Controlled Synchronisation of Continuous PWA Systems*

Nathan van de Wouw[1], Alexey Pavlov[2], and Henk Nijmeijer[1]

[1] Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, N.v.d.Wouw@tue.nl
[2] NTNU, Department of Engineering Cybernetics, N7491, Trondheim, Norway, Alexey.Pavlov@itk.ntnu.no
[3] Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, H.Nijmeijer@tue.nl

Abstract

In this paper, the controlled synchronisation problem for identical continuous piecewise affine (PWA) systems is addressed. Due to the switching nature of these systems, strategies common for controlled synchronisation can not be used. In this paper, an observer-based output-feedback control design solving the master-slave synchronisation problem for two PWA systems is proposed. The design of these dynamic controllers is based on the idea of, on the one hand, rendering the slave system convergent by means of feedback (which makes all its solutions converge to each other) and, on the other hand, guaranteeing that the closed-loop slave system has a bounded solution corresponding to zero synchronisation error. This implies that all solutions of the closed-loop slave system converge to this bounded solution with zero synchronisation error. The results are illustrated by application to a master-slave synchronisation problem of two mechanical systems with one-sided restoring characteristics.

1 Introduction

Synchronisation of dynamical systems has received considerable interest because of the wide variety of systems in which synchronisation can occur or is desirable, e.g. in secure communication [1], biological systems [2], (electro-)mechanical systems [3], such as rotor dynamic systems or cooperating robots [4]. Many more illustrative examples can be found in [5]. Different

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kinds of synchronisation [6] can be defined; namely, *natural synchronisation*, which is established without interaction between the systems involved, *self-synchronisation*, which occurs due to a proper coupling between the systems while this coupling is inherently in the system and not enforced externally, and *controlled synchronisation*, which implies synchronisation enforced by active control. This paper deals with the controlled synchronisation problem for continuous PWA systems. Another distinction between different types of synchronisation can be made in terms of the variables being synchronised. One can speak of phase synchronisation, see e.g. [7], when the responses of the systems only comply in terms of a certain phase variable. Here, we will discuss full-state synchronisation, which implies the exact correspondence of all the states of the system. Other notions of synchronisation are partial synchronisation [8, 9, 10], in which only part of the state of the systems synchronise, and generalised synchronisation, in which correspondence of certain functionals of the state is established [6].

The controlled synchronisation problem can be divided into the master-slave synchronisation problem and the mutual synchronisation problem. In the master-slave variant, which is considered here, the slave system is unidirectionally coupled (by means of control) to the master system, whereas in mutual synchronisation a bilateral coupling ensures adaptation of the systems with respect to each other [11]. Many results on controlled synchronisation exist, e.g. [6, 12, 13, 14, 4], where both state-feedback and (observer-based) dynamic output-feedback control strategies are proposed. In [14, 15], the controlled synchronisation problem is considered in the scope of the regulator problem and in [16, 17] the strong link between the synchronisation problem and the observer design problem is illuminated. Robustness issues with respect to differences between the systems to be synchronised are treated in [13, 18, 19]. Furthermore, the synchronisation of chaotic oscillators has received a huge amount of attention, e.g. in [1, 13] and many other publications.

Currently, PWA systems are receiving wide attention due to the fact that the PWA framework [20] provides a means to describe dynamic systems exhibiting switching between a multitude of linear dynamic regimes. Such switching can be due to piecewise-linear characteristics such as dead-zone, saturation, hysteresis or relays. A common strategy in achieving synchronisation is the stabilisation of the error dynamics between the systems to be synchronised. One could then think of translating the controlled synchronisation problem for PWA systems into some stabilisation problem for PWA systems and subsequently applying known results for the stabilization of PWA systems, see for example [21, 22, 23, 24, 25]. As we will illuminate in the next section, the switching nature of the vector-field of PWA systems seriously complicates such an approach. *Some* PWA systems can be represented in the form of a Lur'e system (as is the case for the famous Chua circuit), for which the master-slave synchronisation problem is considered in [12, 26, 27, 13]. It is also worth mentioning the work in [28] on state-feedback tracking control

of bimodal PWA systems, since master-slave synchronisation and tracking are closely related problems.

Here, we propose a different approach towards the controlled synchronisation problem for general continuous PWA systems. In this approach, the notion of convergence plays a central role. A system, which is excited by an input, is called convergent if it has a unique globally asymptotically stable solution that is bounded on the whole time axis. Obviously, if such a solution does exist, all other solutions, regardless of their initial conditions, converge to this solution, which can be considered as a steady-state solution [29, 30]. Similar notions describing the property of solutions converging to each other are studied in literature. The notion of contraction has been introduced in [31] (see also references therein). An operator-based approach towards studying the property that all solutions of a system converge to each other is pursued in [32, 33]. In [34], a Lyapunov approach has been developed to study both the global uniform asymptotic stability of all solutions of a system (in [34], this property is called incremental stability) and the so-called incremental input-to-state stability property, which is compatible with the input-to-state stability approach (see e.g. [35]). In the scope of synchronisation we use the convergence property in the following way. The design of the synchronising controllers is based on the idea of, on the one hand, rendering the closed-loop slave system convergent by means of feedback (which means that all its solutions converge to each other) and, on the other hand, guaranteeing that the closed-loop slave system has a bounded solution corresponding to zero synchronisation error. This implies that all solutions of the closed-loop slave system converge to the synchronising solution.

The paper is structured as follows. In Section 2, the problem of master-slave synchronisation of PWA systems is stated and it is illuminated that the common approach of synthesising synchronising controllers by providing asymptotically stable error dynamics does not lead to tractable solutions due to the switching nature of PWA systems. The notions of convergence and input-state convergence are introduced and sufficient conditions for these properties for PWA systems are proposed in Section 3. The latter properties are used in Section 4 to design state- and (observer-based) dynamic output feedback controllers achieving synchronisation. An illustrative example of the master-slave synchronisation of two mechanical systems with one-sided restoring characteristics is presented in Section 5 and Section 6 gives concluding remarks.

2 Problem Formulation

Consider the state space $\mathbb{R}^n$ to be divided into polyhedral cells Λ_i, $i = 1, \ldots, l$, by hyperplanes given by equations of the form $\boldsymbol{H}_j^T \boldsymbol{x} + h_j = 0$, for some $\boldsymbol{H}_j \in \mathbb{R}^n$ and $h_j \in \mathbb{R}$, $j = 1, \ldots, k$. We will consider the master PWA system to be of the form:

$$\begin{aligned} \dot{\boldsymbol{x}}^m &= \boldsymbol{A}_i \boldsymbol{x}^m + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}^m(t) \;\text{ for }\; \boldsymbol{x}^m \in \Lambda_i,\; i = 1, \ldots, l, \\ \boldsymbol{y}^m &= \boldsymbol{C}\boldsymbol{x}^m \end{aligned} \tag{1}$$

and the slave system of the form

$$\begin{aligned} \dot{\boldsymbol{x}}^s &= \boldsymbol{A}_i \boldsymbol{x}^s + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}^s \;\text{ for }\; \boldsymbol{x}^s \in \Lambda_i,\; i = 1, \ldots, l, \\ \boldsymbol{y}^s &= \boldsymbol{C}\boldsymbol{x}^s. \end{aligned} \tag{2}$$

Here $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}^{q\times n}$, $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b}_i \in \mathbb{R}^n$, $i = 1, \ldots, l$, are constant matrices and vectors, respectively. The vectors $\boldsymbol{x}^m \in \mathbb{R}^n$ and $\boldsymbol{x}^s \in \mathbb{R}^n$ represent the state of the master and slave system, respectively, the vectors $\boldsymbol{y}^m \in \mathbb{R}^q$ and $\boldsymbol{y}^s \in \mathbb{R}^q$ are the corresponding measured outputs, $\boldsymbol{u}^m(t)$ is a time-dependent input of the master system and the vector $\boldsymbol{u}^s \in \mathbb{R}^m$ is the control input for the slave system. The hyperplanes $\boldsymbol{H}_j^T \boldsymbol{x} + h_j = 0$, $j = 1, \ldots, k$, are the switching surfaces for both systems. Now, we adopt the following assumptions:

Assumption 1 *The right-hand sides of (1) and (2) are continuous in the corresponding states.*

It is known (see e.g. [15]) that this assumption can be formalized in the (necessary and sufficient) requirement that for any two cells Λ_i and Λ_j having a common boundary $\boldsymbol{H}_{ij}^T \boldsymbol{x} + h_{ij} = 0$ the corresponding matrices $\boldsymbol{A}_i$ and $\boldsymbol{A}_j$ and the vectors $\boldsymbol{b}_i$ and $\boldsymbol{b}_j$ satisfy the equalities

$$\begin{aligned} \boldsymbol{G}_{ij}\boldsymbol{H}_{ij}^T &= \boldsymbol{A}_i - \boldsymbol{A}_j \\ \boldsymbol{G}_{ij} h_{ij} &= \boldsymbol{b}_i - \boldsymbol{b}_j, \end{aligned} \tag{3}$$

for some vector $\boldsymbol{G}_{ij} \in \mathbb{R}^n$.

Assumption 2 *The input $\boldsymbol{u}^m(t)$ of the master system and the corresponding solutions $\boldsymbol{x}^m(t)$ are bounded for $t \geq 0$.*

The problem considered in this work is formulated as follows:

Master-slave synchronisation Design a control law for $\boldsymbol{u}^s$ for the slave system that, based on information on the measured outputs $\boldsymbol{y}^s$ and $\boldsymbol{y}^m$ and the input $\boldsymbol{u}^m(t)$ of the master system, renders $\boldsymbol{x}^s(t) \to \boldsymbol{x}^m(t)$ as $t \to \infty$ and the states of the closed-loop slave system are bounded.

As mentioned in the introduction, a common strategy in achieving synchronisation is the stabilisation of the error dynamics between the master and slave systems. The topic of stabilisation of PWA systems is currently receiving wide attention. One could then think of translating the controlled synchronisation problem into some stabilisation problem and subsequently applying known results for the stabilization of PWA systems. Yet, this common way of solving the problem does not lead to tractable solutions. In order to illustrate this, let us consider the master-slave synchronisation problem for

systems (1), (2) as formulated above for the simplest case in which the entire states of both the master and slave system are measured, i.e. $\boldsymbol{y}^m = \boldsymbol{x}^m$ and $\boldsymbol{y}^s = \boldsymbol{x}^s$, and see how this problem would be approached in a conventional way. The first step in this approach would be to decompose the control law into a feedforward part $\boldsymbol{u}^m(t)$ and a feedback part $\boldsymbol{u}_{fb}(\boldsymbol{x}^s, \boldsymbol{x}^m(t))$:

$$\boldsymbol{u}^s(\boldsymbol{x}^s, \boldsymbol{x}^m(t), \boldsymbol{u}^m(t)) = \boldsymbol{u}^m(t) + \boldsymbol{u}_{fb}(\boldsymbol{x}^s, \boldsymbol{x}^m(t)). \tag{4}$$

This results in the following closed-loop slave system:

$$\dot{\boldsymbol{x}}^s = \boldsymbol{A}_i\boldsymbol{x}^s + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}^m(t) + \boldsymbol{B}\boldsymbol{u}_{fb}(\boldsymbol{x}^s, \boldsymbol{x}^m(t)) \quad \text{for} \quad \boldsymbol{x}^s \in \Lambda_i,\ i = 1, \ldots, l. \tag{5}$$

The feedforward ensures that a solution $\boldsymbol{x}^s(t)$, $t > t_0$, of the closed-loop slave system (5) will match the solution of the master system if it matches the solution of the master system at $t = t_0$, i.e. if $\boldsymbol{x}^s(t_0) = \boldsymbol{x}^m(t_0)$ (and if $\boldsymbol{u}_{fb}(\boldsymbol{x}^m(t), \boldsymbol{x}^m(t)) = \boldsymbol{0}$). In other words, the feedforward generates the solution of the master system $\boldsymbol{x}^m(t)$ in the slave system (2).

Subsequently, asymptotic synchronisation is assured by designing the feedback part $\boldsymbol{u}_{fb}(\boldsymbol{x}^s, \boldsymbol{x}^m(t))$ such that $\boldsymbol{u}_{fb}(\boldsymbol{x}^m, \boldsymbol{x}^m) = \boldsymbol{0}$ and the dynamics of the synchronisation error $\boldsymbol{e} = \boldsymbol{x}^s - \boldsymbol{x}^m$ are globally asymptotically stable. These error dynamics follow from (1) and (5):

$$\begin{aligned} \dot{\boldsymbol{e}} = \boldsymbol{A}_i(\boldsymbol{e} + \boldsymbol{x}^m(t)) - \boldsymbol{A}_j\boldsymbol{x}^m(t) + (\boldsymbol{b}_i - \boldsymbol{b}_j) + \boldsymbol{B}\boldsymbol{u}_{fb}, \\ \text{for } (\boldsymbol{e} + \boldsymbol{x}^m(t)) \in \Lambda_i,\ i = 1, \ldots, l, \\ \text{and } \boldsymbol{x}^m(t) \in \Lambda_j,\ j = 1, \ldots, l. \end{aligned} \tag{6}$$

Now, the problem in this approach for PWA systems lies in the fact that the error-dynamics in (6) not only switches when the state $\boldsymbol{x}^s$ of the slave system switches from one polyhedral cell to another but also when the state $\boldsymbol{x}^m(t)$ of the master system switches from one polyhedral cell to another. Consequently, the error dynamics is described by (potentially) l^2 different vector fields (which vector field applies depends on $\boldsymbol{e}$ and $\boldsymbol{x}^m(t)$, see (6)). Moreover, one should realise that these dynamics are time-varying. This combined switching and time-varying nature seriously complicates the stability analysis of the equilibrium point $\boldsymbol{e} = \boldsymbol{0}$ of (6) and keeps one from applying standard stability analysis methods for PWA systems. This can be illustrated by considering a Lyapunov-based stability argument using for example a positive-definite quadratic Lyapunov function candidate of the form:

$$V = \boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e} \quad \text{with} \quad \boldsymbol{P} = \boldsymbol{P}^T > 0. \tag{7}$$

The time-derivative of this function V obeys:

$$\begin{aligned} \dot{V} =& \boldsymbol{e}^T\left(\boldsymbol{A}_i\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_i\right)\boldsymbol{e} \\ &+ \left((\boldsymbol{b}_i - \boldsymbol{b}_j)^T + \boldsymbol{u}_{fb}^T\boldsymbol{B}^T + (\boldsymbol{x}^m)^T(t)(\boldsymbol{A}_i - \boldsymbol{A}_j)^T\right)\boldsymbol{P}\boldsymbol{e} \\ &+ \boldsymbol{e}^T\boldsymbol{P}\left((\boldsymbol{b}_i - \boldsymbol{b}_j) + \boldsymbol{B}\boldsymbol{u}_{fb} + (\boldsymbol{A}_i - \boldsymbol{A}_j)\boldsymbol{x}^m(t)\right), \\ &\text{for } (\boldsymbol{e} + \boldsymbol{x}^m(t)) \in \Lambda_i,\ i = 1, \ldots, l \text{ and } \boldsymbol{x}^m(t) \in \Lambda_j,\ j = 1, \ldots, l. \end{aligned} \tag{8}$$

Clearly, the design of the feedback law $\boldsymbol{u}_{fb}$ guaranteeing the negative-definiteness of $\dot{V}$, satisfying (8), becomes a rather cumbersome task. The latter exposition aims at clarifying that the master-slave synchronisation problem for PWA systems, on the one hand, can not be tackled by applying known techniques for master-slave synchronisation of smooth systems and, on the other hand, it is significantly more complex than the stabilisation problem for PWA systems.

This example motivates our study of the master-slave synchronisation problem for PWA systems. Here, we will propose a new approach to this problem based on the notion of convergent systems [29, 30], which is introduced in the next section.

3 Convergent Systems

In this section, we will briefly discuss the definition of convergence, certain properties of convergent systems and sufficient conditions for convergence of non-smooth, continuous piecewise affine systems. The definitions presented here extend the definition given in [29]. Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \tag{9}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\boldsymbol{f}(\boldsymbol{x}, t)$ is locally Lipschitz in $\boldsymbol{x}$ and piecewise continuous in t.

Definition 1. *System (9) is said to be*

- convergent *if there exists a solution $\bar{\boldsymbol{x}}(t)$ satisfying the following conditions:*
  (i) *$\bar{\boldsymbol{x}}(t)$ is defined and bounded for all $t \in \mathbb{R}$,*
  (ii) *$\bar{\boldsymbol{x}}(t)$ is globally asymptotically stable.*
- uniformly convergent *if it is convergent and $\bar{\boldsymbol{x}}(t)$ is globally uniformly asymptotically stable.*
- exponentially convergent *if it is convergent and $\bar{\boldsymbol{x}}(t)$ is globally exponentially stable.*

The solution $\bar{\boldsymbol{x}}(t)$ is called a *steady-state solution*. As follows from the definition of convergence, any solution of a convergent system "forgets" its initial condition and converges to some steady-state solution. This, in turn, implies that any two solutions $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ converge to each other, i.e. $|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)| \to \boldsymbol{0}$ as $t \to +\infty$.

In the scope of our problem setting of controlled synchronisation, the time-dependency is due to some input determined by the time-dependent trajectory of the master system (e.g. $\boldsymbol{x}^m(t)$ and $\boldsymbol{u}^m(t)$ in (5)). Below we will consider convergence properties for systems with inputs. So, instead of systems of the form (9), we consider systems of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{w}), \tag{10}$$

with state $\boldsymbol{x} \in \mathbb{R}^n$ and input $\boldsymbol{w} \in \mathbb{R}^d$. The function $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{w})$ is locally Lipschitz in $\boldsymbol{x}$ and continuous in $\boldsymbol{w}$. In the sequel, we will consider the class $\overline{\mathbb{PC}}_d$ of piecewise continuous inputs $\boldsymbol{w}(t) : \mathbb{R} \to \mathbb{R}^d$ which are bounded on $\mathbb{R}$. Below we define the convergence property for systems with inputs.

Definition 2. *System (10) is said to be* (uniformly, exponentially) convergent *if it is (uniformly, exponentially) convergent for every input $\boldsymbol{w} \in \overline{\mathbb{PC}}_d$. In order to emphasize the dependency on the input $\boldsymbol{w}(t)$, the steady-state solution is denoted by $\bar{\boldsymbol{x}}_w(t)$.*

In the scope of synchronisation problems, inputs are usually defined not on the whole time axis $\mathbb{R}$, but only for $t \geq 0$. For this case, we can formulate the following property.

Property 1 ([15]). *If a convergent system (10) is excited by an input $\boldsymbol{w}(t)$, that is defined and bounded only for $t \geq 0$ (rather than for $t \in \mathbb{R}$), then any two solutions $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ of (10) satisfy $\mid \boldsymbol{x}_1(t) - \boldsymbol{x}_2(t) \mid \to \boldsymbol{0}$ as $t \to \infty$.*

The next definition extends the uniform convergence property to the input-to-state stability framework.

Definition 3. *System (10) is said to be input-to-state convergent if it is uniformly convergent and for every input $\boldsymbol{w} \in \overline{\mathbb{PC}}_d$ system (10) is input-to-state stable (ISS) with respect to the steady-state solution $\bar{\boldsymbol{x}}_w(t)$, i.e. there exist a $\mathcal{KL}$-function $\beta(r, s)$ and a $\mathcal{K}_\infty$-function $\gamma(r)$ such that any solution $\boldsymbol{z}(t)$ of system (10) corresponding to some input $\hat{\boldsymbol{w}}(t) := \boldsymbol{w}(t) + \boldsymbol{\Delta w}(t)$ satisfies*

$$|\boldsymbol{x}(t) - \bar{\boldsymbol{x}}_w(t)| \leq \beta(|\boldsymbol{x}(t_0) - \bar{\boldsymbol{x}}_w(t_0)|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} |\boldsymbol{\Delta w}(\tau)|). \tag{11}$$

In general, the functions $\beta(r, s)$ and $\gamma(r)$ may depend on the particular input $\boldsymbol{w}(t)$.

Similar to the conventional ISS property, the property of input-to-state convergence is especially useful for studying convergence properties of interconnected systems. One can easily show that the parallel interconnection of (exponentially, uniformly, input-to-state) convergent systems is again an (exponentially, uniformly, input-to-state) convergent system. A series connection of two input-to-state convergent systems, see Figure 1, is an input-to-state convergent system, as stated in the next property.

Property 2 ([36, 37]). *Consider the system*

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{w}), & \boldsymbol{x}_1 \in \mathbb{R}^{n_1} \\ \dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(\boldsymbol{x}_2, \boldsymbol{w}), & \boldsymbol{x}_2 \in \mathbb{R}^{n_2}. \end{cases} \tag{12}$$

Suppose the $\boldsymbol{x}_1$-subsystem, with $(\boldsymbol{x}_2, \boldsymbol{w})$ as inputs, is input-to-state convergent and the $\boldsymbol{x}_2$-subsystem, with $\boldsymbol{w}$ as an input, is input-to-state convergent. Then, system (12) is input-to-state convergent.

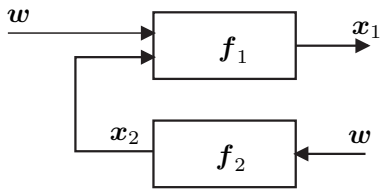

Fig. 1: Series connection of two systems with inputs.

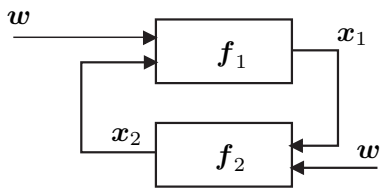

Fig. 2: Bidirectionally interconnected systems with inputs.

The next property deals with bidirectionally interconnected input-to-state convergent systems, see Figure 2.

Property 3 ([36, 37]). *Consider the system*

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{w}), & \boldsymbol{x}_1 \in \mathbb{R}^{n_1} \\ \dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{w}), & \boldsymbol{x}_2 \in \mathbb{R}^{n_2}. \end{cases} \tag{13}$$

Suppose the $\boldsymbol{x}_1$-subsystem with $(\boldsymbol{x}_2, \boldsymbol{w})$ as inputs is input-to-state convergent. Assume that there exists a class $\mathcal{KL}$ function $\beta_{x_2}(r, s)$ such that for any input $(\boldsymbol{x}_1, \boldsymbol{w}) \in \overline{\mathbb{PC}}_{n_1+d}$ any solution of the $\boldsymbol{x}_2$-subsystem satisfies

$$|\boldsymbol{x}_2(t)| \leq \beta_{x_2}(|\boldsymbol{x}_2(t_0)|, t - t_0).$$

Then the interconnected system (13) is input-to-state convergent.

Remark 1. Property 3 can be used for establishing the separation principle for input-to-state convergent systems. This will be used in Section 4 to design synchronising output feedback controllers. In that context, system (13) represents a system in closed loop with a state-feedback controller and an observer generating state estimates for this controller. The $\boldsymbol{x}_2$-subsystem corresponds to the observer error dynamics.

Now, we present sufficient conditions for exponential convergence and input-to-state convergence for the class of continuous PWA systems considered here. Consider a PWA system of the form:

$$\dot{\boldsymbol{x}} = \tilde{\boldsymbol{A}}_i \boldsymbol{x} + \tilde{\boldsymbol{b}}_i + \tilde{\boldsymbol{B}} \boldsymbol{w} \;\text{ for }\; \boldsymbol{x} \in \tilde{\Lambda}_i, \; i = 1, \ldots, l, \tag{14}$$

with $\tilde{\boldsymbol{A}}_i$, $\tilde{\boldsymbol{b}}_i$, $i = 1, \ldots, l$, and $\tilde{\boldsymbol{B}}$ some matrices of appropriate dimensions and $\tilde{\Lambda}_i$, $i = 1, \ldots, l$, are the polyhedral cells defined in Section 2.

Theorem 1 ([36],[38]). *Consider system (14) and assume that the right-hand side of (14) is continuous in $\boldsymbol{x}$. If there exists a positive definite matrix $\boldsymbol{P} = \boldsymbol{P}^T > 0$ such that*

$$\boldsymbol{P}\tilde{\boldsymbol{A}}_i + \tilde{\boldsymbol{A}}_i^T \boldsymbol{P} < 0, \quad i = 1, \ldots, l, \tag{15}$$

then system (14), with $\boldsymbol{w}$ as input, is exponentially convergent and input-to-state convergent.

In [38], the following technical lemma is proven, which will be used in the next section to construct output-feedback synchronising controllers for the master-slave system (1), (2). We denote the right-hand side of (14) by $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{w})$, i.e. $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{w}) := \tilde{\boldsymbol{A}}_i\boldsymbol{x} + \tilde{\boldsymbol{b}}_i + \tilde{\boldsymbol{B}}\boldsymbol{w}$ for $\boldsymbol{x} \in \tilde{\Lambda}_i,\ i = 1,\dots,l$.

Lemma 1 ([36],[38]). *Under the conditions of Theorem 1 it holds that*

$$(\boldsymbol{x}_1 - \boldsymbol{x}_2)^T\boldsymbol{P}(\boldsymbol{f}(\boldsymbol{x}_1,\boldsymbol{w}) - \boldsymbol{f}(\boldsymbol{x}_2,\boldsymbol{w})) \leq -\alpha(\boldsymbol{x}_1 - \boldsymbol{x}_2)^T\boldsymbol{P}(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{16}$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, $\boldsymbol{w} \in \mathbb{R}^d$, for some $\alpha > 0$ and for the matrix $\boldsymbol{P}$ satisfying (15).

4 State- and output-feedback design

Let us now propose a convergence-based design of synchronising controllers that avoid explicitly investigating the stability of the synchronisation error-dynamics and thus avoids a cumbersome stability analysis as illustrated in section 2. Since we take into account that, for both the master and the slave system, we do not have the entire state available for measurement, we will present an observer-based output-feedback control design.

The main idea of this convergence-based controller design is to find a controller that guarantees two properties:

a. the closed-loop slave system has a bounded solution along which the synchronisation error $(\boldsymbol{x}^s - \boldsymbol{x}^m(t))$ is identically zero,
b. the closed-loop slave system is uniformly convergent.

Condition **b** guarantees that any two solutions of the closed-loop slave system converge to each other (see Property 1). Together with condition **a**, this guarantees that all solutions of the closed-loop slave system converge to the bounded solution along which the synchronisation error is identically zero, i.e. the synchronisation control goal is achieved.

Let us first adopt the perspective that the entire state vectors of both systems can be measured, i.e. $\boldsymbol{C} = \boldsymbol{I}$ in (1) and (2), where $\boldsymbol{I}$ is an $n \times n$-identity matrix. Then, we propose the following synchronising control law for the slave system, incorporating a linear synchronisation error feedback law:

$$\boldsymbol{u}(\boldsymbol{x}^s, \boldsymbol{x}^m(t), \boldsymbol{u}^m(t)) = \boldsymbol{u}^m(t) + \boldsymbol{K}\left(\boldsymbol{x}^s - \boldsymbol{x}^m(t)\right), \tag{17}$$

with $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ a constant feedback gain matrix to be designed. The following lemma poses conditions (in the form of LMIs) under which asymptotic synchronisation is achieved with controller (17).

Lemma 2. *Consider the master-slave system (1), (2), with $\boldsymbol{C} = \boldsymbol{I}$, satisfying Assumptions 1 and 2. If the LMI*

$$\begin{aligned} \boldsymbol{\mathcal{P}}_c = \boldsymbol{\mathcal{P}}_c^T > 0, \\ \boldsymbol{A}_i\boldsymbol{\mathcal{P}}_c + \boldsymbol{\mathcal{P}}_c\boldsymbol{A}_i^T + \boldsymbol{B}\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T\boldsymbol{B}^T < 0,\ \ i = 1,\dots,l, \end{aligned} \tag{18}$$

is feasible, then system (2) with the controller (17), with $\boldsymbol{K} := \boldsymbol{\mathcal{Y}}\boldsymbol{\mathcal{P}}_c^{-1}$ *and* $\boldsymbol{x}^m(t)$ *and* $\boldsymbol{u}^m(t)$ *as inputs, is input-to-state convergent. Moreover, the synchronisation error* $(\boldsymbol{x}^s(t) - \boldsymbol{x}^m(t))$ *converges to zero as* $t \to \infty$.

Proof. The closed-loop slave system has the form

$$\dot{\boldsymbol{x}}^s = (\boldsymbol{A}_i + \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{x}^s + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}^m(t) - \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}^m(t) \quad \text{for} \quad \boldsymbol{x}^s \in \Lambda_i,\ i = 1, \ldots, l. \tag{19}$$

Since the right-hand side of system (2) is continuous, the right-hand side of the closed-loop slave system (19) is also continuous. Since the LMI (18) is feasible, for the matrix $\boldsymbol{K} := \boldsymbol{\mathcal{Y}}\boldsymbol{\mathcal{P}}_c^{-1}$ it holds that

$$\boldsymbol{\mathcal{P}}_c^{-1}(\boldsymbol{A}_i + \boldsymbol{B}\boldsymbol{K}) + (\boldsymbol{A}_i + \boldsymbol{B}\boldsymbol{K})^T\boldsymbol{\mathcal{P}}_c^{-1} < 0, \quad i = 1, \ldots, l.$$

Therefore, the closed-loop slave system (19) satisfies the conditions of Theorem 1 with the matrix $\boldsymbol{P} := \boldsymbol{\mathcal{P}}_c^{-1} > 0$. Hence, system (19) with $(\boldsymbol{u}^m(t), \boldsymbol{x}^m(t))$ as inputs is input-to-state convergent and exponentially convergent. The fact that $\boldsymbol{x}^s(t) \equiv \boldsymbol{x}^m(t)$ is a solution of the closed-loop slave system (19) implies, by Property 1, that $\boldsymbol{x}^s(t) - \boldsymbol{x}^m(t)$ converges to zero as $t \to \infty$. □

Remark 2. If system (19) would have an extra input $\boldsymbol{B}\boldsymbol{K}\boldsymbol{\Delta}\boldsymbol{x}(t)$, then under the conditions of Lemma 2 the closed-loop slave system is input-to-state convergent with respect to the inputs $\boldsymbol{x}^m(t)$, $\boldsymbol{u}^m(t)$ and $\boldsymbol{\Delta}\boldsymbol{x}(t)$. This fact will be used later on.

Remark 3. The closed-loop slave system (19) satisfies the conditions of Theorem 1, which by Lemma 1 implies that the quadratic Lyapunov function $V(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2}(\boldsymbol{x}_1 - \boldsymbol{x}_2)^T\boldsymbol{P}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ satisfies the inequality $\dot{V} \leq -2\alpha V$, for any two solutions $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ of the closed-loop slave system. By taking $\boldsymbol{x}_1 = \boldsymbol{x}_s$ and $\boldsymbol{x}_2 = \boldsymbol{x}_m$, one can easily see that V is a quadratic Lyapunov function of the form (7) for the synchronisation error-dynamics (6). Here, we have constructed this quadratic Lyapunov function using the convergence property. Still, the proof of the existence of such a Lyapunov function and the corresponding feedback law from expression (8) directly is significantly more difficult.

The linear static error-feedback design in (17) can easily be extended to more sophisticated linear dynamic error feedback laws, while LMIs similar to those in (18) can be formulated to guarantee input-to-state convergence and exponential convergence of the resulting closed-loop slave system.

Let us now turn to the case in which only the respective outputs $\boldsymbol{y}^m$, $\boldsymbol{y}^s$ of the master and slave systems are measured. We will propose an observer-based output-feedback design for the slave system. Here, we adopt an approach also taken in [4]. See Figure 3 for a block diagram of the entire controlled master-slave system. The first step is the design of observers for the (switching) PWA slave system (2). Note that the same observer design can be used for the master system, since the systems are identical. Hereto, we use an observer design proposed in the next lemma:

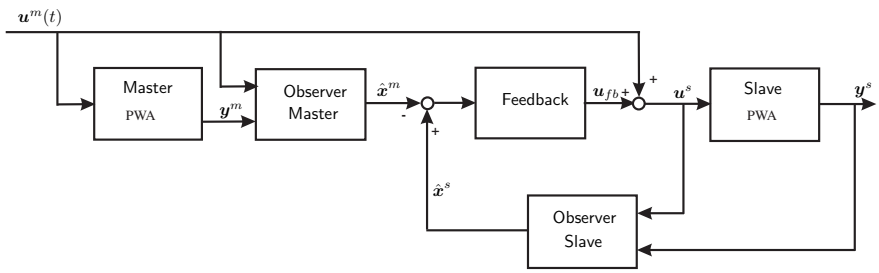

Fig. 3: Block diagram of the controlled master slave system.

Lemma 3. *Consider slave system (2) satisfying Assumption 1. If the LMI*

$$\begin{aligned} \boldsymbol{\mathcal{P}}_o = \boldsymbol{\mathcal{P}}_o^T > 0, \\ \boldsymbol{\mathcal{P}}_o \boldsymbol{A}_i + \boldsymbol{A}_i^T \boldsymbol{\mathcal{P}}_o + \boldsymbol{\mathcal{X}} \boldsymbol{C} + \boldsymbol{C}^T \boldsymbol{\mathcal{X}}^T < 0, \ i = 1, \dots l, \end{aligned} \tag{20}$$

is feasible, then the system

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}^s &= \boldsymbol{A}_i \hat{\boldsymbol{x}}^s + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}^s + \boldsymbol{L}(\hat{\boldsymbol{y}}^s - \boldsymbol{y}^s), \quad \hat{\boldsymbol{x}}^s \in \Lambda_i, \\ \hat{\boldsymbol{y}}^s &= \boldsymbol{C}\hat{\boldsymbol{x}}^s, \qquad i = 1, \dots, l, \end{aligned} \tag{21}$$

with $\boldsymbol{L} := \boldsymbol{\mathcal{P}}_o^{-1}\boldsymbol{\mathcal{X}}$*, is an observer for system (2) with globally exponentially stable error dynamics. The observer dynamics (21) is input-to-state convergent with respect to the inputs* $\boldsymbol{u}^s$ *and* $\boldsymbol{y}^s$*. Denote* $\boldsymbol{\Delta x}^s := \hat{\boldsymbol{x}}^s - \boldsymbol{x}^s$ *(the observer error). Moreover, the observer error dynamics*

$$\boldsymbol{\Delta}\dot{\boldsymbol{x}}^s = \boldsymbol{g}(\boldsymbol{x}^s + \boldsymbol{\Delta x}^s, \boldsymbol{u}^s) - \boldsymbol{g}(\boldsymbol{x}^s, \boldsymbol{u}^s), \tag{22}$$

where $\boldsymbol{g}(\boldsymbol{x}^s, \boldsymbol{u}^s) := \boldsymbol{A}_i\boldsymbol{x}^s + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}^s + \boldsymbol{L}\boldsymbol{C}\boldsymbol{x}^s$ *for* $\boldsymbol{x}^s \in \Lambda_i$*,* $i = 1, \dots, l$*, is such that for any bounded* $\boldsymbol{x}^s(t)$ *and any feedback* $\boldsymbol{u}^s = \boldsymbol{u}^s(\boldsymbol{\Delta x}^s, t)$ *all solutions of system (22) satisfy*

$$|\boldsymbol{\Delta x}^s(t)| \le c e^{-a(t-t_0)} |\boldsymbol{\Delta x}^s(t_0)|, \tag{23}$$

where the numbers $c > 0$ *and* $a > 0$ *are independent of* $\boldsymbol{x}^s(t)$ *and* $\boldsymbol{u}^s = \boldsymbol{u}^s(\boldsymbol{\Delta x}^s, t)$*.*

Proof. Let us first prove the second part of the lemma. Consider the function $\boldsymbol{g}(\boldsymbol{x}^s, \boldsymbol{u}^s)$. After unifying the terms containing $\boldsymbol{x}^s$, we obtain $\boldsymbol{g}(\boldsymbol{x}^s, \boldsymbol{u}^s) := (\boldsymbol{A}_i + \boldsymbol{L}\boldsymbol{C})\boldsymbol{x}^s + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}^s$ for $\boldsymbol{x}^s \in \Lambda_i$, $i = 1, \dots, l$. Since the right-hand side of system (2) is continuous, $\boldsymbol{g}(\boldsymbol{x}^s, \boldsymbol{u}^s)$ is also a continuous piecewise-affine function. Moreover, since the LMI (20) is feasible, for $\boldsymbol{P} := \boldsymbol{\mathcal{P}}_o$ and $\boldsymbol{L} := \boldsymbol{\mathcal{P}}_o^{-1}\boldsymbol{\mathcal{X}}$ it holds that

$$\boldsymbol{P}(\boldsymbol{A}_i + \boldsymbol{L}\boldsymbol{C}) + (\boldsymbol{A}_i + \boldsymbol{L}\boldsymbol{C})^T \boldsymbol{P} < 0, \quad i = 1, \dots, l.$$

Hence, by Theorem 1 system (21) is input-to-state convergent with respect to the inputs $\boldsymbol{u}^s$ and $\boldsymbol{y}^s$. Applying Lemma 1 to the function $\boldsymbol{g}(\boldsymbol{x}^s, \boldsymbol{u}^s)$, we obtain

$$\boldsymbol{\Delta x}^{s^T} \boldsymbol{P}(\boldsymbol{g}(\boldsymbol{x}^s + \boldsymbol{\Delta x}^s, \boldsymbol{u}^s) - \boldsymbol{g}(\boldsymbol{x}^s, \boldsymbol{u}^s)) \leq -a \boldsymbol{\Delta x}^{s^T} \boldsymbol{P} \boldsymbol{\Delta x}^s \tag{24}$$

for all $\boldsymbol{x}^s$, $\boldsymbol{\Delta x}^s$ and $\boldsymbol{u}^s$ and some constant $a > 0$ independent of $\boldsymbol{x}^s$, $\boldsymbol{\Delta x}^s$ and $\boldsymbol{u}^s$. Consider the function $V(\boldsymbol{\Delta x}^s) := 1/2 \boldsymbol{\Delta x}^{s^T} \boldsymbol{P} \boldsymbol{\Delta x}^s$. The derivative of this function along solutions of system (22) satisfies

$$\frac{dV}{dt} = \boldsymbol{\Delta x}^{s^T} \boldsymbol{P}(\boldsymbol{g}(\boldsymbol{x}^s + \boldsymbol{\Delta x}^s, \boldsymbol{u}^s) - \boldsymbol{g}(\boldsymbol{x}^s, \boldsymbol{u}^s)) \leq -2aV(\boldsymbol{\Delta x}^s).$$

This inequality, in turn, implies that there exists $c > 0$ depending only on the matrix $\boldsymbol{P}$ such that if $\boldsymbol{x}^s(t)$ is defined for all $t \geq t_0$ then the solution $\boldsymbol{\Delta x}^s(t)$ is also defined for all $t \geq t_0$ and satisfies (23). It remains to show that system (21) is an observer for system (2). Denote $\boldsymbol{\Delta x}^s := \hat{\boldsymbol{x}}^s - \boldsymbol{x}^s(t)$. Since $\boldsymbol{x}^s(t)$ is a solution of system (2), $\boldsymbol{\Delta x}^s(t)$ satisfies equation (22). By the previous analysis, we obtain that $\boldsymbol{\Delta x}^s(t)$ satisfies (23). Therefore, the observation error $\boldsymbol{\Delta x}^s$ exponentially tends to zero. □

It should be stressed once more that, since the master and the slave system are identical, the observer (21), with the output injection matrix $\boldsymbol{L}$ satisfying (20), is also an observer for the master system (of course using $\boldsymbol{y}^m$ instead of $\boldsymbol{y}^s$).

Note that this observer guarantees exponentially stable observer error dynamics and does not require knowledge on the moment of switching of the system. We note that if system (2) can be represented as a Lur'e system, one can also use the circle criterion-based observer design from [39], see also [40]. These observer designs are more general than (21). For general PWA systems, one can also extend the observer design in (21) based on the ideas from [39] and [40]; however, we will not pursue such an extension in this paper.

Lemma 2 shows how to design a state feedback controller which, based on $\boldsymbol{x}^m$ and $\boldsymbol{x}^s$, achieves the synchronisation goal. Lemma 3 provides observer designs to asymptotically reconstruct $\boldsymbol{x}^m$ and $\boldsymbol{x}^s$ from the measured outputs $\boldsymbol{y}^m$ and $\boldsymbol{y}^s$. In fact, a combination of these controller and observers gives us an output feedback synchronising controller as stated in the following theorem.

Theorem 2. *Consider the master-slave system (1), (2) satisfying Assumptions 1 and 2. Suppose the LMIs (18) and (20) are feasible. Denote* $\boldsymbol{K} := \boldsymbol{\mathcal{Y}}\boldsymbol{\mathcal{P}}_c^{-1}$ *and* $\boldsymbol{L} := \boldsymbol{\mathcal{P}}_o^{-1}\boldsymbol{\mathcal{X}}$. *Then, the slave system (2) in closed loop with the controller*

$$\begin{aligned}
\dot{\hat{\boldsymbol{x}}}^m &= \boldsymbol{A}_i \hat{\boldsymbol{x}}^m + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}^m + \boldsymbol{L}(\hat{\boldsymbol{y}}^m - \boldsymbol{y}^m), \;\; \hat{\boldsymbol{x}}^m \in \Lambda_i, \quad i = 1, \ldots, l, \\
\hat{\boldsymbol{y}}^m &= \boldsymbol{C}\hat{\boldsymbol{x}}^m, \\
\dot{\hat{\boldsymbol{x}}}^s &= \boldsymbol{A}_i \hat{\boldsymbol{x}}^s + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}^s + \boldsymbol{L}(\hat{\boldsymbol{y}}^s - \boldsymbol{y}^s), \;\; \hat{\boldsymbol{x}}^s \in \Lambda_i, \quad i = 1, \ldots, l, \\
\hat{\boldsymbol{y}}^s &= \boldsymbol{C}\hat{\boldsymbol{x}}^s, \\
\boldsymbol{u}^s &= \boldsymbol{K}\,(\hat{\boldsymbol{x}}^s - \hat{\boldsymbol{x}}^m) + \boldsymbol{u}^m,
\end{aligned} \tag{25}$$

is input-to-state convergent with respect to the inputs $\boldsymbol{y}^m(t)$ *and* $\boldsymbol{u}^m(t)$*. Moreover, the synchronisation error* $(\boldsymbol{x}^s(t) - \boldsymbol{x}^m(t))$ *converges to zero as* $t \to \infty$*, i.e. synchronisation is achieved asymptotically.*

Proof. In the coordinates $(\hat{\boldsymbol{x}}^m, \boldsymbol{\Delta x}^s, \boldsymbol{x}^s)$, the equations of the closed-loop slave system are given by the first equation in (25) and

$$\dot{\boldsymbol{x}}^s = (\boldsymbol{A}_i + \boldsymbol{BK})\boldsymbol{x}^s + \boldsymbol{b}_i + \boldsymbol{BK\Delta x}^s + \boldsymbol{Bu}^m - \boldsymbol{BK}\hat{\boldsymbol{x}}^m, \text{ for } \boldsymbol{x}^s \in \Lambda_i, \quad (26)$$

$$\boldsymbol{\Delta}\dot{\boldsymbol{x}}^s = \boldsymbol{g}(\boldsymbol{x}^s + \boldsymbol{\Delta x}^s, \boldsymbol{u}^s) - \boldsymbol{g}(\boldsymbol{x}^s, \boldsymbol{u}^s), \quad (27)$$

$$\boldsymbol{u}^s = \boldsymbol{K}\left(\boldsymbol{x}^s + \boldsymbol{\Delta x}^s - \hat{\boldsymbol{x}}^m\right) + \boldsymbol{u}^m, \quad (28)$$

with the function $\boldsymbol{g}$ as defined before. By the choice of the control gain $\boldsymbol{K}$ satisfying LMIs (18), system (26) with $(\boldsymbol{\Delta x}^s, \boldsymbol{u}^m, \hat{\boldsymbol{x}}^m)$ as inputs is input-to-state convergent (see Lemma 2). By the choice of the observer gain $\boldsymbol{L}$, for any bounded inputs $\boldsymbol{x}^s$, $\hat{\boldsymbol{x}}^m$, $\boldsymbol{u}^m$ and for the feedback law (28), any solution of system (27), (28) satisfies

$$|\boldsymbol{\Delta x}^s(t)| \leq ce^{-a(t-t_0)}|\boldsymbol{\Delta x}^s(t_0)|, \quad (29)$$

where the numbers $c > 0$ and $a > 0$ are independent of $\boldsymbol{x}^s(t)$, $\hat{\boldsymbol{x}}^m(t)$ and $\boldsymbol{u}^m(t)$ (see Lemma 3). Applying Property 3, we obtain that the system defined by (26), (27) and (28) is input-to-state-convergent with respect to the inputs $\boldsymbol{u}^m(t)$ and $\hat{\boldsymbol{x}}^m(t)$.

Next, we consider another interconnected system consisting of the series connection of the system defined by (26), (27) and (28) and the observer for the master system. By choice of the observer gain $\boldsymbol{L}$, the observer for the master system is input-to-state convergent with respect to the inputs $\boldsymbol{u}^m$ and $\boldsymbol{y}^m$. Therefore, by Property 2, this series connection is input-to-state convergent with respect to the inputs $\boldsymbol{u}^m$ and $\boldsymbol{y}^m$. Notice that $(\hat{\boldsymbol{x}}^m(t), \hat{\boldsymbol{x}}^s(t), \boldsymbol{x}^s(t)) = (\boldsymbol{x}^m(t), \boldsymbol{x}^m(t), \boldsymbol{x}^m(t))$ is a solution of the closed-loop slave system. It is bounded due to Assumption 2. Therefore, by Property 1, $(\hat{\boldsymbol{x}}^m(t), \hat{\boldsymbol{x}}^m(t), \boldsymbol{x}^s(t)) \to (\boldsymbol{x}^m(t), \boldsymbol{x}^m(t), \boldsymbol{x}^m(t))$ as $t \to \infty$ and, as a consequence, $\boldsymbol{x}^s(t) - \boldsymbol{x}^m(t) \to \boldsymbol{0}$ as $t \to \infty$, i.e. asymptotic synchronisation is achieved. □

Remark 4. The results in Theorem 2 for the master-slave synchronisation of two PWA systems can be readily exploited to design synchronising dynamic output-feedback controllers for an interconnected system of PWA systems, where the connectivity has a tree-like structure, see Figure 4. In that case, we address the problem of synchronising all systems by coupling (through active control) each system to its neighbour up in the tree. For the sake of brevity, we will omit a formal statement here, however, a sketch of the idea is given below. For every slave system in the tree, the control design can be decomposed into a feedforward part $\boldsymbol{u}_{ff}$ and a feedback part $\boldsymbol{u}_{fb}$. The feedback control design for every slave system in the tree involves a

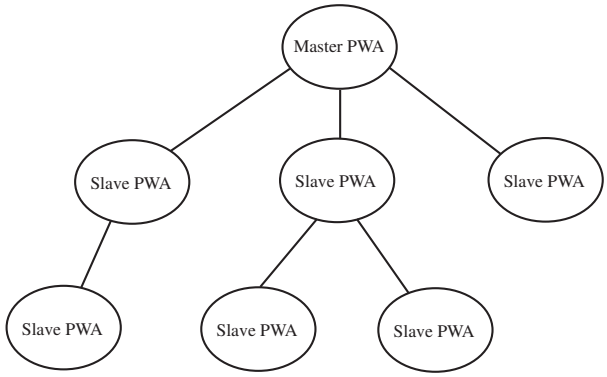

Fig. 4: Interconnected system of PWA systems with a tree-like connectivity structure.

linear synchronisation error feedback of the form $\boldsymbol{u}_{fb}^{s_i} = \boldsymbol{K}(\hat{\boldsymbol{x}}^{s_i} - \hat{\boldsymbol{x}}^m)$ for slave systems i directly coupled to the master system and $\boldsymbol{u}_{fb}^{s_j} = \boldsymbol{K}(\hat{\boldsymbol{x}}^{s_j} - \hat{\boldsymbol{x}}^{s_i})$ for any slave system j coupled to slave system i up in the tree (the gain matrices $\boldsymbol{K}$ and output injection matrices $\boldsymbol{L}$ may differ as long as the LMIs (18) and (20) are satisfied). This control design renders each closed-loop slave system input-to-state convergent. Note that the interconnected chains of closed-loop slave systems are also input-to-state convergent due to the fact that a series connection of two input-to-state convergent system is also input-to-state convergent, see Property 2. Consequently, all solutions of the total interconnected system converge to each other as $t \to \infty$. The feedforward designs for the slave systems should now ensure that this solution coincides with the synchronising solution. Suitable choices for the feedforward design are for example $\boldsymbol{u}_{ff}^{s_j} = \boldsymbol{u}^m(t)$ for all slave systems *or* $\boldsymbol{u}_{ff}^{s_i} = \boldsymbol{u}^m(t)$ for slave systems i directly coupled to the master system and $\boldsymbol{u}_{ff}^{s_j} = \boldsymbol{u}^{s_i}$ for any slave system j coupled to slave system i up in the tree (where $\boldsymbol{u}^{s_i}$ is the total control input for slave system i). Then, the closed-loop interconnected system exhibits a solution for which all systems are synchronised. Consequently, all closed-loop systems in the tree-like interconnection structure asymptotically synchronise.

5 An Illustrative Example

In this section, an example illustrating the theory presented in this paper is given. The example concerns the master-slave synchronisation of two identical PWA systems. Every system is a two-degree-of-freedom (2DOF) mechanical system with one-sided restoring characteristics as depicted in Figure 5. An engineering motivation for studying such models can be recognised in syn-

chronising control of wire bonders; in this context the mass-spring-damper combinations reflect the dynamics of wire bonder and the one-sided spring reflects the flexibility of the workpiece. Each system consists of two masses m_1 and m_2 interconnected by a linear spring-damper combination with stiffness k and damping coefficient c_1. Mass m_1 also experiences a damping force due to a linear damper attached to the fixed world with damping coefficient c_2. Moreover, mass m_2 experiences a gap $d > 0$ before hitting the one-sided spring with stiffness coefficient k_{nl}. Mass m_1 of the master system is driven by a time-dependent forcing $u^m(t) = A\sin(\omega t)$ while the slave system is actuated by a control force u^s. The displacements of masses m_1 and m_2 of the master system are denoted by z_1^m and z_2^m, respectively, and their respective velocities by $\dot{z}_1^m$ and $\dot{z}_2^m$. The displacements of the masses m_1 and m_2 of the slave system are denoted by z_1^s and z_2^s, respectively, and their respective velocities by $\dot{z}_1^s$ and $\dot{z}_2^s$. Moreover, for both systems only the position z_1 of mass m_1 is available for measurement. The master system can be written in the form (1) and the slave system can be written in the form (2), with $n = 4$, $l = 2$, $m = k = q = 1$ and the state vectors defined by $\boldsymbol{x}^i = \begin{bmatrix} z_1^i & \dot{z}_1^i & z_2^i & \dot{z}_2^i \end{bmatrix}$, $i \in \{m, s\}$. Moreover, $\Lambda_1^i = \{\boldsymbol{x}^i \mid x_3^i \geq d\}$, $\Lambda_2 = \{\boldsymbol{x}^i \mid x_3^i < d\}$, $i \in \{m, s\}$, $\boldsymbol{b}_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{k_{nl}d}{m_2} \end{bmatrix}^T$, $\boldsymbol{b}_2 = \boldsymbol{0}$, $\boldsymbol{B} = \begin{bmatrix} 0 & \frac{1}{m_1} & 0 & 0 \end{bmatrix}^T$, $\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$,

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k}{m_1} & \frac{c_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{c_1}{m_1} & -\frac{k+k_{nl}}{m_2} & -\frac{c_1}{m_2} \end{bmatrix}, \tag{30}$$

and

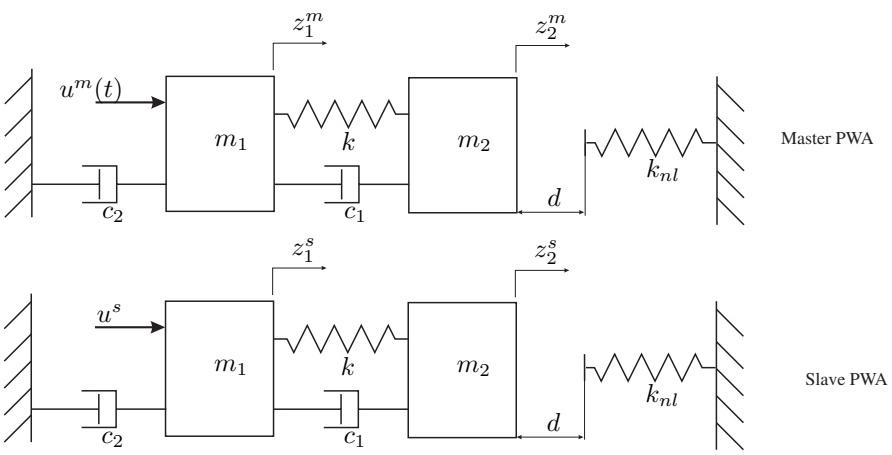

Fig. 5: Master-slave system consisting of two two-degree-of-freedom mass-spring-damper systems with one-sided restoring characteristics.

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k}{m_1} & \frac{c_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{c_1}{m_1} & -\frac{k}{m_2} & -\frac{c_1}{m_2} \end{bmatrix}. \tag{31}$$

Now we adopt the controller design as in (25), see Theorem 2. We adopt the following parameter setting: $m_1 = m_2 = 1$, $c_1 = c_2 = 2$, $k = k_{nl} = 10$, $d = 0.01$, $A = 0.02$ and $\omega = 1.5$. A solution to the LMIs (18) is represented by

$$\boldsymbol{\mathcal{P}}_c = \begin{bmatrix} 115.7.0 & -376.5 & 1.8 & 31.8 \\ -376.5 & 1395.9 & -17.1 & -226.8 \\ 1.8 & -17.1 & 51.7 & -65.8 \\ 31.8 & -226.8 & -65.8 & 782.8 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 111.5 & 30.1 & 19.4 & 7.4 \end{bmatrix} \tag{32}$$

and a solution to the LMIs (20) is represented by

$$\boldsymbol{\mathcal{P}}_o = \begin{bmatrix} 1575.2 & -158.6 & -737.6 & -24.2 \\ -158.6 & 69.1 & -140.5 & 13.9 \\ -737.6 & -140.5 & 1639.5 & -40.0 \\ -24.2 & 13.9 & -40.0 & 100.7 \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} 20.9676 \\ 110.1598 \\ 18.6509 \\ -4.4789 \end{bmatrix}. \tag{33}$$

Herewith, all conditions of Theorem 2 are satisfied and synchronisation of the systems is achieved. A simulation with the initial conditions $\boldsymbol{x}^m(0) = \begin{bmatrix} 0 & 0.01 & 0 & 0.01 \end{bmatrix}^T$ for the master system, $\hat{\boldsymbol{x}}^m(0) = \begin{bmatrix} 0.01 & 0.01 & 0.01 & 0.01 \end{bmatrix}^T$ for the observer of the master system, $\boldsymbol{x}^s(0) = \begin{bmatrix} 0.005 & 0 & 0.01 & 0.01 \end{bmatrix}^T$ for the slave system and $\hat{\boldsymbol{x}}^s(0) = \begin{bmatrix} 0 & 0 & 0 & -0.01 \end{bmatrix}^T$ for the observer of the slave system is performed. In Figures 6-9, the resulting time series for the states x_1 to x_4 of both the master and the slave system are compared. Moreover, in Figures 10 and 11,

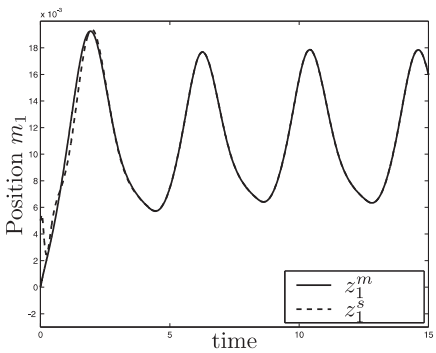

Fig. 6: Position of mass m_1 for master and slave systems: Synchronisation is achieved.

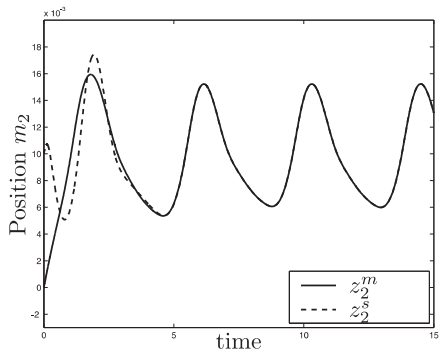

Fig. 7: Position of mass m_2 for master system and slave system: Synchronisation is achieved.

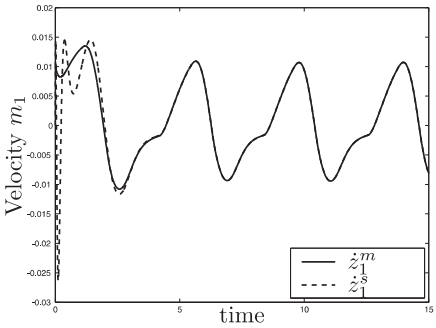

Fig. 8: Velocity of mass m_1 for master and slave systems: Synchronisation is achieved.

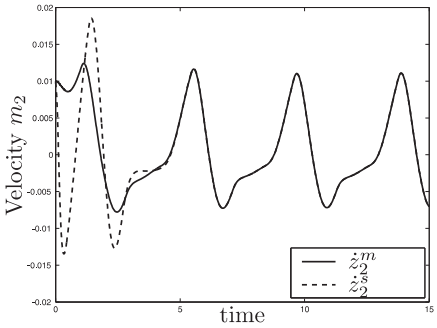

Fig. 9: Velocity of mass m_2 for master system and slave system: Synchronisation is achieved.

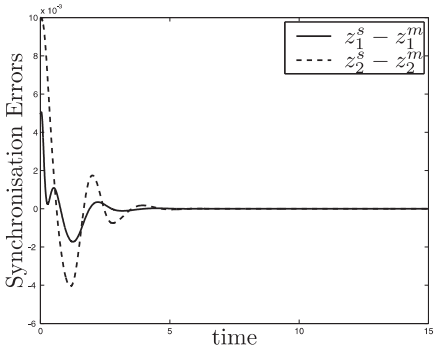

Fig. 10: Synchronisation errors in positions of masses m_1 and m_2.

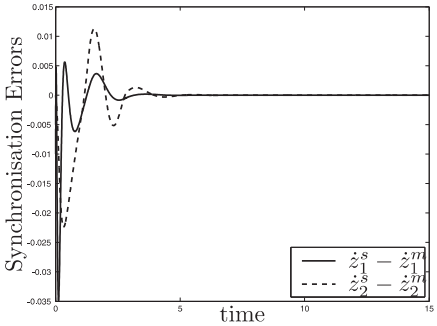

Fig. 11: Synchronisation errors in velocities of masses m_1 and m_2.

the corresponding synchronisation errors are displayed. Clearly, asymptotic synchronisation is achieved.

6 Conclusions

In this paper, the controlled synchronisation problem for identical continuous piecewise affine (PWA) systems is addressed. It is shown that due to the switching nature of these systems conventional strategies for controlled synchronisation, which are commonly based on stabilising the synchronisation error dynamics, lead to highly complex stabilisation problems. This complexity is due to, firstly, the fact that the error dynamics switches not only on the state of the slave system but also on the state of the master system, and, secondly, the fact that the error dynamics is time-dependent, where the

time-dependency is due to the time-dependent state trajectories of the master system (which are a priori unknown).

We consider the master-slave synchronisation problem for two PWA systems with an arbitrary number polyhedral cells. An observer-based output-feedback control design solving this problem is proposed. The design of these dynamic controllers is based on the idea of, on the one hand, rendering the closed-loop slave system convergent by means of feedback (which means that all its solutions converge to each other) and, on the other hand, guaranteeing that the closed-loop slave system has a bounded solution corresponding to zero synchronisation error. This implies that all solutions of the closed-loop slave system converge to this bounded solution with zero synchronisation error. This convergence-based approach avoids explicitly dealing with building up a stabilisation argument for the time-dependent switching synchronisation error dynamics.

This result can be used to address the controlled synchronisation problem for interconnected PWA systems with a tree-like structure. The results are illustrated by application to the master-slave synchronisation of two mechanical systems with one-sided restoring characteristics.

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