of the tracking error that overcomes this problem and formulate Lyapunov-based conditions for the global asymptotic stability of the hybrid reference trajectory. Using these conditions, we design hysteresis-based controllers that solve the hybrid tracking problem for two exemplary systems, including the well-known bouncing ball problem.

**Index Terms**—Asymptotic stability, control system analysis, hybrid systems, tracking control.

## I. INTRODUCTION

**H**YBRID systems, such as for example robotic systems with impacts, digitally controlled physical systems, electrical circuits with switches, and models of chemical reactors with valves, can be characterized by the interaction between continuous-time dynamics and discrete events, cf. [1]–[3]. Due to this interaction, hybrid systems can show more complex behavior than can occur in ordinary differential equations (ODEs) or discrete-time systems. Consequently, conventional control approaches are often not directly applicable.

Most existing results in the literature on hybrid control systems deal with the stability of time-independent sets (especially with equilibrium points), such that the stability can be analyzed using Lyapunov functions, see, e.g., [1], [2], [4]–[10]. Essentially, such a set is asymptotically stable when a Lyapunov function decreases both during flow and jumps (i.e., discrete events), see, e.g., [1], [4], and [5]. Extensions of these results allow for Lyapunov functions that increase during jumps, as long as this increase is compensated by a larger decrease during flow, or

vice versa. Such results are reviewed in [7] and [11]. Using these Lyapunov-based stability results, several control strategies have been developed to stabilise time-independent sets for hybrid systems, see, e.g., [1], [9].

Few results exist where controllers are designed to make a system track a given, *time-varying*, reference trajectory, that exhibits both continuous-time behavior and jumps. In this paper, we consider reference trajectories that are solutions of the plant for a given, time-dependent reference input. When the jump times of the plant trajectories can be guaranteed to coincide with jumps of the reference trajectory, then stable behavior of the Euclidean tracking error is possible and several tracking problems have been solved in this setting, see, e.g., [12]–[16]. In [17], observer problems are considered for a class of hybrid systems where a similar condition is exploited, namely, that the jumps of the plant and the observer coincide. When jump times of the plant trajectory  $x$  and reference trajectory  $r$  can be ensured to coincide, standard Lyapunov tools are applicable to study the evolution of  $x - r$  along trajectories. However, requiring the jump times of plant and reference trajectories (or plant and observer) to coincide is a strong condition that limits the applicability of these results. For example, this can not be ensured for general hybrid systems with state-triggered jumps, such as models of mechanical systems with unilateral constraints, cf. [10], [18].

In hybrid systems with state-triggered jumps, the jump times of the plant and the reference trajectory are in general not coinciding. To illustrate this behavior, we consider the trajectories of a scalar hybrid system with state  $x \in [0, 1]$ , where the continuous-time evolution is given by

$$\dot{x} = 1 + u(t), \quad x \in [0, 1] \quad (1a)$$

where  $u$  is a bounded control input and jumps occur according to:

$$x^+ = 0, \quad x = 1. \quad (1b)$$

Now, consider the signal  $r = t \bmod 1$  as a reference trajectory, where  $\bmod$  denotes the modulus operator, and observe that  $r$  is the solution of (1) from the initial condition  $r_0 = 0$  with  $u(t) \equiv 0$ . Suppose that a control signal  $u$  is constructed such that a plant trajectory  $x$  tracks the reference trajectory  $r$  (in fact, such a controller will be designed in Section V-A), then we expect behavior as given in Fig. 1, where the state  $x$  and reference trajectory  $r$  converge to each other away from the jump times, and the jump times show a vanishing mismatch. During the time interval caused by this jump-time mismatch, the Euclidean error  $|x - r|$  is large, as shown in Fig. 1(b). Since this behavior also occurs for arbitrarily small initial errors  $|x - r|$ , the Euclidean

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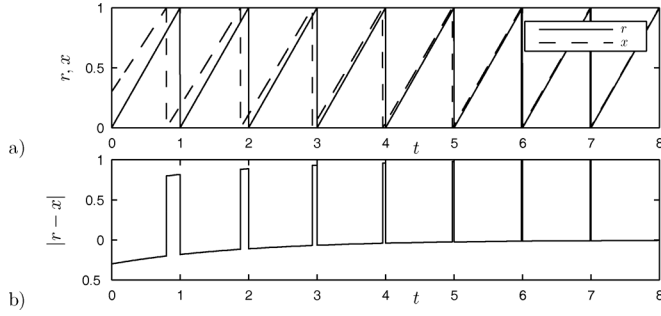


Fig. 1. (a) Exemplary trajectories of (1). (b) Euclidean tracking error  $|r - x|$ .

error displays unstable behavior in the sense of Lyapunov. This “peaking behavior” was observed in [10], [12], [13], [17], [19], and is expected to occur in all hybrid systems with state-triggered jumps when considering tracking or observer design problems. However, although the Euclidean error may display undesirable properties, from a control engineering point of view, the trajectories shown in Fig. 1 are considered to exhibit desirable behavior. Therefore, it seems that the evaluation of the tracking error using non-Euclidean distance functions might be advantageous for a class of hybrid systems, such as the example in (1). For this reason, we formulate a different notion of tracking in this paper, that considers the behavior shown in Fig. 1 as a proper solution, since the jump times of the plant converge to the jump times of the reference and the distance between the plant and reference trajectories converges to zero during time intervals without jumps. This tracking notion is less restrictive than notions requiring stability of the Euclidean error (cf. [12]–[15]), such that the class of hybrid systems that can be considered is widened significantly.

Several approaches have already been presented to formalize tracking notions where controllers that solve the resulting tracking problem are allowed to induce behavior as shown in Fig. 1. However, in these approaches it is not clear how to formulate conditions under which such tracking problems are solved. In [19], [20], the tracking of a billiard system is considered using the concept of “weak stability,” which implies that the position of the ball is always required to be close to the reference trajectory, but the error in velocity is not studied for a small time interval near the jump instances. In addition, the convergence of jump times is required. In [21], this approach is extended to a larger class of hybrid systems. However, since no requirements are imposed close to the jump times, such a tracking problem definition needs knowledge of complete trajectories. Alternatively, the notion of weak stability is employed in [22], [23] for unilaterally constrained mechanical systems with reference trajectories where all impacts, if they occur, show accumulation points (i.e., Zeno behavior), followed by a time interval where the constraint is active. In the very recent conference papers [24], [25], tracking control problems for billiard systems are formulated by requiring asymptotic stability of a set of trajectories, consisting of the reference trajectory and its mirror images, when reflected in the boundaries of the billiard. This independent research effort resulted in a related control problem formulation and controller design approach as those given for the bouncing ball example in Section V-B of the current paper. In this paper, we aim to present a general

framework for addressing tracking problems for a relatively generic class of hybrid systems (not focussing on a class of mechanical systems with unilateral constraints as in [24] and [25]). Alternatively, in [19], it is suggested to employ the stability concept of Zhukovsky (see [26]). Using this stability concept, the plant trajectory is compared with the reference trajectory after a rescaling of time for the plant trajectory, i.e., the error  $x(\rho(t)) - r(t)$  should behave asymptotically stable, where a function  $\rho(t)$  is used with  $\lim_{t \rightarrow \infty} (\rho(t) - t) = 0$ . As a second alternative, a Hausdorff-type metric between the graphs of the reference and plant trajectory is suggested in [27]. Both the rescaling function  $\rho$  for Zhukovsky stability and the Hausdorff-type metric require complete knowledge of the trajectories, and, consequently, it is not clear how these concepts can be used to solve the design problem of tracking controllers.

In order to study tracking problems with non-matching jump times, we propose an alternative approach using a non-Euclidean distance between the plant and reference states, where convergence of this distance measure corresponds to the desired notion of tracking. Since this distance measure incorporates information on the “closeness” of the reference state and plant state at each time instant, the tracking problem can be formulated based on the time evolution of the distance measure evaluated along trajectories of the closed-loop system. This fact is instrumental in our approach, as it allows us to derive sufficient conditions under which the tracking problem is solved, that are formulated using the instantaneous state, its time-derivatives, and the jumps that can occur. Since such information is encoded directly in the hybrid system description, this property is an advantage of our approach when compared to the analysis of [19]–[21], where convergence of jump times is proven using complete trajectories. In addition to this new formulation of the tracking problem for hybrid systems, we present sufficient conditions that guarantee that this problem is solved. In this manner, we will provide a general framework for the formulation and analysis of tracking problems for hybrid systems. Although we do not address the synthesis problems of tracking controllers in its full generality, we are convinced that the results of this paper provide an indispensable stepping stone towards such a synthesis procedure. In fact, the applicability of the presented framework for the design of tracking controllers will be demonstrated for two exemplary systems, including a mechanical system with a unilateral constraint.

The main contributions of this paper can be summarized as follows. First, the proposed reformulation of the tracking notion using a non-Euclidean tracking error measure allows to state and analyze tracking problems for a large class of hybrid systems, and these tracking problems are not rendered infeasible by the “peaking” of the Euclidean tracking error. Second, existing Lyapunov-type stability conditions, both with and without an additional average dwell-time condition, are extended to allow non-Euclidean distance functions, yielding sufficient conditions for the global asymptotic stability of time-invariant sets. This result allows to formulate conditions that ensure that the new tracking problem is solved. Third, in two examples we show that the new tracking error measure can be used to design controllers that solve the tracking problem.

This paper is organized as follows. In Section II, the hybrid system model and the corresponding solution concept are introduced. Subsequently, in Section III, requirements are formulated for the design of appropriate tracking error measures, and the tracking problem is formulated. Section IV contains Lyapunov-type conditions that are sufficient for the tracking problem to be solved. The results of this paper are illustrated with examples on controller synthesis for the hybrid tracking problem in Section V, and conclusions are formulated in Section VI.

### Notation

$\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space;  $\mathbb{R}$  the set of real numbers;  $\mathbb{R}_{\geq 0}$  the set of nonnegative real numbers;  $\mathbb{N}$  the set of natural numbers including 0. Let  $\text{Int}(S)$  denote the interior of a set  $S \subset \mathbb{R}^n$ ,  $\partial S$  the boundary of the set,  $\text{cl}(S)$  its closure,  $\text{co}(S)$  the smallest closed convex hull containing  $S$ , and  $\mu(S)$  denotes its Lebesgue measure. Let  $f : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  denote a set-valued mapping from  $\mathbb{R}^m$  to subsets of  $\mathbb{R}^n$ . Given vectors  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^m$ ,  $|x|$  denotes the Euclidean vector norm,  $\text{col}(x, y)$  denotes  $(x^T y^T)^T$  and  $\nabla_x$  denotes  $\partial/\partial x$ . A function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  is said to belong to class- $\mathcal{K}_\infty$  ( $\alpha \in \mathcal{K}_\infty$ ) if it is continuous, zero at zero, strictly increasing and unbounded. Given a Lipschitz function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\partial_C V$  denotes the generalized differential of Clarke, i.e.,  $\partial_C V = \text{co}(\lim \nabla_x V(x_i) | x_i \rightarrow x, x_i \notin \Omega_V)$ , where  $\Omega_V$  denotes the set of measure zero where  $\nabla_x V$  is not defined.

## II. MODELING OF HYBRID CONTROL SYSTEMS

In this paper, we employ the framework of hybrid inclusions described in [1], allowing the continuous-time dynamics (flow) to be time-dependent, such that the hybrid system is given by

$$\dot{x} \in F(t, x, u), \quad x \in C \subseteq \mathbb{R}^n \quad (2a)$$

$$x^+ \in G(x), \quad x \in D \subseteq \mathbb{R}^n \quad (2b)$$

with  $F : \mathbb{R}_{\geq 0} \times C \times \mathcal{U} \rightrightarrows \mathbb{R}^n$ ,  $G : D \rightrightarrows \mathbb{R}^n$ , where  $\dot{x} \in F(t, x, u)$  describes the continuous-time (flow) dynamics that is feasible when states  $x$  are in the flow set  $C \subseteq \mathbb{R}^n$ , and jumps can occur according to  $x^+ \in G(x)$  when states are in the jump set  $D \subseteq \mathbb{R}^n$ . The control inputs  $u \in \mathcal{U} \subset \mathbb{R}^m$  are assumed to be contained in the compact set  $\mathcal{U}$ . Moreover, only trajectories from initial conditions in  $C \cup D$  are considered. Note that the reset map  $G$  in (2b) is not dependent on time or the actuator inputs  $u$ , and models purely state-triggered jumps. We adopt the convention that  $G(x) = \emptyset$  when  $x \notin D$ .

The following technical assumption is imposed on the data of the hybrid system (2).

*Assumption 1:* For any bounded set  $C_1 \subset \text{cl}(C)$ , the set  $F(t, x, u)$  is non-empty, measurable and essentially bounded for all  $(t, x, u) \in \mathbb{R}_{\geq 0} \times C_1 \times \mathcal{U}$  and  $G(x) \subset C \cup D$  is non-empty and bounded for all  $x \in D$ .

In order to define solutions of the hybrid system (2), we assume that the input  $u$  satisfies  $u \in U(t, x) \subseteq \mathcal{U}$ , which allows, first, to evaluate solutions of the hybrid systems when the input is a time signal  $U(t, x) = U(t)$ , and second, to consider discontinuous, state-dependent control laws, e.g., sliding mode controllers. We consider solutions  $\varphi$  of the hybrid system (2) in

the sense of [1], such that  $\varphi$  is defined on a hybrid time domain  $\text{dom } \varphi \subset [0, \infty) \times \mathbb{N}$  as follows. A hybrid time instant is given as  $(t, j) \in \text{dom } \varphi$ , where  $t$  denotes the continuous time lapsed, and  $j$  denotes the number of experienced jumps. The arc  $\varphi$  is a solution of (2) associated to  $u$  when jumps satisfy (2b) and  $\varphi$  is a Filippov solution of (2a) during flow, cf. [28]. This implies  $\varphi(t, j+1) \in G(\varphi(t, j))$  for all  $(t, j) \in \text{dom } \varphi$  such that  $(t, j+1) \in \text{dom } \varphi$  and  $^1(d/dt)\varphi(t, j) \in \bar{F}(t, \varphi(t, j), U(t, \varphi(t, j)))$  for almost all  $t \in I_j := \{t | (t, j) \in \text{dom } \varphi\}$  and all  $j$  such that  $I_j$  has non-empty interior, where

$$\begin{aligned} \bar{F}(t, x, U(t, x)) \\ = \bigcap_{\delta \geq 0} \bigcap_{\mu(N)=0} \text{co}\{F(t, \tilde{x}, U(t, \tilde{x})), |\tilde{x} - x| \leq \delta, \tilde{x} \notin N\} \end{aligned}$$

represents the convexification of the vector field as defined by Filippov, where sets  $N$  of Lebesgue measure zero are excluded. The solution  $\varphi$  is said to be complete if  $\text{dom } \varphi$  is unbounded, which, for example, holds for all trajectories of (2) if  $C \cup D$  is invariant under the dynamics of (2). The hybrid time domain  $\text{dom } \varphi$  is called unbounded in  $t$ -direction when for each  $T \geq 0$  there exist a  $j$  such that  $(T, j) \in \text{dom } \varphi$ . In this paper, we only consider maximal solutions, i.e., solutions  $\varphi$  for which the domain  $\text{dom } \varphi$  can not be extended.

Analogous to the common approach in tracking control for ODEs, we consider reference trajectories  $r$  that are unique solutions to (2), i.e., solutions to  $\dot{r} \in F(t, r, u_{\text{ref}}(t))$ ,  $r \in C$ ,  $r^+ \in G(r)$ ,  $r \in D$ , for a given input signal  $u = u_{\text{ref}}(t) \in \mathcal{U}$  and initial condition  $r_0$ . We design a control law for  $u$  to obtain asymptotic tracking, in an appropriate sense, of the reference trajectory  $r$  by the resulting closed-loop plant. We consider feedback controllers that are static, where  $u = u_c(t, r, x) \in \mathcal{U}$ , or dynamic, where the (possibly hybrid) controller has an internal state  $\eta \in \mathbb{R}^p$  and is described by

$$\begin{aligned} \dot{\eta} &\in F_c(t, r, x, \eta), & (r, x, \eta) &\in C_c \\ \eta^+ &\in G_c(r, x, \eta), & (r, x, \eta) &\in D_c \\ u &= u_c(t, r, x, \eta) \end{aligned} \quad (3)$$

and assume that this controller satisfies Assumption 1. In order to study the stability of the closed-loop system, we create an extended hybrid system with state  $q = \text{col}(r, x, \eta)$ . The dynamics of this extended hybrid system is given by

$$\dot{q} \in F_e(t, q), \quad q \in C_e := (C^2 \times \mathbb{R}^p) \cap C_c \quad (4a)$$

$$q^+ \in G_e(q) = \begin{cases} \text{col}(G(r), x, \eta) \\ q \in (D \times (C \cup D) \times \mathbb{R}^p) \cap (C_c \cup D_c) \\ \text{col}(r, G(x), \eta) \\ q \in ((C \cup D) \times D \times \mathbb{R}^p) \cap (C_c \cup D_c) \\ \text{col}(r, x, G_c(r, x, \eta)) \\ q \in D_c \end{cases} \quad (4b)$$

where

$$\begin{aligned} F_e(t, q) \\ := \text{col}(F(t, r, u_{\text{ref}}(t)), F(t, x, u_c(t, r, x, \eta)), F_c(t, r, x, \eta)) \end{aligned}$$

<sup>1</sup>We employ, with a slight abuse of notation, the convention that  $\bar{F}(t, \varphi(t, j), U)$  denotes  $\{f \in \mathbb{R}^n | f = \bar{F}(t, \varphi(t, j), u), u \in U\}$ .

and we denote the domain of  $G_e$  in (4b) with  $D_e$ . We refer to  $F_e, G_e, C_e$  and  $D_e$  as the data of system (4).

The main advantage of considering this extended hybrid system (4) is that a joint hybrid time domain is created, where hybrid times  $(t, j) \in \text{dom } q$  denote the continuous time  $t$  lapsed, and  $j$  gives the total number of jumps that occurred in  $x, r$  and  $\eta$ . Hence, one can compare the reference state  $r$  with the plant state  $x$  for each time instant  $(t, j) \in \text{dom } q$ . Let  $I_{i,i}$  denote the unit matrix of dimension  $i \times i$ , and let  $O_{i,j}$  be a zero matrix of dimension  $i \times j$ . Defining

$$\begin{aligned}\bar{r}(t, j) &:= (I_{n,n} \ O_{n,n} \ O_{n,p}) \ q(t, j) \\ \bar{x}(t, j) &:= (O_{n,n} \ I_{n,n} \ O_{n,p}) \ q(t, j) \\ \bar{\eta}(t, j) &:= (O_{p,n} \ O_{p,n} \ I_{p,p}) \ q(t, j)\end{aligned}\quad (5)$$

allows to introduce a tracking error  $d(r, x)$  and formulate a tracking problem by requiring, first, that  $d(\bar{r}(t, j), \bar{x}(t, j))$  remains small provided that the initial error  $d(\bar{r}(0, 0), \bar{x}(0, 0))$  is small, and second, that  $d(\bar{r}(t, j), \bar{x}(t, j))$  converges to zero for  $t + j \rightarrow \infty, (t, j) \in \text{dom } q$ , provided that this limit exists. In fact, we will formulate the tracking problem by requiring asymptotically stable behavior of  $d(\bar{r}(t, j), \bar{x}(t, j))$ , and, in two exemplary systems, we will design controllers that guarantee that the domain  $\text{dom } q$  is unbounded in  $t$ -direction, such that the limit  $t \rightarrow \infty$  exists and, for example, no accumulation of jump times (known as Zeno behavior, cf. [1]) will occur.

*Remark 1:* If two distinct trajectories  $x$  and  $r$  from (2) are considered, then, in general,  $\text{dom } r \neq \text{dom } x$ . In this case, if one would define a time-dependent tracking error at time  $(t, j) \in \text{dom } x$ , then it would not be clear what time  $(t_r, j_r) \in \text{dom } r$  is appropriate to use in a comparison of  $x$  and  $r$ . Such problems are avoided by studying the extended dynamics in (4), where the functions  $\bar{x} : \text{dom } q \rightarrow C \cup D$ ,  $\bar{r} : \text{dom } q \rightarrow C \cup D$  and  $\bar{\eta} : \text{dom } q \rightarrow \mathbb{R}^p$  are reparameterizations of the functions  $x : \text{dom } x \rightarrow C \cup D, r : \text{dom } r \rightarrow C \cup D$  and  $\eta : \text{dom } \eta \rightarrow \mathbb{R}^p$ , respectively.

### III. TRACKING CONTROL PROBLEM FORMULATION FOR HYBRID SYSTEMS

In Section III-A, we introduce distance functions  $d(r, x)$  suitable to compare the plant trajectory with the reference trajectory, such that asymptotically stable behavior of  $d(\bar{r}(t, j), \bar{x}(t, j))$  corresponds to appropriate tracking. Subsequently, in Section III-B we introduce asymptotic stability with respect to the tracking error  $d(r, x)$ , and formalize the hybrid tracking problem.

#### Definition of the Tracking Error Measure

In hybrid systems where jumps of the plant are state-triggered, as in (2), asymptotically stable behavior of the Euclidean error  $|x - r|$  is generally impossible to achieve due to the peaking phenomenon, see Fig. 1, which, even when  $x$  and  $r$  converge to each other away from the jump instances, occurs when jump times of the reference and plant trajectories show a small, possibly asymptotically vanishing, mismatch. To illustrate this in the exemplary system (1), observe that if  $|x - r|$  converges to zero during continuous-time evolution and jumps are not exactly coinciding, then, by the structure of the jump map (1b),

$|x - r| \approx 1$  directly after a jump of either  $x$  or  $r$ . Since this peaking phenomenon renders all tracking problems infeasible that require stable behavior of the Euclidean error  $|x - r|$ , in this paper, we present a novel approach to compare trajectories of the plant with a reference trajectory. For hybrid systems with state-triggered jumps given in (2), we will show that the exact properties of the jumps can be used to compare a reference trajectory with a plant trajectory, when one of them just experienced a jump, and the other did not. A distance function between two trajectories that enables such a comparison incorporates the structure of the jumps, as described in (2b), and hence is tailored to the specific hybrid system. In this paper, we employ distance functions, denoted as  $d(r, x)$ , to formulate and solve the tracking problem.

We consider distance functions  $d(r, x)$  that are not sensitive to jumps of the plant and the reference trajectory, i.e.,  $d(r, x) = d(g_r, x)$  for  $r \in D, g_r \in G(r)$  and  $d(r, x) = d(r, g_x)$  for  $x \in D, g_x \in G(x)$ . In this manner, stability with respect to the distance function  $d(r, x)$  is not influenced by the jumps of the plant or the reference trajectory. As we will show below, a distance function  $d(r, x)$  is an appropriate measure to compare a reference trajectory  $r$  with a plant trajectory  $x$  when it satisfies the following conditions. We adopt the notation  $G^0(x) = \{x\}$ ,  $\forall x \in C \cup D$  and  $G^{k+1}(x) = G(G^k(x))$ ,  $k = 0, 1, 2, \dots$ . Recall that  $G(x) = \emptyset$  when  $x \notin D$ .

*Definition 1:* Consider a hybrid system  $\mathcal{H}$  given by (2) that satisfies Assumption 1. A nonnegative function  $d : \text{cl}((C \cup D) \times (C \cup D)) \rightarrow \mathbb{R}_{\geq 0}$  is called a distance function compatible with  $\mathcal{H}$  when it is continuous and satisfies

$$\begin{aligned}d(r, x) = 0 &\Leftrightarrow (\exists k_1, k_2 \in \mathbb{N}, \text{ such that } G^{k_1}(x) \cap G^{k_2}(r) \neq \emptyset) \\ &\quad \forall (r, x) \in \text{cl}(C \cup D)^2\end{aligned}\quad (6a)$$

$$\begin{aligned}\{x \in C \cup D \mid d(r, x) < \beta\} &\text{ is bounded} \\ \forall r \in C \cup D, \beta &\geq 0\end{aligned}\quad (6b)$$

$$d(r, x) = d(r, g_x), \quad \forall x \in D, g_x \in G(x), r \in C \cup D \quad (6c)$$

$$d(r, x) = d(g_r, x), \quad \forall r \in D, g_r \in G(r), x \in C \cup D. \quad (6d)$$

In this paper, we will study stability of the set where  $d(r, x)$  is zero for system (4). Using (6a), in this set,  $(\exists k_1, k_2 \in \mathbb{N}, \text{ such that } G^{k_1}(x) \cap G^{k_2}(r) \neq \emptyset)$  holds true, such that the distance  $d(r, x)$  is zero if and only if either  $r = x$  (such that the right-hand side of the implication in (6a) holds with  $k_1 = k_2 = 0$ ), or  $x$  and  $r$  can be mapped onto each other instantaneously by  $k_1$  jumps of  $x$  and  $k_2$  jumps of  $r$ , and, hence,  $G^{k_1}(x) \cap G^{k_2}(r) \neq \emptyset$ . For example, if jumps of (2b) cannot directly follow each other, i.e., when  $D \cap G(D) = \emptyset$ , and  $G$  is invertible, then (6a) becomes  $d(r, x) = 0 \Leftrightarrow (x = r \vee x = G(r) \vee r = G(x))$ ,  $\forall (r, x) \in \text{cl}(C \cup D)^2$ .

The condition (6b) implies that, for every given  $r$ ,  $d(r, x)$  is radially unbounded. This property will be instrumental to prove that convergence of  $d$  to zero implies convergence of  $|x - r|$  to zero, away from the jump instances, i.e., when  $r$  is not close to the sets  $D$  or  $G(D)$ .

Finally, (6c)-(6d) guarantee that  $d(r, x)$  remains constant over jumps, such that the evaluation of the function  $d$  along a trajectory of a closed-loop system (4), i.e., the

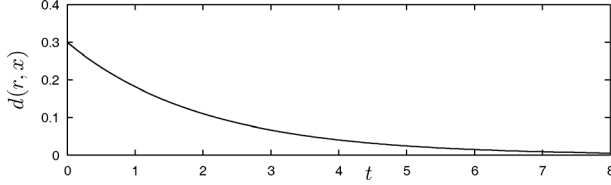


Fig. 2. Tracking error  $d(r, x) = \min(|x - r|, |x - r + 1|, |x - r - 1|)$  evaluated along trajectories of (1).

function  $d(\bar{r}(t, j), \bar{x}(t, j))$ , is a continuous function with respect to  $t$ , and is not affected when  $j$  changes. Consequently,  $d(\bar{r}(t, j), \bar{x}(t, j))$  does not show the “peaking behavior,” as occurs in the Euclidean distance  $|\bar{r}(t, j) - \bar{x}(t, j)|$ . This property is illustrated in Fig. 2, where the function  $d(r, x) = \min(|x - r|, |x - r + 1|, |x - r - 1|)$  is shown when evaluated along the trajectories depicted in Fig. 1. In Section V-A, we will show that this tracking error definition for  $d(r, x)$  is indeed compatible with the hybrid system (1) in the sense of Definition 1.

In this paper, we will formulate a tracking problem that requires asymptotic convergence of  $d(\bar{r}(t, j), \bar{x}(t, j))$  to zero along trajectories. The following theorem states that such a convergence property of  $d(\bar{r}(t, j), \bar{x}(t, j))$  implies, at least for the time instances where  $\bar{r}(t, j)$  is bounded away from the jump set  $D$  and its image  $G(D)$ , that  $|\bar{r}(t, j) - \bar{x}(t, j)|$  converges to zero.

**Theorem 1:** *Let the distance  $d(\bar{r}(t, j), \bar{x}(t, j))$  converge asymptotically to zero along solutions  $q$  of the closed-loop system (4), i.e.,  $\lim_{t+j \rightarrow \infty} d(\bar{r}(t, j), \bar{x}(t, j)) = 0$ , let these solutions be complete, let  $\bar{r}(t, j)$  be bounded for all  $(t, j) \in \text{dom } q$  and let  $d$  be compatible with (2). For each trajectory  $q$  and all  $\epsilon > 0$ , there exists a  $T > 0$  such that  $|\bar{x}(t, j) - \bar{r}(t, j)| < \epsilon$  holds when  $t + j > T$  and*

$$\inf_{y \in D \cup G(D)} |\bar{r}(t, j) - y| > \epsilon. \quad (7)$$

*Proof:* See Appendix A. ■

We note that in Section V, we present examples of hybrid systems and reference trajectories where the time intervals where (7) is violated can be made arbitrarily small by selecting  $\epsilon > 0$  sufficiently small. Consequently, in these cases, asymptotic convergence of  $d(\bar{r}(t, j), \bar{x}(t, j))$  to zero implies that the time intervals where “peaking” may occur (i.e., the time intervals where the Euclidean distance  $|x - r|$  can be large), become shorter over time. We are convinced that more general (sufficient) conditions can be formulated that guarantee that the time intervals where (7) is violated can be made arbitrarily small by selecting  $\epsilon > 0$  sufficiently small, such that these conditions additionally guarantee that the time intervals where the Euclidean error may display “peaking behavior” vanish asymptotically over time.

Inspired by these observations, in the following section, we will formulate the tracking problem by requiring asymptotically stable behavior of  $d(\bar{r}(t, j), \bar{x}(t, j))$ .

#### A. Tracking Problem Formulation

In this section, we discuss the stability of reference trajectories, and restrict our attention to bounded reference trajectories  $r$  that satisfy the following assumption.

**Assumption 2:** *The reference trajectory  $r$  is bounded,  $\text{dom } r$  is unbounded in  $t$ -direction, and  $r$  is the unique solution of (2) for an input  $u = u_{\text{ref}}(t)$  and given initial condition  $r(0, 0) = r_0$ .*

Below, we formulate the tracking problem which requires all trajectories of (4) to be such that  $d(r, x)$  behaves asymptotically stable. Hereto, we combine the definitions of (asymptotic) stability of trajectories, cf. [26], [29], with existing stability notions for hybrid systems, cf. [1], and employ distance functions  $d(r, x)$  compatible with system (2) to express the distance between  $r$  and  $x$ , as introduced in Definition 1. To create a common hybrid time domain, as mentioned before, we consider solutions of the embedded system (4), such that  $\bar{x}$  and  $\bar{r}$  are both defined on the hybrid time domain  $\text{dom } q$  of trajectories of (4), where  $\bar{x}$  and  $\bar{r}$  are defined in (5). Let us now formalize the stability properties of the reference trajectory.

**Definition 2:** *Given a distance function  $d : (C \cup D) \times (C \cup D) \rightarrow \mathbb{R}_{\geq 0}$  compatible with system (2), a reference trajectory  $r$  satisfying Assumption 2 is*

- *stable with respect to  $d$  if for all  $T_0 \geq 0$  and  $\epsilon > 0$  there exists a  $\delta(T_0, \epsilon) > 0$  such that*

$$\begin{aligned} d(\bar{r}(0, 0), \bar{x}(0, 0)) < \delta(T_0, \epsilon) \Rightarrow \\ d(\bar{r}(t, j), \bar{x}(t, j)) < \epsilon, \quad \forall t + j \geq T_0; \end{aligned} \quad (8)$$

- *asymptotically stable with respect to  $d$  if it is stable and one can choose  $\delta > 0$  such that*

$$\begin{aligned} d(\bar{r}(0, 0), \bar{x}(0, 0)) < \delta \Rightarrow \\ \lim_{t+j \rightarrow \infty, (t,j) \in \text{dom } q} d(\bar{r}(t, j), \bar{x}(t, j)) = 0 \end{aligned} \quad (9)$$

*holds if  $\text{dom } q$  is unbounded;*

- *globally asymptotically stable with respect to  $d$  if it is stable with respect to  $d$  and*

$$\lim_{t+j \rightarrow \infty, (t,j) \in \text{dom } q} d(\bar{r}(t, j), \bar{x}(t, j)) = 0 \quad (10)$$

*holds for all trajectories  $q = \text{col}(r, x, \eta)$  of (4) such that  $\text{dom } q$  is unbounded.*

As a special case of this definition, the (global) asymptotic stability of an equilibrium point  $r_{\text{eq}}$  with respect to  $d$  can be evaluated by using  $r(t, j) = r_{\text{eq}}$ . If the Euclidean distance  $d(r, x) = |x - r|$  would be used, then the given definition reduces to the classical definition of asymptotic stability of trajectories in the sense of Lyapunov, see, e.g., [26].

Using Definition 2, we formalize the tracking problem as follows.

**Problem 1 [(Global) Tracking Problem]:** *Given a hybrid system (2) satisfying Assumption 1, a compatible distance function  $d$  and a reference trajectory  $r$  satisfying Assumption 2, design a controller (3) such that the trajectory  $r$  is (globally) asymptotically stable with respect to  $d$ .*

This tracking problem is not affected by the peaking phenomenon of the Euclidean error, as depicted in Fig. 1(b), since the trajectory  $x$  of the plant is compared with the reference trajectory using a distance function *compatible with* (2), as given in Definition 1. As stated in Theorem 1, convergence of  $d$  to zero implies that, away from the jump instances,  $|x - r|$  converges

to zero. By formulation of the tracking problem using the distance  $d(r, x)$ , the tracking problem in Problem 1 embeds a more intuitive and less restrictive notion of the closeness of jumping trajectories. Note that the tracking problem defined here considers a controller (3) that does not induce jumps of the plant state  $x$  directly, cf. (2b).

*Remark 2: The tracking problem given in Problem 1 only considers asymptotically stable behavior of the tracking error  $d(r, x)$ . However, transient performance requirements of the closed-loop system in terms of decay rates can also be formulated using the distance function  $d(r, x)$  given in Definition 1.*

*Remark 3: Problem 1 implies that  $d(r, x) = 0$  has to be an invariant set for the closed-loop system (2), (3), and for this reason, it requires  $u_c(t, r, x, \eta) = u_{\text{ref}}(t)$  a.e. when  $d(r, x) = 0$  and  $r, x \notin D \cup G(D)$ . Although, for example, PI-controllers naturally do not have this property, a relevant class of static and dynamic hybrid controllers can be considered, including hysteresis-based controllers, of which an example will be given in Section V-A.*

#### IV. SUFFICIENT CONDITIONS FOR STABILITY WITH RESPECT TO $d$

In this section, we will use Lyapunov-like functions to study the behavior of  $d$  and to analyze whether a given controller (3) solves the tracking problem formulated in Problem 1. First, we will show that the solutions of the closed-loop system can be considered as Filippov solutions during continuous-time evolution (Section IV-A). Subsequently, in Section IV-B we present two theorems with Lyapunov-type stability conditions for the stability of a set, and apply these results to obtain sufficient conditions under which the tracking problem formulated in Problem 1 is solved.

##### A. Closed-Loop Solutions

By construction of the extended hybrid system (4), if we adopt Assumption 1 for the plant (2) and controller (3), then the following property directly follows for the extended hybrid system (4).

*1) Property 1: For any bounded set  $C_1 \subset \text{cl}(C_e)$ , the set  $F_e(t, q)$  is non-empty, measurable and essentially bounded for all  $(t, q) \in \mathbb{R}_{\geq 0} \times C_1$ , and  $G_e(q)$  is non-empty and bounded for all  $q \in D_e$ .*

Note, in particular, that this property implies that the solution concept of Filippov can be applied over those segments of the hybrid trajectories of (4) where flow occurs. As noted before, Filippov's solutions are defined using the convexification of the vector field of (4a):

$$\bar{F}_e(t, q) := \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \{\text{co}(F_e(t, \tilde{q})), |q - \tilde{q}| \leq \delta, \tilde{q} \notin N\} \quad (11)$$

such that  $\bar{F}_e(t, q)$  is non-empty, bounded, closed, and convex for all  $t$  and all  $q$  from a bounded set, and upper semi-continuous in  $q$ , as shown in [28, p. 85]. Hence, trajectories satisfy  $(d/dt)\varphi(t, j) \in \bar{F}_e(t, \varphi(t, j))$  for almost all  $t$  and fixed  $j$ .

##### B. Lyapunov-Type Stability Conditions

In order to formulate Lyapunov-type conditions for the tracking problem given in Problem 1, we first present conditions for the asymptotic stability of the set  $\{q = \text{col}(r, x, \eta) \in C_e \cup D_e \mid \rho(q) = 0\}$  for a continuous function  $\rho : C_e \cup D_e \rightarrow \mathbb{R}_{\geq 0}$ . The considered stability properties of this set for the dynamics (4), using  $\rho(q) = d(r, x)$ , directly imply asymptotic stability of the reference trajectory  $r$  for the closed-loop trajectory of the hybrid system (2), (3). Analogously to Definition 2, the set  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$  is said to be stable with respect to a continuous function  $\rho$ , when, for each  $T_0, \epsilon > 0$ , there exists a  $\delta(T_0, \epsilon) > 0$  such that  $\rho(q(t, j)) < \epsilon$ ,  $\forall (t, j) \in \text{dom } q$ ,  $t + j > T_0$ , holds for all trajectories  $q(t, j)$  of system (4) with  $\rho(q(0, 0)) < \delta(T_0, \epsilon)$ . The set is asymptotically stable with respect to the function  $\rho$  when, in addition, there exist a  $\delta > 0$  such that, for all complete solutions  $q$  with initial conditions  $\rho(q(0, 0)) < \delta$ ,  $\rho(q(t, j))$  converges to zero for  $t + j \rightarrow \infty$ . We first formulate a basic Lyapunov-function-based result guaranteeing asymptotic stability with respect to  $\rho$  (Theorem 2). In contrast to existing results on stability of sets, see, e.g., [1], firstly, non-Euclidean errors measures are used, and second, stability of unbounded sets is considered. Furthermore, we present Theorem 3 which allows the increase of the Lyapunov function over jumps, as long as this increase is compensated for by a larger decrease over continuous-time evolution, a characteristic which we will employ in Section V to prove stability of a reference trajectory when a hysteresis-based controller is used.

Recall that Property 1 holds naturally for the system (4) when both the plant (2) and controller (3) satisfy Assumption 1, and that the data of (4) is designed to model the closed-loop dynamics, as discussed in Section II. For this reason, we will now proceed as follows. In Theorems 2 and 3, we present sufficient conditions for the stability of the set  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$  with respect to  $\rho$  for hybrid systems (4) with Property 1. Subsequently, these conditions are used in Theorem 4 to present conditions that guarantee that the tracking problem presented in Problem 1 is solved. Throughout this paper, Lyapunov functions are considered that are Lipschitz functions, which, in addition, are regular as defined in Definition 2.3.4 in [30]. Recall that, without any further reference, we will only consider solutions which are maximal.

*Theorem 2: Consider the hybrid system (4), and let Property 1 hold. In addition, suppose there exist a continuous function  $\rho : \text{cl}(C_e \cup D_e) \rightarrow \mathbb{R}_{\geq 0}$ , a regular and Lipschitz function  $V : \text{cl}(C_e \cup D_e) \rightarrow \mathbb{R}_{\geq 0}$ , functions  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$  and scalar  $c < 0$ , such that*

$$\alpha_1(\rho(q)) \leq V(q) \leq \alpha_2(\rho(q)), \quad \forall q \in \text{cl}(C_e \cup D_e) \quad (12a)$$

$$\max_{\zeta \in \partial_C V, f \in \bar{F}_e(t, q)} \langle \zeta, f \rangle \leq cV(q), \quad \forall q \in \text{cl}(C_e) \quad (12b)$$

$$V(g) \leq V(q), \quad \forall g \in G_e(q), \forall q \in D_e \quad (12c)$$

*with  $\bar{F}_e$  given in (11), then  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$  is a stable set of (4) with respect to  $\rho$ . If, in addition, for all solutions  $\varphi$  of (4) with initial conditions in  $C_e \cup D_e$ ,  $\text{dom } \varphi$  is unbounded in*

*t*-direction, then the set  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$  is globally asymptotically stable with respect to  $\rho$ .

*Proof:* See Appendix B. ■

Theorem 2 is valuable for the design of controllers solving the tracking problem formulated in Problem 1. For example, if static state-feedback controllers are employed, where  $u = u_c(t, r, x)$ , and a Lyapunov function  $V$  is selected, then (12b) directly gives sufficient conditions for the closed-loop differential inclusion (4a) after substitution of the control law. In Section V-B, a bouncing-ball system is presented as an example, where a static control law  $u = u_c(t, r, x)$  is designed in this manner.

For various hybrid control systems, along discrete events (i.e., jumps), the Lyapunov function might increase (i.e., might not satisfy condition (12c) of Theorem 2), while this increase is compensated for by a larger decrease of the Lyapunov function over a sufficiently long continuous-time period without jumps. An exemplary system with this behavior will be given in Section V-A, where the increase of the Lyapunov function is induced by the switches of a hysteresis-based controller. To study the behavior of such systems, in Theorem 3 below, we present sufficient conditions for stability of trajectories that have an average inter-jump time of at least  $\tau$  time units, as defined in the following definition. This definition is formulated by adapting the average dwell time conditions of [11] to the hybrid system framework, allowing trajectories that have a finite number of subsequent jumps at the same continuous-time instant  $t$ .

**Definition 3:** We say that a trajectory  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$  for  $\tau, \kappa > 0$ , if for all  $(t, j) \in \text{dom } \varphi$  and all  $(T, J) \in \text{dom } \varphi$  where  $T + J \geq t + j$ , the relation  $J - j \leq \kappa + (T - t)/\tau$  holds.

Note that  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, 1)$  implies that the trajectory satisfies a minimal inter-jump time restriction of  $\tau$  time units. However, the state of the extended hybrid system (4) is designed such that it embeds the plant state  $x$  and reference state  $r$ . Hence, if the jump times of plant converge to the jump times of the reference trajectory, as shown, e.g., in Fig. 1, then the extended hybrid system (4) directly violates such a dwell time restriction  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, 1)$ . For this reason, we will focus on trajectories satisfying the less restrictive condition  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ , with  $\kappa > 1$ . For the example presented in Section V-A, we will show that explicit expressions for  $\tau$  and  $\kappa$  can be derived, such that trajectories  $\varphi$  of the extended hybrid system (4) indeed satisfy  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ .

We present sufficient conditions for global asymptotic stability of the set  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$  with respect to the distance  $\rho$  in the following theorem, where we require all trajectories  $\varphi$  of a system (4) to satisfy  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ .

**Theorem 3:** Let trajectories  $\varphi$  of (4) from all initial conditions in  $C_e \cup D_e$  satisfy  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ , for some  $\tau, \kappa > 0$ , and let (4) satisfy Property 1. In addition, suppose there exist a continuous function  $\rho : \text{cl}(C_e \cup D_e) \rightarrow \mathbb{R}_{\geq 0}$ , a regular and Lipschitz function  $V : \text{cl}(C_e \cup D_e) \rightarrow \mathbb{R}_{\geq 0}$ , functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and scalars  $c < 0$ ,  $\mu \geq 1$  such that

$$\alpha_1(\rho(q)) \leq V(q) \leq \alpha_2(\rho(q)), \quad \forall q \in \text{cl}(C_e \cup D_e) \quad (13a)$$

$$\max_{\zeta \in \partial_C V, f \in \bar{F}_e(t, q)} \langle \zeta, f \rangle \leq cV(q), \quad \forall q \in \text{cl}(C_e) \quad (13b)$$

$$V(g) \leq \mu V(q), \quad \forall g \in G_e(q), \forall q \in D_e \quad (13c)$$

with  $\bar{F}_e$  given in (11), then  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$  is a stable set of (4) with respect to  $\rho$  if

$$\mu e^{c\tau} < 1 \quad (13d)$$

holds. If, in addition,  $\text{dom } \varphi$  is unbounded in *t*-direction, then  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$  is globally asymptotically stable with respect to  $\rho$ .

*Proof:* See Appendix C. ■

Note that in the proof of this theorem we evaluate an upper bound of the Lyapunov function for all  $t$ . For this reason, in contrast to the approach in [21], our analysis does not require periodicity of reference trajectories.

Using the previously presented Theorems 2 and 3, we now formulate sufficient conditions under which the tracking control problem, i.e., Problem 1, is solved. We note that, for a relevant class of hybrid systems (2), controllers can be designed such that the closed-loop system trajectories  $\varphi$ , which are modelled by the extended hybrid system (4), satisfy  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ , with  $\tau, \kappa > 0$ , where the values for  $\tau$  and  $\kappa$  follow directly from the plant dynamics (2) and the controller (3). In Section V-A, we present an example where the derivation of  $\tau$  and  $\kappa$  is shown.

**Theorem 4:** Consider the global tracking problem for hybrid system (2) with compatible tracking error  $d$  (according to Definition 1) and reference trajectory  $r$  satisfying Assumption 2, let the controller (3) be given, let (2) and (3) satisfy Assumption 1, and let the hybrid time domain  $\text{dom } \varphi$  of any trajectory  $\varphi$  of system (4) be unbounded in *t*-direction. Suppose there exist a regular and Lipschitz function  $V : \text{cl}(C_e \cup D_e) \rightarrow \mathbb{R}_{\geq 0}$ , functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and scalars  $c < 0$ ,  $\mu \geq 1$  such that:

$$\alpha_1(d(r, x)) \leq V(q) \leq \alpha_2(d(r, x)), \quad \forall q \in \text{cl}(C_e \cup D_e) \quad (14a)$$

$$\max_{\zeta \in \partial_C V, f \in \bar{F}_e(t, q)} \langle \zeta, f \rangle \leq cV(q), \quad \forall q \in \text{cl}(C_e) \quad (14b)$$

$$V(g) \leq \mu V(q), \quad \forall g \in G_e(q), \forall q \in D_e \quad (14c)$$

with  $\bar{F}_e$  given in (11), such that  $q = \text{col}(r, x, \eta)$  is the state of the extended hybrid system (4) with data  $F_e, G_e, C_e, D_e$ . If one of the following conditions hold:

- (i) the expression (14c) holds with  $\mu = 1$ , or
- (ii) there exist  $\tau, \kappa > 0$  such that all solutions of (4) satisfy  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ , and

$$\mu e^{c\tau} < 1 \quad (14d)$$

then the global tracking problem (given in Problem 1) is solved.

*Proof:* See Appendix D. ■

## V. TRACKING CONTROLLER DESIGN FOR EXEMPLARY SYSTEMS

In this section, two examples are given where a controller is designed to solve the tracking problem formulated in Problem 1. For these two examples, we show that the design of a distance function compatible with the exemplary hybrid system not only allows to formulate the tracking problem, but in addition, can be used to design a controller solving the tracking problem.

After designing the tracking error measure  $d$  and the controller (3), for the examples presented in the following sections,

we employ Theorem 4 to show that the closed-loop system renders the reference trajectory asymptotically stable.

#### A. Global Tracking for a Scalar Hybrid System

In this section, we consider the tracking problem for the scalar hybrid system (1) with a reference trajectory  $r$ , that is the solution to (1) for  $u(t) = u_{\text{ref}}(t) = (1/2)\cos(t)$  and initial condition  $r(0, 0) = 0.95 \in C \cup D$ . Since  $1 + u_{\text{ref}}(t) > 0$  for all  $t$ , the continuous-time dynamics, described by (1a), yields trajectories that can only leave  $C$  by arriving at  $D$  and thus experience a jump. Integrating (1a) over time  $t$ , we obtain  $v(t) = r(0, 0) + t + (1/2)\sin(t)$ , such that we can write the reference trajectory  $r$  as

$$r(t, j) = v(t) - j \quad (15)$$

with  $(t, j) \in \text{dom } r = \text{cl}(\{(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \mid j = \lfloor v(t) \rfloor\})$ , where  $\lfloor a \rfloor$  denotes the largest integer such that  $\lfloor a \rfloor < a$ . Any jump of  $r$  will induce  $r = 0$ , such that subsequently, flow is possible. A next jump can only occur when  $r$  has increased to 1, which takes at least  $1/\max_t(1 + u_{\text{ref}}(t)) = 2/3$  time units. For this reason, the time domain  $\text{dom } r$  is unbounded in  $t$ -direction and the reference trajectory  $r$  satisfies Assumption 2.

To evaluate the tracking error between a plant trajectory  $x$  and the reference trajectory  $r$ , we employ the distance function

$$d(r, x) = \min(|x - r|, |x - r + 1|, |x - r - 1|). \quad (16)$$

To show that this function satisfies the conditions of Definition 1, first, we show that (6a) holds. We observe that for the reset map (1b),  $G^2(x) = \emptyset$  for all  $x$ ; hence,  $G^{k_1}(x) \cap G^{k_2}(r) \neq \emptyset$  can only hold for  $k_1, k_2 \in \{0, 1\}$ . As a consequence,  $(\exists k_1, k_2 \in \mathbb{N}, \text{ such that } G^{k_1}(x) \cap G^{k_2}(r) \neq \emptyset)$  is equivalent to  $(\{x\} = \{r\} \vee \{x\} = G(r) \vee G(x) = \{r\} \vee G(x) = G(r))$ . Considering the jump map (1b), we observe that  $x^+ \in G(x) = \{0\}$  for  $x = 1$ , such that, equivalently, we can write  $G(s) = \{s - 1\}$ . Substituting this function in the foregoing logical expression yields  $(x - r = 0 \vee x - r + 1 = 0 \vee x - r - 1 = 0)$ , which is equivalent to  $d(r, x) = \min(|x - r|, |x - r + 1|, |x - r - 1|) = 0$ , such that indeed (6a) holds.

The requirement formulated in (6b) holds since  $C \cup D$  is bounded. For  $x \in D = \{1\}$  and arbitrary  $r \in C \cup D = [0, 1]$ , we observe that  $d(r, x) = d(r, 1) = \min(|1 - r|, |r|)$ , and since  $G(x) = \emptyset$  for  $x \in D$ , we also obtain  $d(r, G(x)) = d(r, \emptyset) = \min(|-r|, |1 - r|)$ , such that  $d(r, x) = d(r, G(x))$  holds, as required in (6c). An analogous argument shows that (6d) holds. As continuity of  $d$  is obvious, this shows that the conditions in Definition 1 are satisfied and the distance function  $d(r, x)$  in (16) is compatible with the hybrid system (1).

Now, we will design a controller that solves the global tracking problem as formulated in Problem 1 using the distance function  $d(r, x)$  given in (16). Since during flow, solutions are considered in the sense of Filippov, static controllers  $u = u_c(t, r, x)$  cannot solve the global tracking problem, as can be observed intuitively as follows. Let  $r$  represent an analog 12-hours clock, such that  $r = 0$  corresponds to 0h00 or 12h00, let  $x$  denote a different pointer that should track the hour hand of the clock, such that the jumps in  $x$  and  $r$  of plant

(1) correspond to the 12-hour time jump of the clock. If  $x$  is just behind  $r$ , asymptotically stable behavior requires “speeding up” the pointer to converge to the reference time  $r$ . However, if  $x$  is just ahead of  $r$ ,  $x$  should run slower than the reference hour hand, and thus slow down and “wait” for the reference trajectory  $r$ . However, at some point, say, if the difference between  $r$  and  $x$  is 6 hours,  $x$  should arbitrarily decide to “wait for  $r$ ,” or “accelerate to catch up with  $r$ .” If the evolution of  $x$  is described by a Filippov solution, then such a decision is not possible at all points: the Filippov solution will always have a trajectory, where  $\dot{x} = \dot{r}$ , that corresponds to an anti-phase synchronized state, meaning that the difference between  $r$  and  $x$  remains 6 hours.

Essentially, this behavior is induced by the fact that Brockett’s necessary conditions for stabilizability hold both for smooth differential equations and Filippov systems, cf. e.g., [31]. Since the observed phenomenon can be translated directly to system (1), global tracking is not feasible using a static controller  $u = u(t, r, x)$ . As observed e.g., in [32], one approach to avoid this problem, which we will employ in this paper, is to introduce a hysteresis-based controller with discrete variable  $\eta \in \{-1, 0, 1\}$  that will ensure the global tracking problem to be solved.

The rationale behind the controller design is as follows. First, we observe that the distance (16) is given by the minimum of three functions  $\rho_1$ ,  $\rho_2$  and  $\rho_3$ , where  $\rho_1(r, x) = |x - r|$ ,  $\rho_2(r, x) = |x - r + 1|$ , and  $\rho_3(r, x) = |x - r - 1|$ . For each of these functions, a controller  $u_i$ ,  $i = 1, 2, 3$  is designed that enforces converging behavior of  $\rho_i$  to zero during flow, i.e., controls  $x - r$ , or  $x - r + 1$  or  $x - r - 1$  to zero. Subsequently, the function

$$\tilde{V}(r, x, \eta) = \frac{1}{2}(x - r + \eta)^2 \quad (17)$$

is used to determine in which part of the state space, which of the three control inputs  $u_i$ ,  $i = 1, 2, 3$ , is applied. We design the updates of the hysteretic state  $\eta \in \{-1, 0, 1\}$  such that  $\tilde{V}(r, x, \eta)$  becomes zero if either  $x - r - 1 = 0$ ,  $x - r = 0$  or  $x - r + 1 = 0$ . For this purpose, updates of  $\eta$  are triggered by a violation of  $\tilde{V}(r, x, \eta) \leq (\mu/2)d(r, x)^2$ , with  $\mu > 1$ , which may occur when the plant or reference trajectory experiences a jump. At these time instances,  $\eta$  is reset to ensure  $\tilde{V}(r, x, \eta) \leq (\mu/2)d(r, x)^2$  again. The parameter  $\mu > 1$  determines the hysteretic domain. Using this reasoning, the following hysteresis-based controller is designed:

$$\dot{\eta} = 0, \quad (r, x, \eta) \in C_c := \left\{ (r, x, \eta) \mid \tilde{V}(r, x, \eta) \leq \frac{\mu}{2}d(r, x)^2 \right\} \quad (18a)$$

$$\eta^+ = \arg \min_{i \in \{-1, 0, 1\}} \tilde{V}(r, x, i), \quad (r, x, \eta) \in D_c \quad (18b)$$

with

$$D_c := \left\{ (r, x, \eta) \mid \tilde{V}(r, x, \eta) = \frac{\mu}{2}d(r, x)^2 \right\} \\ \cup \left\{ (r, x, \eta) \mid x = 0, \tilde{V}(r, 0, \eta) \geq \frac{\mu}{2}d(r, 0)^2 \right\} \\ \cup \left\{ (r, x, \eta) \mid r = 0, \tilde{V}(0, x, \eta) \geq \frac{\mu}{2}d(0, x)^2 \right\}$$



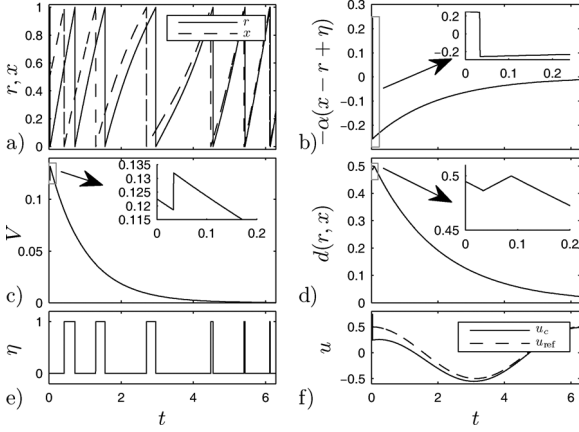


Fig. 3. Simulation results for system (1) with controller (18), where  $u_{\text{ref}}(t) = 0.5 \cos(t)$  and reference trajectory  $r$  is the solution from initial condition  $r(0, 0) = 0.95$ . The plant trajectory emanates from initial condition  $x(0, 0) = 0.455$ . (a) Reference and plant trajectory. (b) Feedback action. (c) Lyapunov function evaluated along the trajectory. (d) Error measure  $d(r, x)$ . (e) Hysteretic variable  $\eta$ . (f) Control input  $u_c(t, r, x, \eta)$ . Details near  $t = 0$  are shown in the insets of panels (b-d).

where  $(r, x, \eta) \in [0, 1]^2 \times \{-1, 0, 1\}$  is used implicitly. The resets of  $\eta$  are designed to ensure that a violation of  $\dot{V}(r, x, \eta) \leq (\mu/2)d(r, x)^2$  is directly corrected by a reset in  $\eta$ , and, as we will show in the proof of Theorem 5, such jumps only occur directly after jumps of  $x$  or  $r$ . We set  $\mu = 1.125$  and design the controller output as

$$u_c(t, r, x, \eta) = u_{\text{ref}}(t) - \alpha(x - r + \eta) \quad (18c)$$

where we select  $\alpha = 1/2$ , which is a parameter that determines the convergence rate of the closed-loop tracking error  $d$ .

The following theorem proves that this controller solves the global tracking problem given in Problem 1 for the reference trajectory  $r$ , by employing case (ii) of Theorem 4.

**Theorem 5:** *Given the hybrid system (1) with reference trajectory (15) and distance function (16), the hybrid controller (18) solves the global tracking problem stated in Problem 1.*

*Proof:* See Appendix E. ■

In Fig. 3, simulation results are shown that illustrate the controller for a reference trajectory  $r$  with initial condition  $r(0, 0) = 0.95$ . The plant shows intuitively correct behavior and converges to the reference away from the jump instances, as predicted by Theorem 1. Note that the introduced hysteresis causes the Lyapunov function  $V$  to increase at the first jump of the reference trajectory. The Lyapunov function decreases monotonically to zero during flow, and, after the first jump, is not affected by jumps of the hybrid system. After the first hysteretic reset, the control feedback action  $-\alpha(x - r + \eta)$ , i.e.,  $u_c - u_{\text{ref}}(t)$ , is always negative, such that the plant trajectory “waits” for the reference trajectory, as explained using the analogy between system (1) and a clock hand. The tracking error evaluated in  $d(r, x)$ , depicted in Fig. 3(d), does not display the “peaking” of the Euclidean tracking error of these trajectories, shown in Fig. 1.

### B. Tracking Control for the Bouncing Ball

We consider the bouncing trajectories of a ball on a table, see Fig. 4(a), as an elementary though representative model in

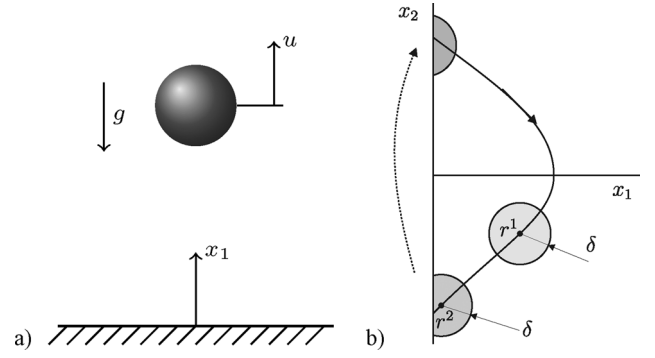


Fig. 4. (a) Bouncing ball system (21). (b) Sets  $\{x \in C \cup D \mid d(r^i, x) < \delta\}$  for two points  $r^1$  and  $r^2$ , where  $d$  is given in (22).

the class of hybrid models for mechanical systems with impacts. Assuming that non-impulsive forces  $u$  can be applied on the ball with unit mass, the flow of the system is described by

$$\dot{x} = \text{col}(x_2, -g + u + \lambda(x_1, x_2)), \quad x_1 \geq 0 \quad (19a)$$

where  $x = \text{col}(x_1, x_2)$  contains the vertical position  $x_1$  and velocity  $x_2$  of the ball, respectively,  $g$  is the gravitational acceleration,  $u$  is a force that can be applied to the system, and the contact force  $\lambda$  between the ball and the table, with  $\lambda(x_1, x_2) = 0$ , for  $x_1 > 0$ , and  $\lambda(x_1, x_2) \in [0, \infty)$ , for  $x_1 = 0$ , avoids penetration of the table by the ball, cf. [18].

Motion according to (19a) is only possible when the distance  $x_1$  between the table and the ball is nonnegative. If the ball arrives at the surface  $x_1 = 0$ , then a Newton-type impact law with restitution coefficient equal to one is assumed, modelled as

$$x^+ = \text{col}(x_1, -x_2), \quad x_1 = 0 \text{ and } x_2 \leq 0. \quad (19b)$$

We consider the following reference trajectory:

$$r(t) = \text{col}\left(\tau - \frac{g}{2}\tau^2, 1 - g\tau\right), \quad \tau = t \bmod \frac{2}{g} \quad (20)$$

where  $\bmod$  denotes the modulus operator. This trajectory is a solution to (19) for initial conditions  $r(0, 0) = \text{col}(0, 1)$  and  $u_{\text{ref}} \equiv 0$ . In this example, we focus on the local tracking problem given in Problem 1 and design a static control law  $u = u_d(t, r, x)$  such that the reference trajectory  $r$  is asymptotically stabilized for the closed-loop plant dynamics. For trajectories near this reference trajectory the contact force  $\lambda$  vanishes for almost all  $t$ , such that the trajectories are described by the system

$$\dot{x} = F(t, x, u) := \text{col}(x_2, -g + u), \quad x \in C := [0, \infty) \times \mathbb{R} \quad (21a)$$

$$x^+ = G(x) := \text{col}(x_1, -x_2), \quad x \in D := \{0\} \times (-\infty, 0]. \quad (21b)$$

In order to define a tracking error measure  $d(r, x)$ , we use the property that the velocity  $x_2$  changes sign at impacts, and the position  $x_1$  is zero, see (21b). Hence, if we want to compare a reference state  $r$  with plant state  $x$  when one of them just experienced a jump, then the distance  $|x + r|$  is appropriate.

Away from jump instances, typically, the conventional distance  $|x - r|$  can be used. To distinguish when the distance  $|x - r|$  or  $|x + r|$  should be considered, we use the minimum of both, such that the novel distance measure is given by

$$d(r, x) = \min(|x - r|, |x + r|). \quad (22)$$

To recognize that this distance function satisfies the conditions posed in Definition 1, first, note that  $G(D) \cap D = \{0\}$ , such that  $G^k(x) = \emptyset, \forall x \in C \cup D \setminus \{0\}$  when  $k \geq 2$ , implying that  $(\exists k_1, k_2 \in \mathbb{N}, \text{ such that } G^{k_1}(x) \cap G^{k_2}(r) \neq \emptyset)$  is equivalent to  $(x = r \vee \{x\} = G(r) \vee G(x) = \{r\})$ . Since  $G(x) = \{-x\}$  for  $x \in D$ , the condition  $(x = r \vee \{x\} = G(r) \vee G(x) = \{r\})$  can be rewritten to  $x - r = 0 \vee x + r = 0$ , such that  $d(r, x) = 0$  directly follows and relation (6a) is satisfied. As required in (6b), for given  $r, \beta$ , the set  $\{x \in C \cup D \mid d(r, x) < \beta\}$  is compact, as it is the union of the bounded sets  $\{x \in C \cup D \mid |x - r| < \beta\}$  and  $\{x \in C \cup D \mid |x + r| < \beta\}$ . Since  $G(x) = -x$  for  $x \in D$ , relation (6c) holds, as  $d(r, G(x)) = d(r, -x) = \min(|r - x|, |r + x|)$ , which equals  $d(r, x)$ . An analogue argument shows that (6d) holds. Since, in addition,  $d$  is continuous, the tracking error measure  $d(r, x)$  is compatible with the hybrid system (19).

In Fig. 4(b), the neighborhoods  $\{x \in C \cup D \mid d(r^i, x) < \delta\}$  of two different points  $r^i, i = 1, 2$ , are shown. Essentially, the tracking error measure  $d$  allows to compare a reference trajectory with a plant trajectory, “as if” both of them already jumped. For example, in Fig. 4(b), the gray domain with positive  $x_2$  is considered close to  $r^2$ , since  $r^2$  will experience a jump soon, and after this jump,  $r^2$  will arrive in this domain.

We design a tracking control law  $u = u_d(t, r, x)$  for system (21) using a reasoning that exploits the design of the tracking error measure  $d$  in (22). Analogously to the design approach in the previous section, observe that  $d(r, x)$  in (22) is given by the minimum between the two functions,  $\rho_1 = |x - r|$  and  $\rho_2 = |x + r|$ . When the trajectory  $x$  is sufficiently close to  $r$  and neither of them experiences a jump in the near future or past, then the tracking error  $d(r, x)$  given in (22) is given by  $\rho_1 = |x - r|$ . Along solutions of the differential (19a), this error could accurately be controlled towards zero using a controller with PD-type feedback, given by

$$u_1 = -[k_p \ k_d](x - r) \quad (23)$$

where  $k_p, k_d > 0$ . Implementation of this controller yields the error dynamics  $\ddot{x}_1 - \ddot{r}_1 = -k_p(x_1 - r_1) - k_d(\dot{x}_1 - \dot{r}_1)$ , such that  $\text{col}(x_1 - r_1, \dot{x}_1 - \dot{r}_1) = \text{col}(0, 0)$  is an asymptotically stable equilibrium point of the flow dynamics with  $u$  given in (23).

However, if either the reference trajectory or the plant just experienced a jump,  $d(r, x)$  as given in (22) is given by  $\rho_2 = |x + r|$ . The continuous-time behavior of  $x + r$  is stable when the closed-loop dynamics satisfy  $\ddot{x}_1 + \ddot{r}_1 = -k_p(x_1 + r_1) - k_d(\dot{x}_1 + \dot{r}_1)$ , which is obtained by selecting the controller as

$$u_2 = 2g - [k_p \ k_d](x + r) \quad (24)$$

with  $k_p, k_d > 0$ . Based on these insights, we propose a controller that switches between (23) and (24). To choose the partitioning of the state space where either (23) or (24) are ap-

plied, the following candidate Lyapunov function  $V(r, x)$  is considered:

$$V(r, x) = \min(V_d(r, x), V_m(r, x)) \quad (25)$$

with

$$V_d(r, x) = \frac{1}{2}(x - r)^T P (x - r), \quad V_m(r, x) = \frac{1}{2}(x + r)^T P (x + r) \quad (26)$$

where a symmetric, positive definite matrix  $P$  and scalar  $c > 0$  are chosen such that

$$A^T P + P A \prec -cP \quad (27)$$

with  $A = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}$ , where for symmetric and real matrices  $M, N$ , we adopt the notation  $M \prec 0$  when  $M$  is negative definite, and  $M \prec N$  when  $M - N \prec 0$ . For strictly positive  $k_p, k_d$ , the matrix  $A$  is Hurwitz, which implies that such  $P$  and  $c$  exist, since [33][Theorem 4.6] implies that for any symmetric, positive definite matrix  $Q$  there exist a symmetric, positive definite  $P$  such that  $A^T P + P A \prec -Q$ , and  $c > 0$  can be chosen sufficiently small, such that  $-Q \prec -cP$ . Based on the Lyapunov function candidate  $V$ , the following control law  $u = u_d(t, r, x)$  is designed, such that  $\rho_1$  or  $\rho_2$  decreases along continuous-time solutions described by (19a):

$$u_d(t, r, x) = \begin{cases} -[k_p \ k_d](x - r), & V_d \leq V_m \\ 2g - [k_p \ k_d](x + r), & V_d > V_m. \end{cases} \quad (28)$$

In the next theorem we show that the control law (28) indeed solves the tracking problem formulated in Problem 1.

**Theorem 6:** Consider the bouncing ball system (21), tracking error  $d$  given in (22), and reference trajectory  $r$  given in (20) for  $u_{\text{ref}}(t) \equiv 0$ . Application of the control law  $u_d(t, r, x)$  as defined in (28), with  $V_d, V_m$  given in (26) and  $k_p, k_d > 0$ , to the hybrid system (21) makes the reference trajectory  $r$  asymptotically stable with respect to  $d$ . In addition, the set  $\{x \in C \cup D \mid V(r(0, 0), x) \leq K\}$  is contained in the basin of attraction of  $r$ , where  $K$  is chosen to satisfy

$$K < \min_{(t, j) \in \text{dom } r} V(r(t, j), 0). \quad (29)$$

*Proof:* See Appendix F. ■

The control law  $u_d(t, r, x)$  given in (26), (28) with  $[k_p \ k_d] = [1 \ 0.5]$  and  $P = \begin{bmatrix} 2.25 & 0.5 \\ 0.5 & 2 \end{bmatrix}$  is applied to system (21) with  $g = 9.81$ . The trajectory from the initial condition  $x(0, 0) = \text{col}(4, (3g)/4)$  is shown with the dashed line in Fig. 5(a)-(b). Clearly, the hybrid trajectory  $x$  converges to  $r$  during flow, and the jump instances of  $r$  and  $x$  converge to each other. The Euclidean distance  $|x - r|$  and the distance  $d(r, x)$  between both trajectories are shown in Fig. 5(c) and (d), respectively. Although the Euclidean distance displays the unstable “peaking” behavior, the tracking error expressed using the distance  $d(r, x)$  remains continuous over trajectories, and converges to zero. Hence, the *local* tracking problem as formulated in Problem 1 is solved, and the trajectories shown in Fig. 5(a). show desirable tracking behavior. Indeed, as predicted by Theorem 1,  $|x - r|$  decreases to zero for  $r$  away from  $D \cup G(D)$ , where  $r_1 = 0$ .

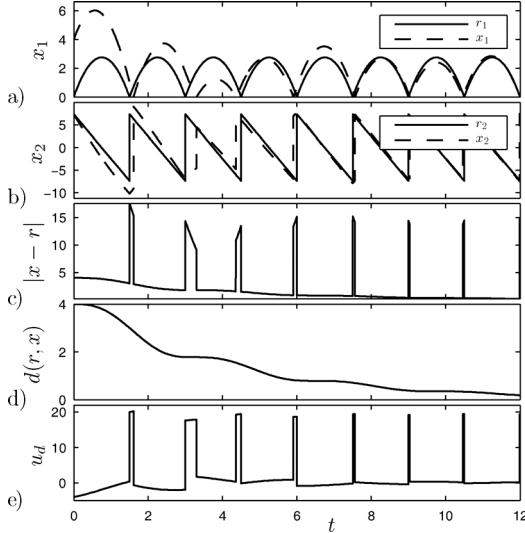


Fig. 5. (a), (b) Reference trajectory  $r$  and plant trajectory  $x$  of (21), where  $g = 9.81$  and the control law (28) is applied with  $[k_p, k_d] = [1, 0.5]$ . (c) Tracking error expressed in Euclidean distance  $|x - r|$ , (d) distance function  $d(r, x)$  given in (22) and (e) control input  $u_d(t, r, x)$ .

Although the controller designs for the examples in Sections V-A and V-B have been tailored to the specific examples, we care to highlight that we employed a common rationale for the controller design. First, the tracking error measure  $d(r, x)$  is designed as the minimum of functions  $\rho_i$ ,  $i = 1, 2, 3$  corresponding to  $|x - r|$ ,  $|x - \tilde{G}(r)|$ ,  $|\tilde{G}(r) - x|$ , where the functions  $\tilde{G}$  are designed such that  $\tilde{G}(x) = G(x)$  for  $x \in D$ . (For the bouncing ball example given in (21), the functions  $\rho_2$  and  $\rho_3$  coincide due to the structure of  $G$  and  $D$ .) Subsequently, for each function  $\rho_i$ ,  $i = 1, 2, 3$ , a control law  $u_i$  is designed that would stabilize the set where  $\rho_i = 0$  along flowing solutions. Finally, a Lyapunov candidate function is employed to determine which control law  $u_i$  is applied in what part of the state space.

## VI. CONCLUSION

In this paper, the problem of (global) tracking of time-dependent reference trajectories is studied for hybrid systems with state-triggered jumps. We formulated the tracking problem in such a way that it corresponds to the intuitive notion of tracking for hybrid systems: the plant trajectories tend asymptotically to the reference trajectory, such that away from the time instances where the reference trajectory jumps, the Euclidean tracking error becomes small. To formalize this notion of tracking, the tracking error is evaluated using a novel, non-Euclidean distance measure. It is shown in this paper that such distance functions have three advantages. First, it facilitates the formulation of a tracking problem that is feasible for a large class of hybrid system, including mechanical systems with impacts, and does not require the jumps of the plant to coincide with the jumps of the reference trajectory. Second, the formulated tracking problem can be analyzed by evaluating Lyapunov functions along closed-loop trajectories, and is feasible for a large class of reference trajectories, which are not required to be periodic. Third, as shown in the examples, the new tracking error measure can be used to design controllers

solving the tracking problem. Using exemplary systems, including the well-known bouncing ball system, we illustrate that the tracking problem is feasible for hybrid systems with state-triggered jumps and that the presented results support the design of tracking controllers for such hybrid systems.

Further research should be directed to the development of a synthesis procedure for generic hybrid systems that leads to, first, a tracking error measure that is tailored to the hybrid system under study and, second, a control law that solves the tracking problem formulated in this paper. The requirements on the distance measure and the stability analysis presented in this paper form important stepping stones towards such a generic synthesis procedure.

## APPENDIX

*Proof of Theorem 1:* Consider an arbitrary trajectory  $q(t, j)$  of system (4) satisfying the hypotheses of the theorem and define  $\bar{x}, \bar{r}$  as in (5). Select  $R > 0$  such that  $|\bar{r}(t, j)| \leq R$ ,  $\forall(t, j)$ .

The requirement (6a) implies that, given  $|r| \leq R$ , the non-negative function  $d(r, x)$  is zero if and only if

$$(r, x) \in \mathcal{A} := \{\text{col}(r, x) \in \text{cl}(C \cup D)^2 \mid \exists k_1, k_2 \in \mathbb{N}, G^{k_1}(x) \cap G^{k_2}(r) \neq \emptyset, |r| \leq R\}.$$

The set  $\mathcal{A}$  is closed, since  $\mathcal{A} = \{\text{col}(r, x) \in \text{cl}(C \cup D)^2 \mid d(r, x) = 0, |r| \leq R\}$  due to (6a), and  $d$  is continuous. Boundedness of  $\mathcal{A}$  follows from (6b), such that  $\mathcal{A}$  is compact. Let  $\rho_{\mathcal{A}}(p) := \inf_{y \in \mathcal{A}} |p - y|$  give the distance to the set  $\mathcal{A}$ . We will construct a  $\mathcal{K}_{\infty}$ -function  $\alpha$  such that

$$\alpha(\rho_{\mathcal{A}}(\text{col}(r, x))) \leq d(r, x). \quad (30)$$

For this purpose, analogous to the proof of Lemma 4.3 in [33], let  $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be given as:

$$\phi(s) := \inf_{\text{col}(r, x) \in \text{cl}(C \cup D)^2, |r| \leq R, \rho_{\mathcal{A}}(\text{col}(r, x)) \geq s} d(r, x) \quad (31)$$

which is non-decreasing. Since (6b) is satisfied, we observe that  $\phi(0) = 0$ ,  $\phi(s) > 0$  for  $s > 0$  and  $\phi(s) \rightarrow \infty$  for  $s \rightarrow \infty$ . Hence, we can construct a strictly increasing and continuous function  $\alpha$  such that  $\alpha(s) \leq \phi(s)$  and  $\alpha \in \mathcal{K}_{\infty}$ . Since, in addition, (31) implies  $\phi(\rho_{\mathcal{A}}(\text{col}(r, x))) \leq d(r, x)$ , we observe that (30) is satisfied.

By assumption,  $d(r, x)$  converges to zero along the closed-loop trajectory  $q(t, j) = \text{col}(\bar{r}(t, j), \bar{x}(t, j), \bar{\eta}(t, j))$ . Hence, for all  $\epsilon > 0$  there exists a time  $T > 0$  such that

$$d(\bar{r}(t, j), \bar{x}(t, j)) \leq \alpha(\epsilon), \quad \forall(t, j) \in \text{dom } q, t + j > T. \quad (32)$$

The  $\mathcal{K}_{\infty}$ -function  $\alpha$  is invertible, such that (32) and (30) imply

$$\rho_{\mathcal{A}}(\text{col}(\bar{r}(t, j), \bar{x}(t, j))) \leq \epsilon, \quad \forall(t, j) \in \text{dom } q, t + j > T. \quad (33)$$

We will conclude the proof of this theorem by proving that

$$\rho_{\mathcal{A}}(\text{col}(\bar{r}(t, j), \bar{x}(t, j))) \leq \epsilon \quad (34)$$

implies  $|\bar{r}(t, j) - \bar{x}(t, j)| \leq \epsilon$  when (7) holds. We can rewrite  $\rho_A$  as:

$$\rho_A(\text{col}(r, x)) = \inf_{\text{col}(p_r, p_x) \in \mathcal{A}} \left| \begin{pmatrix} r - p_r \\ x - p_x \end{pmatrix} \right| \quad (35)$$

$$= \inf_{\substack{p_r, p_x \in \text{cl}(C \cup D), k_1, k_2 \in \mathbb{N} \\ G^{k_1}(p_r) \cap G^{k_2}(p_x) \neq \emptyset, |p_r| \leq R}} \left| \begin{pmatrix} r - p_r \\ x - p_x \end{pmatrix} \right| \quad (36)$$

$$= \inf_{k_1, k_2 \in \mathbb{N}} \rho_{k_1 k_2}(r, x), \quad (37)$$

with  $\rho_{k_1 k_2}(r, x) := \inf_{\substack{p_r, p_x \in \text{cl}(C \cup D), |p_r| \leq R \\ G^{k_1}(p_r) \cap G^{k_2}(p_x) \neq \emptyset}} \left| \begin{pmatrix} r - p_r \\ x - p_x \end{pmatrix} \right|$ .

Since we adopted the convention that  $G(x) = \emptyset$  when  $x \notin D$ , for  $k_1 \geq 1$ ,  $G^{k_1}(p_r) \neq \emptyset$  can only hold when  $p_r \in D$ . Since, in addition,  $\left| \begin{pmatrix} r - p_r \\ x - p_x \end{pmatrix} \right| \geq |r - p_r|$ , with (7) we observe that  $\rho_{k_1 k_2}(\bar{r}(t, j), \bar{x}(t, j)) > \epsilon$  when  $k_1 \geq 1$ . Hence, (7) and (34) imply that the infimum of (37) is attained with  $k_1 = 0$ .

For  $k_1 = 0$  and  $k_2 \geq 1$ ,  $G^{k_1}(p_r) \cap G^{k_2}(p_x) \neq \emptyset$  implies  $G^{k_1}(p_r) = \{p_r\} \subseteq G^{k_2}(p_x) \subseteq G(D)$ , such that (7) implies  $|\bar{r}(t, j) - p_r| > \epsilon$ . Hence,  $\rho_{k_1 k_2}(\bar{r}(t, j), \bar{x}(t, j)) > \epsilon$ , when  $k_1 = 0, k_2 \geq 1$ . Consequently, the combination of (7) and (34) implies that the infimum in (37) is attained with  $k_1 = k_2 = 0$ , yielding

$$\begin{aligned} \rho_A(\text{col}(\bar{r}(t, j), \bar{x}(t, j))) &= \inf_{\substack{p_r \in \text{cl}(C \cup D), |p_r| \leq R \\ p_x = p_r}} \left| \begin{pmatrix} \bar{r}(t, j) - p_r \\ \bar{x}(t, j) - p_x \end{pmatrix} \right| \\ &= |\bar{r}(t, j) - \bar{x}(t, j)|. \end{aligned} \quad (38)$$

With (33), we conclude that  $|\bar{r}(t, j) - \bar{x}(t, j)| \leq \epsilon$  holds for all  $(t, j) \in \text{dom } q, t + j > T$  where (7) is satisfied, proving the theorem. ■

*Proof of Theorem 2:* To prove stability of the set  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$ , consider the Lyapunov function  $V$ , evaluated along a trajectory  $\varphi$  of (4), i.e.,  $V(\varphi(t, j))$ . Inequality (12c) implies that  $V$  is non-increasing over jumps and, according to Theorem 2.2 of [34], (12b) guarantees

$$\frac{d}{dt} V(\varphi(t, j)) \leq cV(\varphi(t, j)), \quad \text{for almost all } (t, j) \in \text{dom } \varphi. \quad (39)$$

Integration of this expression with respect to time  $t$  and using (12c) yields

$$V(\varphi(t, j)) \leq e^{ct} V(\varphi(0, 0)), \quad \forall (t, j) \in \text{dom } \varphi \quad (40)$$

such that the bounds in (12a) imply

$$\rho(\varphi(t, j)) \leq \alpha_1^{-1}(e^{ct} \alpha_2(\rho(\varphi(0, 0)))), \quad \forall (t, j) \in \text{dom } \varphi \quad (41)$$

which, together with  $c < 0$ , directly implies  $\rho(\varphi(t, j)) \leq \alpha_1^{-1}(\alpha_2(\rho(\varphi(0, 0))))$ ,  $\forall (t, j) \in \text{dom } \varphi$ , proving stability with respect to  $\rho$ . If, additionally,  $\text{dom } \varphi$  is unbounded in  $t$ -direction, then asymptotic convergence with respect to  $\rho$  is obtained. Since this argument holds for all trajectories  $\varphi$ , global asymptotic stability with respect to  $\rho$  is obtained, thereby proving the theorem. ■

*Proof of Theorem 3:* We consider the Lyapunov function  $V$ , evaluated along a trajectory  $\varphi$  of (4), i.e.,  $V(\varphi(t, j))$ , to

prove stability of the set  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$ . Let  $s_0 = 0$ , and let  $s_j, j = 1, 2, \dots$  denote the continuous time  $t$  when  $\varphi$  experiences the  $j$ -th jump, that is, where  $(s_j, j)$  and  $(s_j, j-1) \in \text{dom } \varphi$ . According to Theorem 2.2 of [34], (13b) guarantees

$$\frac{d}{dt} V(\varphi(t, j)) \leq cV(\varphi(t, j)), \quad \text{for almost all } (t, j) \in \text{dom } \varphi. \quad (42)$$

Integration of this expression with respect to time  $t$  over the time intervals  $[s_j, s_{j+1}] \times j$ , with  $s_j \neq s_{j+1}$ , yields  $V(\varphi(s_{j+1}, j)) \leq e^{c(s_{j+1}-s_j)} V(\varphi(s_j, j))$ ,  $\forall (t, j) \in \text{dom } \varphi$ , which holds trivially when  $s_j = s_{j+1}$ . Using this expression and (13c), we obtain

$$V(\varphi(t, j)) \leq \mu^j e^{ct} V(\varphi(0, 0)) \quad (43)$$

Substituting  $j < \kappa + t/\tau$ , which follows from  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ , and using  $\mu \geq 1$ , we obtain  $V(\varphi(t, j)) \leq \mu^\kappa \mu^{t/\tau} e^{ct} V(\varphi(0, 0))$ ,  $\forall (t, j) \in \text{dom } \varphi$ . Using  $\mu^{t/\tau} e^{ct} = ((\mu e^{c\tau})^{1/\tau})^t$ , we find  $V(\varphi(t, j)) \leq \mu^\kappa ((\mu e^{c\tau})^{1/\tau})^t V(\varphi(0, 0))$ ,  $\forall (t, j) \in \text{dom } \varphi$ . Combining (13d) and  $\tau > 0$ , we observe that  $(\mu e^{c\tau})^{1/\tau} < 1$ , such that the previous expression implies that  $V$  converges to zero along trajectories. Using (13a) yields

$$\rho(\varphi(t, j)) \leq \alpha_1^{-1} \left( \mu^\kappa \left( (\mu e^{c\tau})^{1/\tau} \right)^t \alpha_2(\rho(\varphi(0, 0))) \right), \quad \forall (t, j) \in \text{dom } \varphi \quad (44)$$

proving convergence of  $\rho$  to zero. Since  $\varphi$  is chosen arbitrarily, globally asymptotically stability of  $\{q \in C_e \cup D_e \mid \rho(q) = 0\}$  with respect to  $\rho$  is obtained, proving the theorem.

*Proof of Theorem 4:* Let  $V, \alpha_1, \alpha_2, c, \mu, r, u_{\text{ref}}(t)$  be as given in the formulation of the theorem and Assumption 2. If, additionally, requirement (i) or (ii) is satisfied, then application of Theorem 2 or 3, respectively, implies that the set  $\{q = \text{col}(r, x, \eta) \in C_e \cup D_e \mid d(r, x) = 0\}$  is globally asymptotically stable with respect to  $d(r, x)$  for the dynamics (4). If we select  $q(0, 0) = \text{col}(r(0, 0), x(0, 0), \eta(0, 0))$ , then we observe that  $\bar{r}(t, j)$ , as defined in (5), represents the reference trajectory  $r$  on the hybrid time domain  $\text{dom } q$ , since this reference trajectory satisfies Assumption 2 and is the unique solution of the hybrid system from initial condition  $r(0, 0)$ . Additionally, we observe that  $\bar{x}(t, j)$ , as defined in (5), represents the closed-loop plant trajectory from the initial condition  $x(0, 0)$ . Since the set  $\{q = \text{col}(r, x, \eta) \in C_e \cup D_e \mid d(r, x) = 0\}$  is globally asymptotically stable with respect to  $d(r, x)$ , the trajectories  $\bar{r}$  and  $\bar{x}$  satisfy all the conditions imposed in Definition 1 for any initial condition of the plant trajectory, i.e., for any  $x(0, 0) \in C \cup D$ . Hence, the reference trajectory  $r$  is a globally asymptotically stable trajectory of the closed-loop plant dynamics and the tracking problem of Problem 1 is solved. ■

*Proof of Theorem 5:* In this proof, we will show that Theorem 4(ii) is applicable, by firstly, showing that (14a)–(14c) holds for the following Lyapunov function:

$$\begin{aligned} V(r, x, \eta) &= \left\{ \begin{aligned} &\frac{1}{2}(x - r + \eta)^2, & \frac{1}{2}(x - r + \eta)^2 &\leq \frac{\mu}{2}d(r, x)^2 \\ &\frac{1}{2}d(r, x)^2, & \frac{1}{2}(x - r + \eta)^2 &\geq \frac{\mu}{2}d(r, x)^2. \end{aligned} \right. \end{aligned} \quad (45)$$

Subsequently, we prove that the closed-loop trajectories are unbounded in  $t$ -direction, and finally, we derive expressions for  $\tau, \kappa$  that satisfy condition (ii) of Theorem 4.

First, we consider the extended hybrid system (4), where, in this case,

$$F_e(t, q) = \text{col}(1 + u_{\text{ref}}(t), 1 + u_{\text{ref}}(t) - \alpha(x - r + \eta), 0) \quad (46)$$

$$G_e(q) = \begin{cases} \text{col}(0, x, \eta), & \text{for } \text{col}(r, x, \eta) \in C_e \cup D_e, r = 1 \\ \text{col}(r, 0, \eta), & \text{for } \text{col}(r, x, \eta) \in C_e \cup D_e, x = 1 \\ \text{col}(r, x, \arg \min_{i \in \{-1, 0, 1\}} \tilde{V}(r, x, i)), & \text{for} \\ \text{col}(r, x, \eta) \in C_e \cup D_e, \left( \tilde{V}(r, x, \eta) = \frac{\mu}{2} d(r, x)^2 \vee \right. \\ \left. \left( \tilde{V}(r, x, \eta) \geq \frac{\mu}{2} d(r, x)^2 \wedge rx = 0 \right) \right), \end{cases} \quad (47)$$

$$C_e = \left\{ \text{col}(r, x, \eta) \in [0, 1]^2 \times \{-1, 0, 1\} \mid \tilde{V}(r, x, \eta) \leq \frac{\mu}{2} d(r, x)^2 \right\} \quad (48)$$

$$D_e = \left\{ \text{col}(r, x, \eta) \in [0, 1]^2 \times \{-1, 0, 1\} \mid \tilde{V}(r, x, \eta) \right. \\ \left. = \frac{\mu}{2} d(r, x)^2 \vee \left( \left( \tilde{V}(r, x, \eta) \leq \frac{\mu}{2} d(r, x)^2 \right) \wedge (x = 1 \vee r = 1) \right) \right. \\ \left. \vee \left( \tilde{V}(r, x, \eta) \geq \frac{\mu}{2} d(r, x)^2 \wedge rx = 0 \right) \right\}. \quad (49)$$

Using (45), we observe that the relation  $V(r, x, \eta) \leq (\mu/2)d(r, x)^2$  is satisfied and thus also (14a) holds with the  $\mathcal{K}_\infty$  functions  $\alpha_1(d) = (1/2)d^2$  and  $\alpha_2(d) = (\mu/2)d^2$ .

Observe that, for  $q \in \text{cl}(C_e)$ , we obtain  $\partial_C V(q) = \{\nabla V(q)\}$ , and since  $\nabla V(q) = (-x + r - \eta \quad x - r + \eta \quad x - r + \eta)$  we find that (14b) reduces to  $\nabla V(q)F_e(t, q) \leq -cV(q)$ , which is satisfied for  $c = -2\alpha < 0$ , since

$$\nabla V(q)F_e(t, q) = -\alpha(x - r + \eta)^2 = -2\alpha V(q). \quad (50)$$

By verification of the three different jumps described in (47), it follows directly that (14c) is satisfied.

Now, we will derive lower and upper bounds for the right-hand side of (1a) to select  $\tau, \kappa > 0$  such that  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ , and to show that the time domain of trajectories of (4) is unbounded in  $t$ -direction. Observe that  $|x - r + \eta| \leq (\mu/2)$ ,  $\forall \text{col}(r, x, \eta) \in C_e$ , and  $\dot{x} = 1 + u_{\text{ref}}(t) - \alpha(x - r + \eta)$ , with  $u_{\text{ref}}(t) = (1/2)\cos(t)$ , such that

$$1 - \frac{1}{2} - |\alpha(x - r + \eta)| \leq \dot{x} \leq 1 + \frac{1}{2} + |\alpha(x - r + \eta)| \\ \frac{1}{2} - \alpha \frac{\mu}{2} \leq \dot{x} \leq \frac{3}{2} + \alpha \frac{\mu}{2}$$

and using the selection of  $\alpha$  and  $\mu$ , we obtain  $0 < \dot{x} < 2$ . Observe that  $\dot{x} > 0$  directly implies that trajectories of (1a) can only leave  $C$  by arriving at  $D$  and experiencing a jump, which, after one jump, can only be followed by flow. Hence, the time domain of  $x$  is unbounded in  $t$ -direction.

Now, we design  $\tau, \kappa$  such that trajectories  $\varphi$  of the embedded system (4) satisfy  $\varphi \in \mathcal{S}_{\text{avg}}(\tau, \kappa)$ . Observe that jumps in  $x$  (or jumps in  $r$ ) can only occur after it increased from 0 to 1 since the last jump. Using the upper bound  $\dot{x} < 2$ , this will take at least  $1/2$  time units. Hence, every  $1/2$  time units, both  $x$  and  $r$  can jump once according to (1b). The controller input (18c) is

designed such that continuous-time behavior always decreases  $V(q)$ , such that jumps in  $\eta$  only can be triggered by jumps in  $x$  or  $\eta$ , or to be more explicit, jumps of  $\eta$  will not be triggered by reaching the set where  $V(r, x, \eta) = (\mu/2)d(r, x)^2$  when  $x$  and  $r$  are described by (1a). Hence, system (4) can exhibit at most  $\kappa = 4$  jumps during every continuous-time interval of length  $1/2$ , i.e., we can select  $(\tau, \kappa) = (1/8, 4)$  and obtain  $\varphi \in \mathcal{S}_{\text{avg}}(1/8, 4)$ . Since  $r$  satisfies Assumption 2, the time domain of  $x$  is unbounded and  $\varphi \in \mathcal{S}_{\text{avg}}(1/8, 4)$ , we directly obtain that the time domain of  $\varphi$  is unbounded in  $t$ -direction.

Evaluating (14d) with  $c = -2\alpha = -1$  yields  $\mu e^{c\tau} = 1.125e^{-1/8} \approx 0.993 < 1$ , such that Theorem 4(ii) proves that the global tracking problem given in Problem 1 is solved. ■

*Proof of Theorem 6:* This theorem is proven by application of case (i) of Theorem 4 with the Lyapunov function candidate  $V$  defined in (25)–(27). Since we are interested in a local tracking problem, we restrict our attention to the given reference trajectory  $r$  and plant trajectories  $x$  satisfying  $V(r, x) \leq K$ , with  $K$  selected as required in the theorem. Such trajectories are described by system (4) when we select

$$F_e(t, \text{col}(r, x)) \\ = \begin{cases} \text{col}(r_2, -g, x_2, -g - [k_p \ k_d](x - r)), & \text{for } V_d < V_m \\ \text{col}(r_2, -g, x_2, +g - [k_p \ k_d](x + r)), & \text{for } V_d > V_m \end{cases} \quad (51)$$

$$G_e(\text{col}(r, x)) \\ = \begin{cases} \text{col}(-r, x), & \text{for } r \in \{0\} \times (-\infty, 0] \\ \text{col}(r, -x), & \text{for } x \in \{0\} \times (-\infty, 0] \end{cases} \quad (52)$$

$$C_e := \{\text{col}(r, x) \in ([0, \infty) \times \mathbb{R})^2 \mid V(r, x) \leq K\} \quad (53)$$

$$D_e := \left\{ \text{col}(r, x) \in (\{0\} \times (-\infty, 0] \times [0, \infty) \times \mathbb{R}) \right. \\ \left. \cup ([0, \infty) \times \mathbb{R} \times \{0\} \times (-\infty, 0]) \mid V(r, x) \leq K \right\}. \quad (54)$$

We will apply Theorem 4 to prove global asymptotic stability of this system, which directly implies that the local tracking problem defined in Theorem 6 is solved. First, we show that the set  $C_e \cup D_e$  does not contain points where  $V_d(r, x) = V_m(r, x)$ . This follows directly from (29), since, if  $V_d = V_m$ , then

$$V(r, x) = \frac{1}{2}x^T P x + \frac{1}{2}r^T P r \geq \frac{1}{2}r^T P r = V(r, 0) > K.$$

To find a lower bound for  $V(r, x)$ , observe that  $V(r, x) = \min(V_d(r, x), V_m(r, x))$ , with  $V_d, V_m$  given in (26), and both  $V_d$  and  $V_m$  can be bounded from below using the minimum eigenvalue of  $P$ , which we denote with  $\lambda_{\min}(P)$ , such that we observe

$$\alpha_1(d(r, x)) = \frac{1}{2}\lambda_{\min}(P)d(r, x)^2 \leq V(r, x), \\ \forall \text{col}(r, x) \in C_e \cup D_e.$$

We obtain an upper bound for  $V$  by separately studying the domains  $\{\text{col}(r, x) \in C_e \cup D_e \mid |x - r| < |x + r|\}$ ,  $\{\text{col}(r, x) \in C_e \cup D_e \mid |x - r| > |x + r|\}$ , and  $\{\text{col}(r, x) \in C_e \cup D_e \mid |x - r| = |x + r|\}$ , using the upper bounds  $V_d(r, x) \leq (\lambda_{\max}(P)/2)|x - r|^2$  and  $V_m(r, x) \leq (\lambda_{\max}(P)/2)|x + r|^2$ , where  $\lambda_{\max}(P)$  is the maximum eigenvalue of the matrix  $P$ .

First, we observe that the set where  $|x - r| < |x + r|$  contains the points where  $x - r = 0$ , and for this reason,

$V_d(r, x) - V_m(r, x) < 0$  holds at these points. Since the domain  $\{\text{col}(r, x) \in C_e \cup D_e \mid |x - r| < |x + r|\}$  is connected, the function  $V_d(r, x) - V_m(r, x)$  is continuous, and, additionally,  $V_d(r, x) - V_m(r, x) = 0$  cannot occur, we find that  $V_d(r, x) - V_m(r, x) < 0$  holds for all points in this domain, and hence, we can employ the upper bound  $V(r, x) = V_d(r, x) \leq (\lambda_{\max}(P)/2)d(r, x)^2$ . Via analogous reasoning, we obtain  $V(r, x) = V_m(r, x) \leq (\lambda_{\max}(P)/2)d(r, x)^2$  for the domain  $\{\text{col}(r, x) \in C_e \cup D_e \mid |x - r| > |x + r|\}$ . Finally, in the domain where  $|x - r| = |x + r|$ , we observe that

$$V(r, x) \leq \min(V_d(r, x), V_m(r, x)) \\ \leq \min\left(\frac{\lambda_{\max}(P)}{2}|x - r|^2, \frac{\lambda_{\max}(P)}{2}|x + r|^2\right)$$

and since  $d(r, x) = |x - r| = |x + r|$ , we directly obtain  $V(r, x) \leq (\lambda_{\max}(P)/2)d(r, x)^2$ . Therefore, in  $C_e \cup D_e$  we observe that (14a) is satisfied with  $\alpha_1(d(r, x)) = (1/2)\lambda_{\min}(P)d(r, x)^2$  and  $\alpha_2(d(r, x)) = (1/2)\lambda_{\max}(P)d(r, x)^2$ .

It can directly be observed that for the function  $V$  given in (25), the requirement (14c) holds with equality for  $\mu = 1$ . Hence, it only remains to be shown that (14b) holds, and in addition, that the time domain of the trajectories in  $C_e \cup D_e$  is unbounded in  $t$ -direction.

According to (28),  $u$  is discontinuous only on the surface  $V_d(r, x) = V_m(r, x)$ , which, as observed already, is not contained in  $C_e \cup D_e$ . Hence, first,  $V$  is continuously differentiable and, second,  $u$  is continuous for all  $x \in \{x \in C \cup D \mid V(r(t, j), x) \leq K\}$  at any time  $(t, j) \in \text{dom } r$ , and one can write  $\bar{F}_e(t, q) = F_e(t, q)$ .

Since  $\nabla V(\text{col}(r, x))$  is given by

$$\nabla V(\text{col}(r, x)) = \begin{cases} [-(x - r)^T P (x - r)^T P], & \text{for } V_d < V_m \\ [(x + r)^T P (x + r)^T P], & \text{for } V_d > V_m, \end{cases} \quad (55)$$

we obtain

$$\nabla V(\text{col}(r, x))F_e(t, \text{col}(r, x)) \\ = \begin{cases} \frac{1}{2}(x - r)^T (A^T P + P A)(x - r), & \text{for } V_d < V_m \\ \frac{1}{2}(x + r)^T (A^T P + P A)(x + r), & \text{for } V_d > V_m. \end{cases} \quad (56)$$

Hence, (27) yields

$$\dot{V} \leq \begin{cases} -\frac{1}{2}c(x - r)^T P (x - r), & \text{for } V_d < V_m \\ -\frac{1}{2}c(x + r)^T P (x + r), & \text{for } V_d > V_m \end{cases}$$

for all  $(t, j) \in \text{dom } r$  and all  $x \in \{x \in C \cup D \mid V(r(t, j), x) \leq K\}$ , which, given (25), directly implies  $\dot{V} \leq cV(r, x)$ , such that (14b) holds. This implies that  $V(r, x)$  decreases during flow. Since  $F_e$  is not explicitly dependent on time, we can apply Proposition 2.4 of [35] which yields that all trajectories have an unbounded hybrid time domain. For any  $(t, j) \in \text{dom } r$ , the set  $\{x \in C \cup D \mid V(r(t, j), x) \leq K\}$  does not contain the origin, which follows directly from  $V(r(t, j), 0) = (1/2)r(t, j)^T P r(t, j)$  and the choice of  $K$  as given in (29). Hence, after each jump in  $x$ , the trajectory  $x$  has to experience flow and travel at least a distance  $\epsilon > 0$ , which

will take at least a continuous-time duration  $\tau > 0$ , such that the time domain of  $\text{col}(r, x)$  is unbounded in  $t$ -direction.

Hence, Theorem 4(i) is applicable, such that the reference trajectory  $r$  is locally asymptotic stable for the dynamics (4). Since in the above analysis, convergence is shown for all initial conditions in  $\{x \in C \cup D \mid V(r(0, 0), x) \leq K\}$ , this domain is contained in the basin of attraction of the reference trajectory  $r$ , thereby proving the theorem. ■

## REFERENCES

- [1] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Syst. Mag.*, vol. 29, no. 2, pp. 28–93, Apr. 2009.
- [2] A. J. van der Schaft and J. M. Schumacher, *An Introduction to Hybrid Dynamical Systems*. London, U.K.: Springer-Verlag, 2000, vol. 251, Lecture Notes in Control and Information Sciences.
- [3] J. Lunze and F. Lamnabhi-Lagarigue, Eds., *Handbook of Hybrid Systems Control*. Cambridge, U.K.: Cambridge Univ. Press, 2009.
- [4] A. N. Michel and B. Hu, "Towards a stability theory of general hybrid dynamical systems," *Automatica*, vol. 35, no. 3, pp. 371–384, 1999.
- [5] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov functions for hybrid systems – Part I: Existence is equivalent to robustness," *IEEE Trans. Autom. Control*, vol. 52, no. 7, pp. 1264–1277, Jul. 2007.
- [6] J. Lygeros, K. H. Johansson, S. N. Simić, J. Zhang, and S. S. Sastry, "Dynamical properties of hybrid automata," *IEEE Trans. Autom. Control*, vol. 48, no. 1, pp. 2–17, Jan. 2003.
- [7] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM Rev.*, vol. 49, no. 4, pp. 545–592, 2007.
- [8] D. Liberzon, *Switching in Systems and Control*. Boston, MA: Birkhäuser, 2003, Systems & Control: Foundations & Applications.
- [9] W. P. M. H. Heemels, B. De Schutter, J. Lunze, and M. Lazar, "Stability analysis and controller synthesis for hybrid dynamical systems," *Phil. Trans. R. Soc. A*, vol. 368, no. 1930, pp. 4937–4960, 2010.
- [10] R. I. Leine and N. van de Wouw, *Stability and Convergence of Mechanical Systems with Unilateral Constraints*. Berlin, Germany: Springer-Verlag, 2008, vol. 36, Lecture Notes in Applied and Computational Mechanics.
- [11] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, no. 11, pp. 2735–2744, 2008.
- [12] B. Brogliato, S. Niculescu, and M. Monteiro-Marques, "On tracking control of a class of complementary-slackness hybrid mechanical systems," *Syst. Control Lett.*, vol. 39, no. 4, pp. 255–266, 2000.
- [13] R. G. Sanfelice, J. J. B. Biemond, N. van de Wouw, and W. P. M. H. Heemels, "Tracking control for hybrid systems via embedding of known reference trajectories," in *Proc. Amer. Control Conf.*, 2011, pp. 869–874.
- [14] R. I. Leine and N. van de Wouw, "Uniform convergence of monotone measure differential inclusions: With application to the control of mechanical systems with unilateral constraints," *Int. J. Bifurcat. Chaos*, vol. 18, no. 5, pp. 1435–1457, 2008.
- [15] R. G. Sanfelice, A. R. Teel, and R. Sepulchre, "A hybrid systems approach to trajectory tracking control for juggling systems," in *Proc. 46th IEEE Conf. Decision Control*, 2007, pp. 5282–5287.
- [16] N. van de Wouw and R. I. Leine, "Impulsive control of mechanical motion systems with uncertain friction," in *Proc. 50th IEEE Conf. Decision Control*, 2011, pp. 4176–4182.
- [17] W. P. M. H. Heemels, M. K. Camlibel, J. M. Schumacher, and B. Brogliato, "Observer-based control of linear complementarity systems," *Int. J. Robust Nonlinear Control*, vol. 21, no. 10, pp. 1193–1218, 2011.
- [18] C. Glocker, *Set-Valued Force Laws*, F. Pfeiffer, Ed. Berlin, Germany: Springer-Verlag, 2001, vol. 1, Lecture Notes in Applied Mathematics.
- [19] L. Menini and A. Tornambè, "Asymptotic tracking of periodic trajectories for a simple mechanical system subject to nonsmooth impacts," *IEEE Trans. Autom. Control*, vol. 46, no. 7, pp. 1122–1126, Jul. 2001.
- [20] S. Galeani, L. Menini, A. Potini, and A. Tornambè, "Trajectory tracking for a particle in elliptical billiards," *Int. J. Control*, vol. 81, no. 2, pp. 189–213, 2008.
- [21] S. Galeani, L. Menini, and A. Potini, "Robust trajectory tracking for a class of hybrid systems: An internal model principle approach," *IEEE Trans. Autom. Control*, vol. 57, no. 2, pp. 344–359, Feb. 2012.
- [22] B. Brogliato, S.-I. Niculescu, and P. Orhant, "On the control of finite-dimensional mechanical systems with unilateral constraints," *IEEE Trans. Autom. Control*, vol. 42, no. 2, pp. 200–215, Feb. 1997.

- [23] I. C. Morărescu and B. Brogliato, "Trajectory tracking control of multi-constraint complementarity Lagrangian systems," *IEEE Trans. Autom. Control*, vol. 55, no. 6, pp. 1300–1313, Jun. 2010.
- [24] F. Forni, A. R. Teel, and L. Zaccarian, "Tracking control in billiards using mirrors without smoke, Part I: Lyapunov-based local tracking in polyhedral regions," in *Proc. 50th IEEE Conf. Decision Control*, 2011, pp. 3283–3288.
- [25] F. Forni, A. R. Teel, and L. Zaccarian, "Tracking control in billiards using mirrors without smoke, Part II: Additional Lyapunov-based local and global results," in *Proc. 50th IEEE Conf. Decision Control*, 2011, pp. 3289–3294.
- [26] G. A. Leonov, *Strange Attractors and Classical Stability Theory*. St. Petersburg, Russia: St. Petersburg Univ. Press, 2008.
- [27] J. J. Moreau, "Approximation en graphe d'une évolution discontinue," *RAIRO Anal. Numérique*, vol. 12, no. 1, pp. 75–84, 1978.
- [28] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*. Dordrecht, The Netherlands: Kluwer, 1988, vol. 18, Mathematics and its applications (Soviet Series).
- [29] J. L. Willems, *Stability Theory of Dynamical Systems*, R. W. Brockett and H. H. Rosenbrock, Eds. Nashville, TN: Thomas Nelson and Sons, 1970, Studies in dynamical systems.
- [30] F. H. Clarke, *Optimization and Nonsmooth Analysis*. Philadelphia, PA: SIAM, 1990, vol. 5, Classics in Applied Mathematics.
- [31] F. H. Clarke, Y. S. Ledyaev, E. Sontag, and A. I. Subbotin, "Asymptotic controllability implies feedback stabilization," *IEEE Trans. Autom. Control*, vol. 42, no. 10, pp. 1394–1407, Oct. 1997.
- [32] R. G. Sanfelice, M. J. Messina, S. E. Tuna, and A. R. Teel, "Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points," in *Proc. Amer. Control Conf.*, 2006, pp. 3352–3357.
- [33] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice-Hall, 2002.
- [34] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Trans. Autom. Control*, vol. 39, no. 9, pp. 1910–1914, Sep. 1994.
- [35] R. Goebel and A. R. Teel, "Solutions to hybrid inclusions via set and graphical convergence with stability theory applications," *Automatica*, vol. 42, no. 4, pp. 573–587, 2006.



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