Extremum seeking for robust fuel-efficient control of diesel engines

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PROEFSCHRIFT

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Summary

Extremum seeking for robust fuel-efficient control of diesel engines

Over the past decades, pollutant emission of heavy-duty diesel engines, e.g., nitrogen oxides (NO_x: a mixture of NO and NO₂) and particulate matter (PM), has been significantly decreased, driven by increasingly stringent legislated maximum levels. Besides these pollutants, which affect human health and the environment, nowadays, the emission of CO₂ is subject to a legislated maximum as well, aimed at limiting the contribution of CO₂ to global warming.

The emission of CO_2 has a strong proportional relation with the brake specific fuel consumption (BSFC) of a diesel engine, and as a result, fuel-efficient control is key in reducing CO_2 emission. Nowadays, the emission levels are evaluated in real-world driving tests, in addition to the existing laboratory test cycles. This poses a robustness requirement on the engine control system with respect to real-world disturbances, such as varying ambient air conditions, varying fuel composition, production tolerances, component fouling and wear, which affect the engine dynamics and performance.

In state-of-the-art heavy-duty diesel engines, a large number of control inputs are present, e.g., injection settings and valves in the air path, which are used to affect controlled parameters that relate to, e.g., emissions and engine torque. Typically, the number of actuation degrees-of-freedom is larger than the number of controlled parameters, which is known as over-actuation. Over-actuation provides the possibility to allocate the actuation in such a way that the BSFC is minimal, while the controlled parameters are according to their references. As such, over-actuation enables fuel-efficient engine control.

Fuel-efficient engine control is however not straightforward. Suppressing the formation of NO_x in the engine, is contrary to achieving a low BSFC, i.e., there exists a BSFC-NO_x trade-off. As a result, fuel-efficient control comprises a multiple-objective optimization problem. In addition, safety constraints on the engine hardware should be accounted for and the control system should offer

robustness with respect to the aforementioned real-world disturbances.

In existing control approaches in industry and academics, the obtained fuel efficiency is based on offline experiments in an engine test cell. Either by tuning actuator position lookup tables, or by identifying models which are then used for online optimization. Being based on offline experiments, the robustness with respect to real-world disturbances of these approaches is limited. In practice, often a conservative tuning needs to be applied, to create a certain robustness margin at the cost of a reduced fuel efficiency. In addition, these offline approaches require a large calibration effort, i.e., a large development cost.

This thesis considers multiple-input extremum seeking (ES) for diesel engine control systems, to optimize fuel efficiency online. ES optimization only requires a measurement of the cost output, and assumes that the system subject to optimization is stable, and possesses a quasi-convex input-to-cost mapping in steady-state. As such, the online optimization does not rely on parametric system models or knowledge of disturbances that influence the cost output. As a result, ES fuel efficiency optimization is inherently robust with respect to realworld disturbances. An important challenge in application of multiple-input ES, which is addressed in this thesis, is output constraint handling. Furthermore, there is no straightforward approach to tune the ES controller parameters that result in high performance, in terms of convergence rate and accuracy. Finally, there exists interaction between the tracking and ES optimization objectives, i.e., for the actuator positions that yield minimal cost, the tracking objectives, related to NO_x emission and engine torque, are generally not satisfied.

This thesis provides two novel approaches to integrate ES in a diesel engine control system. The first approach is a cascaded structure where ES provides inputs to a low-level tracking control system. Besides a safety constraint on the peak in-cylinder pressure, additional constraints on the actuators and tracking performance are in place. These additional constraints are used to preserve tracking performance, under the aforementioned conflicting tracking and optimization objectives. An existing constrained ES approach is extended such that multiple output constraints can be handled. The second approach is a direct combination of tracking control and ES. The key element is a novel adaptive decoupling strategy, that decouples the ES optimization objective from the tracking objective. An analysis of ES parameter tuning is provided, in which an optimal dither perturbation frequency ratio is derived, for multiple-input ES. Furthermore, frequency-domain system identification results are used to derive the existence of a lower bound on the dither amplitude. The same result indicates that input-based ES outperforms dither-based ES, which is also demonstrated in a simulation example.

The proposed ES-based control approaches are successfully demonstrated in experiments on a heavy-duty Euro VI engine, or in simulations with a physicsbased engine model. In-cylinder pressure sensors are used on the Euro VI engine, to obtain a fuel efficiency equivalent cost output. In nominal test conditions, equal or better fuel efficiency is obtained in the experiments, compared to a baseline industrial controller. Considering that the baseline controller is optimized for nominal conditions, this is a promising result for non-nominal, real-world, application of the ES-based fuel-efficient control approaches.

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Chapter 1

Introduction

On December 12^{th} 2015, parties to the United Nations framework convention on climate change (UNFCCC) reached what is generally known as the Paris agreement. This agreement *Paris Agreement* 2015, aims to limit the average global ambient temperature increase to $1.5 \,^{\circ}\text{C}$ above pre-industrial levels. In order to reach this objective, a substantial reduction of the emission of greenhouse gasses (GHGs) is considered to be necessary in the upcoming decades.

One source of GHG emission, in particular of carbon dioxide (CO₂), is road transport by heavy-duty vehicles (HDVs), i.e., trucks, busses, and lorries. Although the long term objective is to develop zero well-to-wheel CO₂ emission road transport, nowadays reality is that long haul road transport by HDVs, relies on diesel engines for their propulsion. As a result, during the transition phase to zero emission road transport, there is a societal demand for reduced GHG emission of HDVs in the near future, which partly relies on increased engine efficiency. This demand is enforced by legislated constraints on the emission of CO₂ of HDVs, which target a significant decrease over the upcoming years. HDV CO₂ legislation is currently enforced in Japan, the United States (US), Canada, China, India, and the European Union (EU) Rodriguez, 2019.

In addition to CO_2 legislation for HDVs, all diesel powered vehicles are subject to legislated constraints on the emission of pollutants, which are harmful to human health and the environment. Pollutant legislation was first introduced in the US and the EU in approximately the year 1990, and has become increasingly stringent since then.

The targeted CO_2 reduction for the near future, requires, among other measures, an increased engine efficiency. The advanced engine technology, which is assumed to be necessary for the projected increase in engine efficiency, increases the number of control inputs to the engine. Thereby, maximizing the engine efficiency, while maintaining the emission of pollutants below the corresponding constraints, is increasingly challenging from a control perspective. The problem is especially challenging since nowadays, HDV emissions are not only evaluated in laboratory test cycles, but during real-world driving as well. Hence, robustness with respect to real-world disturbances is required, such as varying ambient air conditions, varying fuel composition, production tolerances, and component fouling.

To summarize, there exists a societal demand, which is enforced by increasingly stringent emission legislation, for increased HDV diesel engine efficiency, and near zero emission of pollutants. The corresponding control problem of providing optimal fuel efficiency, with robustness against real-world disturbances, while taking safety and pollutant constraints into account, is a challenging one, and will be discussed in more detail in the following sections.

1.1 Emission legislation for heavy-duty vehicles

Loosely speaking, the tailpipe emission of a diesel engine can be subdivided into GHGs and pollutants. The generally acknowledged theory, see, e.g., *Paris Agreement* 2015, is that the emission of GHGs causes global warming. Moreover, the emission of pollutants has a direct negative effect on the air quality, and thereby on human health and the environment.

1.1.1 Tailpipe pollutant emission

Commonly legislated pollutants for all diesel engines are carbon monoxide (CO), hydro carbons (HC), nitrogen oxides (NO_x) , and PM, which is also known as soot. The most recent legislation in the EU, EURO-VI, also prescribes a maximum particulate number (PN).

In particular suppressing the emission of NO_x poses a challenging control problem, since there exists a trade-off between low NO_x emission and high fuel efficiency. Both the allowed emission of NO_x (\approx factor 20) and PM (\approx factor 60) have been significantly decreased over time. Figure 1.1 depicts an overview of the maximum emission levels for NO_x and PM in different parts of the world.

The levels in Figure 1.1 are time averaged values, over legislated test cycles. Note that differences exist in the applied test cycles in different regions in the world, and between HDVs and passenger cars. Nowadays, laboratory test cycles are representative for real-world driving. However, there are numerous realworld disturbances, which are known to affect the performance of a diesel engine, but are constant in a laboratory test. Hence, the obtained laboratory performance is not necessarily obtained during real-world driving. Therefore, contrary to passenger cars, HDVs are subject to in-service conformity (ISC) checks using portable emission measurement system (PEMS) Mendoza-Villafuerte et al., 2017. Thereby, the engine is tested over its entire lifetime instead of only in new condition. Currently, the nominal EURO-VI limits are enforced, multiplied by a



Figure 1.1. Legislated maximum NO_x and particulate matter (PM) emission for heavy-duty trucks over the years, the lines starting from the year of introduction. The dots indicate the latest US, Japan, China, India, and EU legislation levels: EPA-2007 (NO_x 0.27, PM 0.013), and Japan (2016), China-VI, BS-VI, EURO-VI (NO_x 0.4, PM 0.01), respectively, in [g/kWh]. The data is taken from https://www.dieselnet.com/standards/.



Figure 1.2. Legislated CO_2 reduction targets for heavy-duty trucks, relative to a benchmark CO_2 emission that is determined at the time (indicated by dots) when the legislation has become active, i.e., 2019 for the EU target. Baseline levels are not equal for the different parts of the world and evaluation of the legislation also differs. The data is taken from Rodriguez, 2019.

factor 1.5, during ISC checks. In the US, a slightly different approach is applied with "not to exceed" limits that are evaluated during real-world driving.

1.1.2 CO₂ emission

The emission of CO_2 has, opposite to pollutants, a strong proportional correlation with the fuel consumption of the engine. Hence, reducing CO_2 emission is economically attractive, up to the point where the required investment is higher than the reduction in fuel cost. To enforce manufacturers of HDVs beyond this point, legislated CO_2 emission reduction targets are in place.

In early 2019, an agreement on CO_2 emission reduction for heavy-duty vehicles has been reached in the EU Rodriguez, 2019. Such CO_2 legislation has been in place for several years in different parts of the world. Figure 1.2 depicts CO_2 reductions relative to corresponding baseline levels. The baseline levels are not equal, since they are based on the respective regions newly sold vehicle fleet at the time of introduction, indicated with dots in Figure 1.2. Comparing the targeted reductions is however not the objective here. The key observation from Figure 1.2 is that, currently and in the near future, the reduction of CO_2 emission of HDVs is enforced by legislation in a large part of the world. Increased fuel efficiency contributes to this targeted CO_2 reduction.

1.2 The diesel engine control problem

The societal demand for clean and efficient HDV diesel engines, enforced by the legislation discussed in Section 1.1, poses a challenging problem which is formalized in this section. First, the engine is introduced from a control perspective. Second, the control problem is made specific.

1.2.1 Diesel engine system description

Figure 1.3 depicts a schematic representation of the considered type of state-ofthe-art six cylinder HDV diesel engine with its exhaust after-treatment system (EAS). The control inputs to the engine are commonly associated with the airpath and the fuel-path.

Air-path

The air-path comprises the turbocharger, which consists of a compressor with intercooler, driven by a variable geometry turbine (VGT), and the high-pressure, cooled, exhaust gas recirculation (EGR) system, see Figure 1.3.

The turbocharger increases the intake air density such that more fuel can be injected, thereby yielding an increased power output for the same displacement volume. As a result, the relative friction losses in the engine are reduced.

EGR dilutes the intake air with exhaust gas. The relative amount of exhaust gas in the intake manifold is defined as the EGR fraction X_{egr} [%]. By increasing X_{egr} , the combustion temperature, locally in the cylinder, is reduced, which



Figure 1.3. Schematic layout of a state-of-the-art heavy-duty diesel engine with exhaust after-treatment system (EAS). The magenta blocks indicate outputs that are used for control in this thesis.

significantly reduces NO_x formation. EGR however, leads to an increased fuel consumption for the same output power, i.e., BSFC [g/kWh], because: (1) The engine needs to provide pumping work to create the EGR flow, which is known as pumping-loss, (2) less energy is available for the VGT, and (3) the changed gas composition may result in a reduced thermal efficiency. As such, application of EGR induces a BSFC-NO_x trade-off. In addition, a high X_{egr} may lead to excessive PM emission, since it reduces the air-to-fuel equivalence ratio λ [-].

The considered air-path actuators are the EGR valve position [%] and the VGT position [%], see Figure 1.3. Both actuators affect dp [kPa], given by

$$dp = p_{ex} - p_{in},\tag{1.1}$$

where p_{ex} [kPa] and p_{in} [kPa] are the pressure in the exhaust and intake manifolds on the engine, respectively. The combination of dp and the EGR valve position results in a certain X_{egr} . While using these actuators for control, a safety constraint on the compressor rotational speed n_t [rpm] should be considered.

Fuel-path

The fuel-path comprises the common rail system with cylinder individual injectors, see Figure 1.3. Fuel-path control is commonly done on a cycle-to-cycle base, where cycle refers to a four-stroke diesel cycle. In cycle-to-cycle control, the fuel injection profile, which prescribes the fuel-injection pulses as a function of the crank angle [°CA], is determined before the compression and combustion strokes take place. Then, during the exhaust and intake strokes, the fuel injection profile for the next cycle is determined. The injection profile typically starts with a small "pilot" injection, after which the main injection takes places. The rail pressure can be used as an additional fuel-path control input. In this thesis, the duration and start of injection (SOI) of the main injection pulse for each cylinder are considered as the fuel-path actuation inputs.

Fuel-path actuation affects the heat release profile over the crank angle, and thereby the combustion efficiency, the output power, and the emission of pollutants. As such, the BSFC-NO_x trade-off is also affected by fuel-path actuation. A safety constraint is in place for the peak value of the in-cylinder pressure p_{cyl} [bar] over one cycle. In addition, the pressure rise rate $\frac{dp_{cyl}}{dCA}$ [bar/°CA] is upper constrained in some cases as it relates to engine noise.

Exhaust after-treatment system

The EAS, see again Figure 1.3, reduces the engine-out pollutant emission to the legislated tailpipe-out levels. The diesel oxidation catalyst (DOC) is used for oxidation of the incomplete combustion products CO and HC, and to convert NO into NO₂. The diesel particulate filter (DPF) is used to reduce the PM emission level. The selective catalytic reduction (SCR) converts NO_x emission to N₂ and H₂O, by injection of urea, which is a solution of ammonia in water. A possible excess of ammonia resulting from the SCR, is removed by the ammonia oxidation (AMOX) catalyst. Due to the required injection of urea, reducing NO_x emission with the SCR increases the operating cost of the engine. Moreover, the effectiveness of the SCR is temperature dependent. Therefore, to optimize the combined cost of fuel and urea and to guarantee that the tailpipe-out NO_x emission is within limitations, the engine-out NO_x emission needs to be suppressed to some extent, by EGR and appropriate fuel injection timing.

1.2.2 Diesel engine control problem

The high-level diesel engine control objective is to provide power, with a minimal BSFC, while satisfying the legislated pollutant emission limits and safety constraints on the engine hardware. In addition, the control system needs to provide robust performance with respect to real-world disturbances. This section formalizes the control problem and specifies the control signals.

Control objective

Figure 1.4 depicts the considered control structure at a high level. As introduced in the previous section, the emission of pollutants can be suppressed both in the



Figure 1.4. Schematic layout of the considered high-level engine control structure.

engine and the EAS. Given the engine BSFC-NO_x trade-off, and the fact that reducing NO_x emission by using the SCR increases the operating cost and is not always possible, it is not trivial to determine the engine-out NO_x reference r_{NO_x} that results in minimal operating cost, while the tailpipe-out NO_x emission is within the legislated limits. In practice, r_{NO_x} is obtained from an optimal supervisory controller, see, e.g., Donkers et al., 2017, indicated as $C_{\text{supervisory}}$ in Figure 1.4. The EAS controller C_{EAS} reduces the engine-out NO_x emission to the legislated tailpipe-out requirement by adjusting urea injection in the SCR.

The focus of this thesis is the engine control problem, indicated by the grey box in Figure 1.4. The performance parameter vector z is given by

$$\boldsymbol{z} = \begin{bmatrix} M_e & NO_x & \text{BSFC} & \boldsymbol{h}^\top \end{bmatrix}^\top,$$
 (1.2)

where M_e [Nm] is the delivered brake torque, NO_x is the engine-out NO_x emission [g/kWh], the BSFC is as introduced before, and \boldsymbol{h} is a vector of parameters that are subject to constraints. The corresponding control objective is summarized as

$$\min(BSFC)$$
 s.t. $h \le 0$, (1.3a)

$$M_e \to M_{e,dem},$$
 (1.3b)

$$NO_x \to r_{NO_x},$$
 (1.3c)

where (1.3b) and (1.3b) are tracking objectives of M_e and NO_x to their respective reference signals $M_{e,dem}$ and r_{NO_x} . Examples of constrained parameters in h are the peak in-cylinder pressure p_{peak} and the turbine speed n_t .

In Figure 1.4, the real-world disturbances are indicated by the vector \boldsymbol{w} . Examples are varying ambient conditions, varying fuel composition, production tolerances, and component fouling and aging. As such, besides actual disturbances, the vector \boldsymbol{w} also describes system uncertainty. It is important to note that not all real-world disturbances are measured, their effect on the engine performance is generally not exactly known, and that real-world disturbances are typically slowly varying compared to the engine dynamics. The ambient air pressure while driving uphill is an example of a fast varying real-world disturbance, which is orders of magnitude slower than variation in torque demand $M_{e,dem}$.

The vector \boldsymbol{u} in Figure 1.4 contains the control inputs, of which some are already introduced in the previous section. The vector \boldsymbol{y} contains measured and estimated parameters, which are available for feedback control, and as such are referred to as "controlled parameters". Typically, not all the performance parameters in \boldsymbol{z} are available as controlled parameter in \boldsymbol{y} .

Often, the number of control inputs is larger than the number of tracking parameters in z (M_e and NO_x). As a result, the demanded M_e and NO_x can be obtained using different combinations of actuator positions, i.e., the number of actuation degrees-of-freedom (DOFs) is larger than the number of tracking objectives. This is known as "over-actuation", and can often be exploited to optimize a cost output of the system, i.e., the BSFC.

The diesel engine control objective considered in this thesis, concerning the engine controller C_{engine} in Figure 1.4, is formulated as follows.

Diesel engine control objective: Using the available inputs in \boldsymbol{u} and controlled parameters in \boldsymbol{y} , the engine control objective is to satisfy (1.3), robust with respect to, possibly unknown, real-world disturbances in \boldsymbol{w} .

A detailed overview of the control inputs u and controlled parameters y is provided in the following sections.

Control inputs

The considered air-path control inputs are the EGR valve position and the VGT position, similar to Wahlström et al., 2010; Tschanz et al., 2013b; Criens et al., 2015; Salehi et al., 2018. Different engine layouts exists however, e.g., with additional valves to affect the air-path dynamics, such as a back pressure valve (BPV), which is located downstream the VGT, a throttle valve, swirl valves in the intake manifold, or a by-pass channel for the turbine, a so-called "waste gate" which enables application of a fixed geometry turbine instead of a VGT.

As mentioned in Section 1.2.1, the fuel-path has many actuation DOFs, see Luo et al., 2018b, where multi-pulse fuel injection is considered. In this thesis, fuel-path actuation is limited to the SOI and the injection duration of the main injection pulse, similar to Tschanz et al., 2014.

To summarize, the considered control inputs are

$$\boldsymbol{u} = \begin{bmatrix} u_{egr} & u_{vgt} & \boldsymbol{u}_{SOI}^\top & \boldsymbol{u}_{dur}^\top \end{bmatrix}^\top,$$

where u_{egr} is the EGR value position, u_{vgt} is the VGT position, and the cylinder individual SOI and duration parameters are contained in the vectors u_{SOI} ,



Figure 1.5. A schematic p-V diagram of a four-stroke diesel cycle over 720 °CA. Indicated along the curve are the exhaust valve closing (EVC), intake valve opening (IVO), intake valve closing (IVC), start of injection (SOI), end of injection (EOI), and the exhaust valve opening.

 $u_{dur} \in \mathbb{R}^{n_{cyl} \times 1}$, with n_{cyl} the number of cylinders. Observe that the number of control inputs in u is larger than the number of tracking outputs in z, i.e., as mentioned in the previous section, over-actuation is available.

In-cylinder pressure-based estimates

The controlled parameters in \boldsymbol{y} , are summarized in relation to the performance parameters in \boldsymbol{z} , in the next section. Before doing so, this section introduces parameters that can be obtained from in-cylinder pressure p_{cyl} sensors and a crank angle encoder. Although these sensors are not yet the industry standard for HDVs, their potential is studied in the literature; Eriksson and Thomasson, 2017; Willems, 2018 provide an overview.

Given the engine geometry, the crank angle is directly related to the incylinder volume $V_{cyl} \in [V_{tdc}, V_{bdc}]$ [m³], where V_{tdc} and V_{bdc} are the volume at top dead centre (TDC) and bottom dead centre (BDC) of the piston, respectively. As such, p_{cyl} can be expressed as a function of V_{cyl} in a p-V diagram. Figure 1.5 depicts a schematic p-V diagram for a four-stroke diesel cycle.

The surface in the p-V contour is the net indicated mean effective pressure (IMEP) [bar], which is defined as

$$IMEP_n := \frac{1}{V_d} \oint p_{cyl} \mathrm{d}V_{cyl},\tag{1.4}$$

where $V_d = V_{bdc} - V_{tdc}$ [m³] is the displacement volume of one cylinder, see also Heywood, 1988. The net IMEP is commonly subdivided in the gross IMEP [bar] and the pumping mean effective pressure (PMEP) [bar]:

$$IMEP_n = IMEP_{qr} + PMEP_{rr}$$

The gross IMEP is the positive contribution that is obtained during the compression and combustion strokes, while the PMEP is the negative contribution that is obtained during the exhaust and intake strokes, see also Figure 1.5. The PMEP is correlated to the EGR induced pumping loss. The net IMEP is related to the brake engine torque M_e [Nm] Eriksson and Nielsen, 2014, Equation (4.5):

$$M_e = \frac{V_d n_{cyl}}{4\pi} \underbrace{(IMEP_n - FMEP)}_{BMEP} \cdot 10^5 \tag{1.5}$$

with the friction mean effective pressure (FMEP) [bar], and the break mean effective pressure (BMEP) [bar], and where the scaling factor 10^5 is used for unit conversion between [Pa] and [bar].

Besides the IMEP, the heat release can be determined from the measured p_{cyl} and crank angle. Subsequently, combustion phasing parameters can be derived, e.g., CA_{50} [°CA], which is the crank angle relative to TDC, at which half of the heat that is present in the injected fuel, is released, see, e.g., Heywood, 1988.

The IMEP and the heat release are derived a posteriori, using p_{cyl} and crank angle data, obtained during the compression and combustion strokes. In Wilhelmsson et al., 2006, a recursive calculation approach is proposed.

The in-cylinder pressure can also be used to estimate the engine-out NO_x emission. Such a NO_x estimate avoids the common drawbacks of a NO_x sensor, which has slow dynamics, suffers from a transport delay, and in addition requires a certain exhaust gas temperature to be operational. In Asprion et al., 2013; Mentink et al., 2017, a physics-based approach, while in Formentin et al., 2014 a data-based approach is proposed. In Hametner et al., 2014, a local model network is applied to maintain a physics-motivated structure in the NO_x model. However, since the correlation between p_{cyl} and the formation of NO_x is sensitive to real-world disturbances, practical application of estimated NO_x is challenging.

Controlled parameters

The controlled parameter vector \boldsymbol{y} contains measured and estimated parameters, which are used for control. In general, not all the performance parameters in \boldsymbol{z} are available as controlled parameters in \boldsymbol{y} . This section discusses the controlled parameters in \boldsymbol{y} , see Figure 1.4, in relation to each of the performance parameters in the vector \boldsymbol{z} in (1.2). An overview of the controlled parameters considered throughout this thesis is provided in Table 1.1.

Table 1.1. The controlled parameters in the vector \boldsymbol{y} in Figure 1.4, based on corresponding sensors and estimates in the second and third columns, respectively.

Controlled Parameter	Sensors	Estimates (based on)
p_{peak} [bar]	p_{cyl}	-
$IMEP_n$ [bar]	p_{cyl}, CA	-
CA_{50} [°CA]	p_{cyl}, CA	-
$dp \; [\mathrm{kPa}]$	p_{in}, p_{ex}	-
λ [-]	O_2 [ppm]	$\dot{m}_f(u_{dur})$
NO_x [g/kWh]	NO_x [ppm], n_e	$M_e(IMEP_n), \dot{m}_{air}(n_e)$
X_{egr} [%]	-	$\dot{m}_{egr},\dot{m}_{air}(n_e)$
BSFC [g/kWh]	$n_e \ p_{cyl}, \ CA$	$M_e(IMEP_n), \dot{m}_f(\boldsymbol{u}_{dur}, \boldsymbol{u}_{SOI}, p_{cyl}, p_{rail})$

Contrary to the engine speed n_e [rpm], the engine torque M_e is generally not measured. When in-cylinder pressure sensors are available, M_e can be estimated using the net IMEP, see (1.5), and an estimate of FMEP. Alternatively, a correlation with the injection duration u_{dur} is sometimes used.

The engine-out NO_x concentration [ppm] is commonly available as measurement in production type engines. In Tschanz et al., 2014, NO_x concentration is used for feedback control. Contrarily, the controlled parameter in Criens et al., 2015 and this work is the *specific* engine-out NO_x in [g/kWh], which requires NO_x in [ppm] and estimates of the fresh air mass flow \dot{m}_{air} [g/h] and the engine power. Alternatively, a correlation between the EGR fraction X_{egr} and the formation of NO_x is used for feedback, see, e.g., Wahlström et al., 2010; Salehi et al., 2018. The correlation between combustion phasing and NO_x formation can be used, e.g., via CA_{50} . In Luo et al., 2018b, in addition to CA_{50} , the CA_{10} , CA_{30} , and CA_{70} are considered.

Besides the engine torque M_e , the fuel mass flow \dot{m}_f [g/h] is typically not measured in the vehicle either. Hence, the BSFC, in [g/kWh], cannot be measured. In Kupper et al., 2018, a BSFC estimation approach is proposed, which uses a combination of the net IMEP to estimate M_e , and a \dot{m}_f estimate based on u_{dur} . Manifold pressure sensors are commonly available, and can be used to obtain dp in (1.1). A correlation between dp and the BSFC can be used when in-cylinder pressure sensors are absent. The PMEP is approximated by dp as

$$\alpha \cdot dp \approx -PMEP,$$

where $\alpha = 10 \cdot 9.81$ is a unit conversion scaling between [bar] and [kPa]. The EGR induced pumping-loss PMEP, which is introduced in Section 1.2.1, is a loss of energy. As such, dp can be used as an inferred cost parameter for BSFC, see, e.g., Wahlström and Eriksson, 2013. In Wahlström and Eriksson, 2013, a lower constraint on λ is in place to preserve combustion efficiency and to avoid excessive soot formation. The value of λ is determined using an engine-out exhaust gas oxygen concentration measurement, and an estimate of the fuel mass flow \dot{m}_f .

The mentioned safety constraint on the peak in-cylinder pressure p_{peak} [bar] is directly obtained from p_{cyl} . Additional sensors which are commonly available, but not considered for control in this work are, e.g., turbine speed n_t and temperature sensors.

1.3 Diesel engine control for high fuel efficiency

The diesel engine control objective, introduced in Section 1.2.2, poses a challenge since the engine is a nonlinear multiple-input-multiple-output (MIMO) system, the effect of real-world disturbances on the engine performance is not exactly known, and it comprises a combined tracking and optimization objective. This section discusses the main classes of control approaches that are proposed in the literature. In particular the obtained fuel-efficiency, i.e., the inverse BSFC, compared to the optimal fuel-efficiency, and the robustness of the optimization with respect to real-world disturbances are considered. In this perspective, the section is concluded by a motivation for the application of ES for fuel-efficient diesel engine control.

1.3.1 Lookup table-based feedback and feedforward control

The industrial standard in diesel engine control comprises feedforward and feedback control. Figure 1.6 depicts a general schematic representation. The feedforward controller C_{ff} and the reference governor (RG) are based on interpolation of lookup tables, which are obtained by extensive offline calibration in an engine test cell, aimed at optimal performance of the parameters in z in (1.2). Being based on offline calibration, lookup table-based control cannot provide robust performance with respect to real-world disturbances that are unknown, or not being measured. In fact, to provide the necessary robustness for real-world application, optimal performance is generally comprised.

Increased real-world performance of the industrial standard is possible by feedback control of the available controlled parameters. For example, in Luo et al., 2018b, a multivariable proportional-integral (PI) controller with off-diagonal terms is proposed, for cycle-to-cycle multi-pulse fuel injection control of CA_{30} , CA_{50} , and $IMEP_n$. Although the corresponding references remain based on offline calibration, the correlation between the controlled parameters \boldsymbol{y} and the performance \boldsymbol{z} is less sensitive to the real-world disturbances \boldsymbol{w} than a feedforward approach. In fact, closed-loop control of the net IMEP, enables tracking of the torque demand $M_{e,dem}$.

Closed-loop control of NO_x concentration is presented in Tschanz et al., 2013b, Criens et al., 2015. In Tschanz et al., 2014, in addition the emission of PM is controlled in closed loop, using a combined air-path and fuel-path



Figure 1.6. General control system schematics of classical engine control. The reference governor (RG) and the feedforward controller C_{ff} are based on lookup tables, and provide, respectively, the reference signal \mathbf{r} and feedforward \mathbf{u}_{ff} , as a function of the demanded torque $M_{e,dem}$, the reference engine-out NO_x level r_{NO_x} , and the measured engine speed n_e .

controller, which consists of a cascaded structure with a multivariable \mathcal{H}_{∞} fuelpath controller, and a diagonal PI type air-path controller.

In Wahlström et al., 2010; Criens et al., 2015; Salehi et al., 2018, dp, which is correlated to the BSFC, is controlled in feedback, using a diagonal PI type controller, with static decoupling, while Tschanz et al., 2013b proposes a modelbased feedback controller. Determining the required reference for dp that results in a minimal BSFC, which is feasible given the additional requirements on the performance parameters in the vector z in (1.2), is not trivial. Being based on offline calibration, the dp reference cannot guarantee a minimal BSFC in real-world application.

In summary, although robustness can be increased by closed-loop control, lookup table-based control suffers from the inherent drawback that offline calibrations cannot guarantee robust performance of all parameters in z in (1.2), with respect to all possible real-world disturbances. Hence, the obtained BSFC is generally not minimal in real-world application.

1.3.2 Model predictive control

As discussed in Section 1.3.1, the obtained BSFC with a lookup table-based control approach is typically not minimal in real-world application. Economic model predictive control (MPC) is a model-based control approach that offers the possibility to optimize a cost output of a system online, i.e., during operation, by adding the economic cost to the MPC stage cost. For example, a BSFC estimate or correlated parameter, such as dp, can be optimized online in MPC. Thereby, a sub-optimal BSFC, as a result of real-world disturbances can be avoided. In addition, input and output constraints can be handled in MPC, by solving a constrained optimal control problem at each time step.

Figure 1.7 depicts a general schematic representation of engine control with economic MPC. The map $f_J : \boldsymbol{y} \to J$ represents a BSFC estimation, or a selection of BSFC correlated parameters from \boldsymbol{y} .



Figure 1.7. Control system schematics of an economic MPC engine controller.

In Stewart and Borrelli, 2008; Karlsson et al., 2010, constraint handling of MPC is demonstrated for a diesel engine control system. The MPC framework in Stewart and Borrelli, 2008, considers a set of linear time-invariant (LTI) models that correspond to specific engine operating points, i.e., the combinations of engine speed n_e and load M_e . In Wahlström and Eriksson, 2013; Gelso and Dahl, 2016; Gelso and Dahl, 2017, MPC is applied for control using the air-path actuators, with the dp as economic cost in the stage cost. A lower constraint on λ is in place to preserve combustion efficiency and prevent excessive PM emission. In Broomhead et al., 2017, a fuel flow estimate is considered as economic cost, and constraints are included on the emission of NO_x and PM.

Economic MPC offers online fuel efficiency optimization, both in steady state and during transients. However, a dynamic model of the engine is required to predict the fuel efficiency parameter J(t), over the prediction horizon as a function of the inputs u(t). Hence, the robustness of the obtained fuel efficiency with respect to real-world disturbances depends on the accuracy of the parametric model. As such, a mismatch between the optimum of the MPC stage cost and the true optimal fuel efficiency may result. Considering the nonlinear engine dynamics and constraints, MPC requires real-time numerical optimization, which is not trivial on a standard engine control unit (ECU).

1.3.3 Extremum seeking for robust fuel-efficient control

Given the drawbacks of lookup table-based control and economic MPC for diesel engines, there is a need for fuel-efficient engine control algorithms, that are robust with respect to, possibly unknown, real-world disturbances, do not require parametric models, and are implementable on a standard ECU.

ES is a data-driven optimization approach, that can be used online, and does not require a parametric model of the system or knowledge of the disturbances, see, e.g., Tan et al., 2010 for an overview of ES and examples of applications. The only requirements are that, the cost output can be measured, and that the system is stable and possesses a "quasi-convex" steady-state mapping from the input \boldsymbol{u} to the cost J. As such, ES provides robustness of the obtained optimum



Figure 1.8. General extremum seeking (ES) engine control layout. The reference signal, feedforward, or a combination of the two, is modified by ES.

by design.

The considered integration of ES in existing engine tracking control approaches is schematically depicted in Figure 1.8. As in the MPC schematics in Figure 1.7, the map $f_J: \mathbf{y} \to J$ represents a BSFC estimation, or a selection of BSFC correlated parameters from \mathbf{y} . Two types of ES inputs to the control system are suggested in Figure 1.8. An example of adjusting the feedforward signal u_{ff} is presented in Großbichler et al., 2016, where two fuel injection profile parameters are used as ES input. An example of adjusting the reference signal \mathbf{r} is presented in Lewander et al., 2012, where the CA_{50} reference signal to a low-level closed-loop control system is used as ES input. In both examples the cost J is a fuel efficiency estimate.

Examples exist of ES applied in control of different engine types, with the same practical motivation. See, e.g., Scotson and Wellstead, 1990, and more recently Hellström et al., 2013; Mohammadi et al., 2014; Ramos et al., 2017, where online spark timing optimization in spark ignition (SI) engines is considered. In Popovic et al., 2006, ES control for variable cam timing tuning is considered. In Killingsworth et al., 2009, optimization of combustion timing is considered for homogeneous charge compression ignition (HCCI) engines. In Sharafi et al., 2014, the optimal injection duration for a compressed natural gas engine is determined using ES to deal with fuel composition uncertainty. All these examples concern adjusting the feedforward u_{ff} , see Figure 1.8. In Tschanz et al., 2013a, a similar idea as ES is applied to online adapt the X_{egr} reference lookup table, to balance the fuel-path and air-path control effort to control NO_x.

In summary, ES can be used to obtain a robust, fuel-efficient control approach, that is less demanding than MPC in terms of ECU implementation, and does not require parametric models. In addition, ES is complementary to existing lookup table-based control approaches. When combined with the control design in Criens et al., 2015, which is based on loopshaping Franklin et al., 2015 using non-parametric frequency response function (FRF) measurements Pintelon and Schoukens, 2012, a fully data-based control design is obtained.

Given these advantages, this thesis explores possibilities to apply ES for fuel-

efficient control of diesel engines. The application poses the following practical engine-specific challenges.

- C1 Fuel efficiency equivalent cost output design: Since a BSFC measurement is not available in production type heavy-duty diesel engines, it is not trivial how to obtain a cost output J, based on the available measurements in \boldsymbol{y} , which has a robust correlation with the actual BSFC. Ideally, the cost J is insensitive to disturbances that are faster than the ES adaptation rate, e.g., fast variations in the engine operating point.
- **C2** Input selection for fuel efficiency optimization: Given the fact that the engine is a MIMO system, the potential fuel efficiency increase is generally higher when multiple inputs are considered for ES. A challenge is to select ES inputs that have a strong correlation with the BSFC, in such a way that the resulting steady-state optimization problem is quasi-convex. An additional consideration is that depending on the selected inputs, output constraints may need to be accounted for, which is not trivial in ES.

In addition to C1 and C2, fundamental ES challenges are encountered in the considered application. These challenges are summarized in Section 1.4.3, based on the detailed ES introduction in the following section.

1.4 Introduction to extremum seeking

In the previous section, ES is introduced as a data-driven optimization approach that can be used for online optimization of engine systems, without requiring full knowledge of the system dynamics. This section provides a more detailed and generic introduction to ES, and an overview of the fundamental ES challenges that are encountered in the considered application.

1.4.1 Online optimization using extremum seeking

The objective of ES is to find the input to a system, for which, in steady state, the cost output reaches an optimum. The key property of ES is that it only requires the system to be stable, and that the measured cost output, in steady state, is described by a quasi-convex map as a function of a constant input. ES finds the input that corresponds to the optimum of this unknown map, by using an estimate of the systems steady-state behavior at the current input, based on the measured output of the dynamic system. As such, ES is sometimes referred to as a model-free or data-based approach, contrary to typical modelbased optimization approaches.

Krstić and Wang, 2000 presented the first stability proof, i.e., convergence analysis, of ES. The considered scheme in Krstić and Wang, 2000 is an example of continuous derivative-based ES, which fits the general scheme depicted



Figure 1.9. Common layout of continuous derivative-based extremum seeking (ES) for a dynamic nonlinear single-input-single-output (SISO) system \mathcal{H} , with steady-state map $Q_J(u)$.

by Figure 1.9. The scheme considers the single-input-single-output (SISO) nonlinear system \mathcal{H} with steady-state map $Q_J : u \to J$. The ES objective is to steer the input u to the optimum u^* , using the optimizer output \hat{u} . The optimization is based on a derivative estimate of the map $Q_J(u)$ at $u = \hat{u}$, which is obtained from the correlation between the external perturbation signal d and the measured output J. Typically, the so-called "dither" signal d is sinusoidal. Multiplication of the output J with a scaled version of d yields a signal that is, on average over time, an approximation of the true derivative. This multiplication step is known as "demodulation".

Since the derivative estimation is based on continuously time-varying signals, and the optimizer output is continuously varying, the presented ES is commonly referred to as continuous ES. Contrarily, in sampled-data ES, see, e.g., Teel and Popovic, 2001; Khong et al., 2013, the estimation is based on step wise variation of the input u, while the optimizer output \hat{u} remains constant in between the steps. Sampled-data ES is particularly useful for global optimization, since any offline numerical optimization technique can be applied on the sampled data. This thesis considers continuous ES, since the considered dynamics are continuous, and no local minima are encountered in the engine control problems considered in this thesis.

Being a model-free approach, ES is an appealing approach for optimization of complex systems, or systems that are subject to disturbances and system uncertainty, of which the influence on the location of the optimum is unknown. Additional advantages are low implementation effort, and that ES can be applied online. Since no assumptions on the system dynamics, parameter dependencies, and disturbances are made, the cost optimization is inherently robust. In addition to the aforementioned fuel efficiency optimization applications in Section 1.3.3, numerous other ES application examples exist. Van der Meulen et al., 2014 considers optimal slip ratio control for continuous variable transmissions (CVTs), while Marinkov et al., 2014 maximizes the energy conversion of an engine intake turbine. In Haring et al., 2013; Hunnekens et al., 2015; Hazeleger et al., 2018, ES is used for controller parameter tuning in high precision motion systems. Bolder et al., 2012 considers ES for tracking of an unknown sawtooth period in a nuclear fusion application. In Burns et al., 2018, ES is applied for offline controller calibration of air conditioning units.

In online optimization of fuel efficiency of diesel engines, some of the drawbacks of ES are encountered. In particular, the limited convergence rate of classical ES is considered problematic. Classical dither-based ES, such as depicted in Figure 1.9, relies on time scale separation principles. Loosely speaking, these require that the dither signal variation is slow with respect to the system dynamics, and that, in turn, the optimizer output \hat{u} varies slowly with respect to the dither signal d. As such, it is clear that the convergence rate of classical ES is inherently low compared to the system dynamics.

1.4.2 Advances in extremum seeking

Since Krstić and Wang, 2000 presented their stability result, numerous results have been presented that are aimed at improving upon the classical scheme. This section discusses some significant results, that are relevant for the engine application. As motivated in Section 1.3.3, ES is applied to increase robustness of the obtained optimal fuel efficiency with respect to real-world disturbances. The effectiveness of disturbance rejection is increased when the ES convergence rate is increased. In addition, Section 1.3.3 explains the necessity of considering multiple ES inputs and accounting for output constraints.

Increased convergence rate

In "fast" dither-based ES, contrarily to "classical" dither-based ES, the time scale separation between the dither signal and the system requires the dither frequency to be larger than the dominant frequencies in the system dynamics. Thereby, the convergence rate is significantly increased; convergence can be faster than the system dynamics. The key principle is an assumption on the relative degree of the system, which enables estimation of the system's steady-state behavior, based on its input-output relation at high dither frequencies. Fast ES was first introduced in Moase and Manzie, 2011, and in a more general form in Moase and Manzie, 2012 for Wiener-Hammerstein systems. Guay, 2016 considers general nonlinear systems with a strong relative degree one.

A different approach is taken in input-based ES, where an observer is used to estimate the derivative of the system's steady-state map using the actual input u instead of the dither signal d. This relaxes the time scale separation requirement between the dither signal and the optimizer. In Gelbert et al., 2012, an extended Kalman filter is applied. In Hunnekens et al., 2014, a first-order least-squares fit of past time input and output data is used as derivative estimate. Guay and Dochain, 2015 and Haring, 2016, Chapter 2, propose observers and provide convergence results.

Extremum seeking for multiple-input systems

In ES for multiple-input systems, the correlation of each input u_k , k = 1, 2, ..., n, with the scalar output J needs to be distinguished. In dither-based ES, this is commonly achieved by perturbing each input with a different dither frequency, see, e.g., Rotea, 2000; Ariyur and Krstic, 2002. For input-based ES, the input $\boldsymbol{u}(t) \in \mathbb{R}^{n \times 1}$ should satisfy a persistence-of-excitation (PE) condition. Application of a gradient descent optimizer for multiple-input systems is straightforward. Ghaffari et al., 2012 proposes a Newton-like ES approach for a two-input system, which uses an estimate of the inverse Hessian of the map $Q_J(\boldsymbol{u})$ to balance the convergence rate of the two inputs, irrespective of the map $Q_J(\boldsymbol{u})$. The involved estimation of higher-order derivatives poses additional inequality conditions on the ratio between the two dither signals.

Constraint handling

Handling input constraints in ES is possible using saturation and anti-windup measures in the optimizer, see Tan et al., 2013 for a detailed analysis. More challenging are output constraints, for which it is worth noting that constrained ES approaches only offer approximate constraint satisfaction for dynamic systems, being based on feedback of measured outputs.

Similar to offline optimization techniques, Guay et al., 2015 uses barrier functions to transform the constrained problem into an unconstrained one. In Hazeleger et al., 2019 a barrier function approach is presented for sampled-data ES. Fast convergence however, induces overshoot of the constrained optimum, i.e., constraint violation. This is due to the derivative estimation time scale, which essentially is a delay in the ES closed-loop system. By reducing the optimizer gain, the overshoot is decreased, as is the ES convergence rate. Alternatively, a more smooth barrier function yields a smoother combined cost with a smaller gradient near the constrained optimum. However, since thereby the effective ES loop gain is reduced, so is the convergence rate. Moreover, a low gradient complicates derivative estimation in case of output disturbing noise.

An alternative approach is adopted in this research, which is presented in Ramos et al., 2017 for systems with a single constraint output. Instead of a weighted combination of the cost and barrier functions, a weighted combination of the corresponding estimated gradients is used for optimization, where the weighting is a function of the constraint output value with respect to its limit. By doing so, the effect of derivative estimation delay is omitted, such that the convergence rate can be increased without inducing constraint violation.

1.4.3 Challenges in extremum seeking

This section provides an overview of fundamental and practical challenges that are encountered in the application of ES for fuel-efficient control of diesel engines.

- C3 Extremum seeking approach evaluation: Section 1.4.2 introduces fast and input-based ES as alternatives for classical dither-based ES to increase the convergence rate. In practice however, it is often unclear if application of such an advanced approach is possible, and would lead to an increased performance for the application at hand. To be more precise, fast ES poses additional requirements on the system, whereas the assumed performance increase by input-based ES lacks a fundamental analysis.
- C4 Extremum seeking parameter tuning: Application of ES involves tuning of several parameters, such as the amplitude and frequency of the individual dither signals. Appropriate tuning is essential for the performance of ES. As such, a challenge is how to obtain the tuning that results in the highest possible convergence rate, given the type of ES. Existing tuning guidelines are often "existence-type" results, which lead to sub-optimal performance in practice.
- C5 Multiple-output constraint handling: As discussed in Section 1.4.2, the constraint handling approach proposed in Ramos et al., 2017 is favorable over a barrier function approach when fast convergence is desired. However, Ramos et al., 2017 considers a single constraint output. As such, the considered challenge is to obtain a similar approach that can handle multiple output constraints.
- C6 Combined extremum seeking and tracking control: The engine control objective consists of reference tracking, and minimization of the fuel consumption, taking into account constraints. The minimal fuel consumption however, is generally obtained for a combination of inputs which does not satisfy the tracking objectives, i.e., the tracking and optimization objectives are conflicting. In existing approaches for combined tracking and ES, see, e.g., Van der Meulen et al., 2014, the resulting interaction between the objectives is not accounted for. As a result, when the ES convergence rate is increased towards the tracking time scale, a challenge exists in how to preserve tracking performance while minimizing the cost.

1.5 Contributions

In view of the diesel engine control objective, introduced in Section 1.2.2, this thesis explores possibilities to integrate ES in diesel engine control systems, aiming for online fuel efficiency optimization with robustness to real-world disturbances. In doing so, both general contributions in the field of ES, as well as practical contributions concerning diesel engine control are provided. A brief summary of the contributions is provided in this section, where, if applicable, the corresponding challenges introduced in Sections 1.3.3 and 1.4.3, are mentioned in between brackets.

1.5.1 Extremum seeking contributions

In Chapter 2, a unifying analysis of classical and fast dither-based ES is proposed, based on a frequency-domain description of dither-based derivative estimation. The analysis provides a practical interpretation of the requirements for fast ES (**C3**). Moreover, the frequency-domain analysis is used to determine the tuning of the (highest) dither frequency in classical dither-based ES is proposed (**C4**). In addition, the existence of a lower bound on the dither amplitude is derived, which exists regardless of disturbing noise being present (**C4**). The underlying result from the system identification field also provides a fundamental motivation to apply input-based ES instead of dither-based ES (**C3**).

Besides the analysis of ES in the frequency domain, Chapter 2 provides a generalized dither-based derivative estimation framework. This generalized approach estimates derivatives up to an arbitrary order, for systems with an arbitrary number of inputs, thereby unifying existing classical dither-based ES derivative estimation approaches. Using the framework, the optimal ratio between different dither frequencies is derived for multiple-input systems, which in general, enables faster converge of the closed-loop ES system (C4).

In Chapter 3 an extension for handling multiple output constraints is proposed, to the single output constraint handling proposed in Ramos et al., 2017 (C5). Chapter 3 considers an experimental application of ES, where external signals to a low-level tracking control systems are the ES inputs. Besides a safety constraint, constraints on the low-level tracking errors are in place. By this proposed structure, tracking performance is preserved in the presence of interaction with the optimization objective (C6).

Chapter 4 proposes an ES tracking control design, for robust online cost optimization in over-actuated tracking control system. The interaction between the tracking and optimization objectives is dealt with by an adaptive decoupling mechanism based on a projection of estimated gradients of the system's steady-state map (C6). By explicitly accounting for the interaction, tracking and ES can operate in the same time scale, by which the convergence rate of ES can be increased without affecting the tracking performance.

1.5.2 Diesel engine control contributions

The experimental setup considered in Chapter 3 is a state-of-the-art heavyduty EURO-VI diesel engine. In addition to the production type sensors, the engine is equipped with in-cylinder pressure sensors in each cylinder and a high resolution crack angle encoder. As such, the conducted experiments provide a demonstration of the developed multivariable constrained ES methodology on a complex system. A BSFC estimate based on the measured in-cylinder pressure is applied. The estimate proves to be accurate during experiments (**C1**). To obtain the ES cost, a normalization is applied to the estimated BSFC as a function of the engine operating point, to reduce the correlation of the ES cost with fast operating point transients (C1).

The ES minimization in Chapter 3 of the aforementioned cost, uses both airpath and fuel-path actuation simultaneously (C2). Contrarily, the ES engine control examples mentioned in Section 1.3.3 consider fuel-path actuation only. Using constraints on the tracking error of the low-level multivariable engine control system, tracking performance is preserved for the specific engine-out NO_x and the net IMEP, which is correlated with the engine torque, see (1.5). In addition, a safety constraint on the peak in-cylinder pressure p_{peak} is accounted for. As such, the control approach provides robust performance with respect to unknown real-world disturbances, for all the parameters in the vector z in (1.2). Contrarily, most existing approaches are limited to one performance parameter.

The combined ES tracking controller in Chapter 4 is applied for online pumping-loss minimization, in air-path control of the EGR fraction. ES-based air-path control is not commonly applied in the literature (C2). The proposed control design enables tracking with a low-complexity controller, which is based on frequency-domain loop-shaping and easy to obtain non-parametric system models. Similar to economic MPC, the proposed control design enables online cost optimization, albeit of the steady-state cost. Contrary to MPC, the optimization is robust to system uncertainty, and does not require parametric models and real-time numerical optimization.

1.6 Outline of the thesis

The outline of this thesis is as follows. Chapter 2 considers derivative estimation of multiple-input derivative-based ES. Application oriented design guidelines are proposed, based on a frequency-domain analysis and a generalized derivative estimation framework. Chapter 3 presents an experimental study on the application of constrained ES for online fuel efficiency optimization, for robust optimal diesel engine performance. Chapter 4 considers online cost optimization using ES in over-actuated tracking control systems, where interaction between tracking and optimization objectives is dealt with by an adaptive decoupling mechanism. Based on the results of this thesis, the main conclusions and directions for future research are derived in Chapter 5.

Chapter 2

Derivative estimation in multivariable extremum seeking

Abstract – In derivative-based extremum seeking (ES), accurate and fast derivative estimation of the system's steady-state input-output map is essential for fast and accurate convergence to the unknown optimum of the steady-state map. This chapter considers ES derivative estimation for multiple-input systems. Explicit tuning guidelines are provided for dither-based ES, based on a frequency-domain analysis and a generalized derivative estimation framework, which are complementary to existing time scale separation based existence-type results. In addition, the existence of a lower bound on the dither amplitude and the advantage of input-based ES approaches, are fundamentally motivated from a system identification perspective, and demonstrated in a simulation example.

2.1 Introduction

This chapter addresses practical challenges that are encountered in application of derivative-based ES, for cost optimization in unknown dynamical multipleinput systems. Derivative-based ES, see, e.g., Nešić et al., 2010, uses estimates of derivatives of the systems steady-state input-output map in the optimizer, to find the optimal input to the system. The derivative estimator (DE) is an the essential part of the considered type of ES, as it connects the unknown system to the derivative-based optimization approach.

Throughout this thesis, "dither-based ES" refers to the class of ES that estimates the system's steady-state derivative from correlation between an externally added dither signal to the input of the system, and the measured cost output. Often sinusoidal dither signals are used for this purpose, see, e.g., the classical ES scheme considered by Krstić and Wang, 2000. The correlation is typically "extracted" by multiplication of the system output with a periodic "demodulation" signal which has the same frequency as the dither signal. In the classical case, the dither signal frequency is low with respect to the frequencies characterizing the system dynamics. As a result of this time scale separation approach, the system's steady-state behavior is observed in the output at the dither signal frequency.

Generalizations exists of the classical ES scheme. Both multiple-input ES, see, e.g., Walsh, 2000; Rotea, 2000; Ariyur and Krstic, 2002; Moase et al., 2011, and ES using higher-order derivatives, see, e.g., Moase et al., 2009a, Moase et al., 2009b, Nešić et al., 2010, Ghaffari et al., 2012, have been studied in the literature. Significantly less work exists on a combination of both. Ghaffari et al., 2012 present a Newton-like ES approach for a system with two inputs that uses estimated second-order derivatives. In this chapter, a generalized DE framework is presented which combines higher-order derivative estimation, for systems with an arbitrary number of inputs. As such, the proposed framework is the multiple-input extension of the DE presented in Nešić et al., 2010.

Application of ES in practice involves parameter tuning, for which the existing time scale separation analyses, e.g., Krstić and Wang, 2000; Tan et al., 2006, provide existence-type results. This chapter aims to provide tuning guidelines within the design freedom of these existence-type results.

Regarding the dither frequency tuning for multiple-input ES, the generalized DE framework is used to derive the optimal ratio between the different dither frequencies, which results in the fastest possible DE time scale. By increasing the DE time scale, ES convergence is generally increased. In addition, a frequency-domain analysis of dither-based ES is provided, to illustrate the effect of the selected (highest) dither frequency on the derivative estimation error, based on two examples. An analysis in the frequency domain is a natural choice, considering that ES relies on the response of a system to periodic inputs.

The frequency-domain analysis is not limited to classical dither-based ES. The time scale separation requirement of low-frequency dither excitation is a fundamental limitation on the achievable convergence rate in classical ES. To overcome this limitation, the phase of the demodulation signal may be shifted, see, e.g., Krstić, 2000; Haring et al., 2013, to compensate for the system dynamics. Contrarily, in fast ES, see, e.g., Moase and Manzie, 2012; Guay, 2016, the dither frequency is high with respect to the frequencies characterizing the system dynamics. Using additional system knowledge and a relative degree assumption, the observed output, as a result of the high-frequency dither excitation, can be related to the systems steady-state behavior. In practice, it is however not always clear if the system satisfies the requirements to apply such advanced dither-based ES approaches. Considering these advanced ES approaches, the frequency-domain representation can be used as a unifying description that connects classical dither-based ES to phase shifted ES and fast ES. By doing so, the

system requirements and practical implications of advanced dither-based ES are clarified with respect to classical dither-based ES.

In addition to dither-based ES, this chapter addresses an alternative class of ES, which we define as "input-based", see, e.g., the approaches in Hunnekens et al., 2014; Guay and Dochain, 2015; Haring, 2016. Contrarily to dither-based ES, input-based ES derivative estimation is based on the correlation between the input to the system and the cost output, instead of the dither signal and the cost output. A fundamental analysis of the advantage of input-based ES is however challenging and not presented in the aforementioned work. This chapter provides a stepping stone towards such an analysis, using the equivalence between the frequency-domain description of dither-based ES derivative estimation and system identification. In addition, the same observation reveals the existence of a lower bound on the dither amplitude. These observations are illustrated in a simulation study.

This chapter is organized as follows. In Section 2.2, a problem description is provided based on an overview of the different types of derivative-based ES. Section 2.3 introduces the generalized dither-based DE framework with optimal dither frequency ratio. Section 2.4 presents the frequency-domain analysis including the derived contributions. The input-based DE approach is provided in Section 2.5 and the simulation study in Section 2.6. A summary of the main conclusions is given in Section 2.7.

2.2 Problem description

This section provides an overview of ES based on estimated derivatives for multiple-input systems. The main classes of DEs are introduced, at a high level, and the corresponding challenges that are encountered during implementation in practice are discussed.

2.2.1 Class of systems

This chapter considers ES for nonlinear multiple-input dynamic systems. The corresponding class of systems is given by

$$\mathcal{H}:\begin{cases} \dot{\boldsymbol{x}}_{\mathcal{H}}(t) &= f(\boldsymbol{x}_{\mathcal{H}}(t), \boldsymbol{u}(t))\\ J(t) &= h(\boldsymbol{x}_{\mathcal{H}}(t), \boldsymbol{u}(t)). \end{cases}$$
(2.1)

In (2.1), $J \in \mathbb{R}$ is the cost output, $x_{\mathcal{H}} \in \mathbb{R}^{n_{\mathcal{H}} \times n}$ is the state vector, and $u \in \mathbb{R}^{n \times 1}$ is the input. The exact dynamics of the system \mathcal{H} are unknown, however, the following assumption states that the system \mathcal{H} exhibits certain stability properties and possesses a steady-state performance map.

Assumption 2.1. For constant inputs \boldsymbol{u} , the system \mathcal{H} in (2.1) possesses a unique, asymptotically stable equilibrium $\boldsymbol{x}_{\mathcal{H}} = l(\boldsymbol{u})$, i.e., $f(l(\boldsymbol{u}), \boldsymbol{u}) = 0$, such that the cost output J is described by the steady-state performance map Q_J : $\boldsymbol{u} \to J$, given by

$$Q_J(\boldsymbol{u}) = h(l(\boldsymbol{u}), \boldsymbol{u}). \tag{2.2}$$

In addition, the following assumption, adopted from Moase et al., 2011, Assumption 2, describes a "quasi-convex" property of the map $Q_J(\boldsymbol{u})$ in (2.2).

Assumption 2.2. There exists a constant input u^* such that the gradient

$$abla_J(\boldsymbol{u}) = \begin{bmatrix} \frac{\partial Q_J(\boldsymbol{u})}{\partial u_1} & \frac{\partial Q_J(\boldsymbol{u})}{\partial u_2} & \cdots & \frac{\partial Q_J(\boldsymbol{u})}{\partial u_n} \end{bmatrix} = \boldsymbol{0}$$

if and only if $u = u^*$. Moreover, $Q_J(u) > Q_J(u^*)$ for all $u \neq u^*$.

Assumption 2.2 states that $Q_J(\boldsymbol{u}^*)$ is a global minimum. Therefore, the input \boldsymbol{u}^* is defined as the optimum. The assumption for the extremum $Q_J(\boldsymbol{u}^*)$ to be a minimum, is without loss of generality; a maximum is possible by suitable adjustment of the signs in the ES controller and Assumption 2.2.

2.2.2 Derivative-based extremum seeking

The objective of ES is to steer the input \boldsymbol{u} of the system \mathcal{H} in (2.1) to the optimum \boldsymbol{u}^* , such that the cost J is minimal when \mathcal{H} is in steady-state. Figure 2.1 depicts a high level schematic outlay of ES based on estimated derivatives of the map $Q_J(\boldsymbol{u})$ in (2.2). In addition to the system \mathcal{H} in (2.1), Figure 2.1 contains the DE, which provides a vector $\tilde{\boldsymbol{g}}(t)$ that contains estimated derivatives up to order one or more, the optimizer \mathcal{F} with output $\hat{\boldsymbol{u}}$, and the dither signal \boldsymbol{d} , which is introduced in Section 2.2.2.2.

2.2.2.1 Optimizer

The optimizer \mathcal{F} that aims to steer the input $\hat{u}(t)$ to u^* , is described by

$$\dot{\hat{\boldsymbol{u}}}(t) = F(\tilde{\boldsymbol{g}}(t)). \tag{2.3}$$

In classical ES, see Krstić and Wang, 2000, \mathcal{F} is a gradient descent optimizer, which is, for the single-input case $\boldsymbol{u} = u_1$, given by: $F = -c\tilde{g}_{u_1}(t)$ where $\tilde{g}_{u_1}(t)$ is the estimate of the first-order derivative $\frac{dQ_J(u_1)}{du_1}$ and $c \in \mathbb{R}_{>0}$ is the optimizer gain. The majority of ES controllers, including advanced approaches, are presented with a gradient descent optimizer. For multiple-input systems, a different optimizer gain c can be taken for each input. Alternatives exist, see, e.g., Moase et al., 2009a where a Newton-like optimizer is proposed using the estimate of the second-order derivative of the map $Q_J(u)$. Ghaffari et al., 2012 presents Newton-like ES for a system with two inputs.


Figure 2.1. High level schematic representation of extremum seeking based on estimated derivatives of the steady-state performance map $Q_J(\boldsymbol{u})$ in (2.2) of the system \mathcal{H} in (2.1).

2.2.2.2 Derivative estimator

Depending on the type of DE, the input of the system \mathcal{H} is perturbed with a dither signal $d(t) \in \mathbb{R}^{n \times 1}$ to aid the derivative estimation. Often, sinusoidal dither signals are applied. In classical single-input dither-based ES, e.g., as in Krstić and Wang, 2000, the (scalar) output $\tilde{g}(t)$ of the DE is a signal that is, on average over one period of the dither signal d(t), approximately equal to the derivative of $Q_J(\hat{u})$.

Derivative estimation in classical ES, as in Krstić and Wang, 2000, is based on the correlation between the dither signal and the measured output. For multipleinput systems, each scalar signal in d(t) typically has a different frequency, such that their contribution to the scalar output J(t) can be distinguished. In this chapter, ES that uses correlation between d(t) and J(t) in the DE is referred to as "dither-based" ES. Figure 2.2 schematically depicts the class of dither-based ES. The difference between "classical" and "fast" ES is addressed in Section 2.2.2.3.

Opposed to dither-based ES, a class of ES exists, in which the correlation between the input u(t) and the measured output J(t) is used in the DE, see, e.g., Guay and Dochain, 2015. This class, schematically depicted in Figure 2.2, is referred to as "input-based" ES and is discussed in Section 2.5. Input-based ES can operate dither-free in practice, when the input u(t) satisfies a persistenceof-excitation (PE) condition.

2.2.2.3 Time scale separation principles

The convergence analysis and practical intuition of derivative-based ES, rely on the same "time scale separation principles":

- P1 The DE operates in a different time scale as the system \mathcal{H} . This time scale separation principle is used to obtain an estimate of the steady-state behavior of the dynamic system \mathcal{H} , described by the map $Q_J(u)$, based on the measured output J(t).
- P2 The optimizer \mathcal{F} adjusts $\hat{\boldsymbol{u}}(t)$ sufficiently slow with respect to the DE time scale, i.e., $\dot{\boldsymbol{u}}(t)$ is limited and $\hat{\boldsymbol{u}}(t)$ is "quasi-constant". This time scale separation principle ensures that the derivatives can be estimated sufficiently accurate.



Figure 2.2. Schematic representation of three different classes of derivative estimators (DEs) for extremum seeking (ES). The highest and lowest frequencies in the dither signal vector $d(t) \in \mathbb{R}^{n \times 1}$ are denoted by ω_{d_n} and ω_{d_1} , respectively, while the frequency $\omega_{\mathcal{H}}$ indicates the frequency range, or time scale, in which the system \mathcal{H} operates.

The "time scale" of the system \mathcal{H} refers to the transient system dynamics that characterize a dynamic system. For linear time-invariant (LTI) systems, the time scale can be interpreted as the frequency range in which the poles and zeros appear, e.g., the natural frequency of a mass-spring-damper system. In Figure 2.2, the time scale of the system \mathcal{H} is indicated by $\omega_{\mathcal{H}}$. For dither-based ES, the DE time scale is equal to the range of dither signal frequencies $[\omega_{d_1}, \omega_{d_n}]$, where ω_{d_1} and ω_{d_n} are the lowest and the highest dither frequencies, respectively.

When P1 and P2 are satisfied, the DE provides an accurate derivative estimate of the steady-state map $Q_J(\boldsymbol{u})$. Combined with the quasi-convex property in Assumption 2.2, in a closed-loop ES system with an optimizer of the form (2.3), the input $\hat{\boldsymbol{u}}(t)$ convergences to a neighborhood of \boldsymbol{u}^* .

For most ES approaches, e.g., classical dither-based ES and input-based ES, P1 can be specified into the requirement that the DE time scale is slower than the time scale of the system \mathcal{H} , indicated by $\omega_{d_n} \ll \omega_{\mathcal{H}}$ in Figure 2.2. An exception are the ES approaches which are known as "fast ES", in which the DE time scale and the time scale of the system \mathcal{H} are separated by requiring $\omega_{d_1} \gg \omega_{\mathcal{H}}$. Fast ES was first introduced in Moase and Manzie, 2011. Application of fast ES is however not always possible, mainly because additional knowledge of the dynamics of the system \mathcal{H} is required, which will be made specific in Section 2.4.5.2.

2.2.3 Practical challenges in derivative-based extremum seeking

In many practical applications of ES, fast convergence of $\hat{u}(t)$ towards \hat{u}^* is desirable. For example, in the application of ES for fuel efficiency optimization in diesel engines, the ability to deal with varying system behavior and realworld disturbances is improved by faster convergence. However, the time scale separation requirement P2 implies that the convergence rate $\dot{\hat{u}}(t)$ is upper limited by the DE time scale, which, in turn, is upper limited by the time scale of the system \mathcal{H} . The latter is not the case for fast ES, however, application of fast ES is not always possible as noted in Section 2.2.2.3. As a result, the combination of P1 and P2 implies that, for all but fast ES, the convergence rate of ES is inherently slow, relative to the dynamics of the system \mathcal{H} .

Provided with existence-type results for the parameter tuning, it is not trivial to satisfy the time scale requirements, without avoiding an unnecessary reduction of the DE time scale, which, in turn, yields a reduced ES convergence rate. Extending single-input ES to the multiple-input case is possible, however, the parameter tuning is more challenging than for single-input ES.

As such, this chapter addresses the following problems:

- (i) How to select the dither signals in dither-based ES for multiple-input systems, possibly using higher-order derivatives, such that the DE is as fast as possible given a certain system time scale.
- (ii) What are the practical requirements for using an alternative DE approach. To be precise, input-based and fast ES are considered.

Thereby, this chapter addresses important practical considerations in ES application for multiple-input systems. In addition, a generalized dither-based DE framework is provided.

2.3 A generalized dither-based derivative estimation framework

This section¹ introduces a generalized DE framework for classical dither-based ES for multiple-input systems. The generalization consists of combining derivative estimation up to an arbitrary order N, for systems with an arbitrary number of inputs n, and as such it is the multiple-input extension of Nešić et al., 2010. For N = 1, n = 1, the proposed framework is equivalent to derivative estimation in the classical ES scheme in Krstić and Wang, 2000.

In the second part of this section, the framework is used to define the optimal ratio between the dither frequencies, which in general enables the fastest

¹This section is based on Van der Weijst et al., 2017.

possible ES convergence rate, by minimizing the DE time scale. As such, a general, explicit tuning guideline is proposed, for multiple-input ES, which is complementary to the existing time scale separation existence-type results, see, e.g., Krstić and Wang, 2000; Tan et al., 2006.

2.3.1 Taylor-based derivative estimation framework

The considered dither signal is

$$\boldsymbol{d}(t) = \begin{bmatrix} d_1(t) & d_2(t) & \dots & d_n(t) \end{bmatrix}^{\top}, \qquad (2.4a)$$

where

$$d_k(t) = a_k \cos(\omega_{d_k} t), \qquad (2.4b)$$

with $a_k, \omega_{d_k} \in \mathbb{R}_{>0}, k = 1, 2, \ldots, n$, the amplitude and frequency, respectively, of the individual dither signals. The dither signal is not restricted to be sinusoidal. In fact, the presented analysis can be done for any zero-mean periodic dither signal, e.g., a block wave signal, which is also considered for ES in Tan et al., 2008. The input of the system \mathcal{H} becomes

$$\boldsymbol{u}(t) = \hat{\boldsymbol{u}}(t) + \boldsymbol{d}(t).$$

In many existing works on dither-based ES, a Taylor series description is used to analyze the derivative estimation, e.g., in Krstić and Wang, 2000; Tan et al., 2006, in which convergence results are presented for classical dither-based ES. In this section, the general form of Taylor's theorem is considered with derivatives up to order N, for systems with an arbitrary number of inputs n.

To apply Taylor's theorem, the following assumption is adopted.

Assumption 2.3. The static map $Q_J(\boldsymbol{u})$ is N+1 times continuously differentiable with respect to all the inputs u_1, u_2, \ldots, u_n in \boldsymbol{u} .

Under Assumption 2.3 and for constant \hat{u} , Taylor's theorem states that

$$Q_J(\boldsymbol{u}(t)) = Q_J(\hat{\boldsymbol{u}} + \boldsymbol{d}(t)) = \sum_{r=0}^N \left(\frac{1}{r!} \left(\sum_{k=1}^n d_k(t) D_{u_k} \right)^r Q_J(\hat{\boldsymbol{u}}) \right) + R_N, \quad (2.5)$$

where R_N is the remainder term, which is a function of derivatives of order larger than N, and

$$D_{u_k}^r := \left(\frac{\partial}{\partial u_k}\right)^r = \frac{\partial^r}{\partial u_k^r}.$$

Taking \hat{u} constant in the analysis, is in accordance with time scale separation principle P2 in Section 2.2.2.3.

Each term in the summation (2.5) consists of a product of harmonics, based on the dither frequencies, a dither amplitude scaling, and a derivative of the map $Q_J(\boldsymbol{u})$. Following this structure, (2.5) is rewritten as

$$Q_J(\hat{\boldsymbol{u}} + \boldsymbol{d}(t)) = \boldsymbol{p}^\top(t) A \boldsymbol{g}(\hat{\boldsymbol{u}}) + R_N, \qquad (2.6)$$

where $\boldsymbol{p}(t)$ contains (products of) the harmonics with the dither frequencies, $A \in \mathbb{R}^{n_g \times n_g}$ is a constant diagonal matrix with (products of) the dither amplitudes $a_{d_1}, a_{d_2}, \ldots, a_{d_n}$, and the derivative vector $\boldsymbol{g} \in \mathbb{R}^{n_g \times 1}$. Since the product $\boldsymbol{p}^{\top}(t)A\boldsymbol{g}(\hat{\boldsymbol{u}})$ is scalar, there is no unique expression for $\boldsymbol{p}(t)$, A, and $\boldsymbol{g}(\hat{\boldsymbol{u}})$. The size n_g follows from the multinomial theorem:

$$n_g = \sum_{l=0}^{N} \frac{(l+n-1)!}{l!(n-1)!}.$$
(2.7)

Note that, 0! := 1, which appears in (2.7) for n = 1. The size n_g is minimal in the sense that all terms in (2.5) are described only once in (2.6), (2.8).

Example 2.4. For n = 2, N = 2, it holds that $n_g = 6$. The notation (2.6) corresponds with (2.5), for $\mathbf{d}(t)$ according to (2.4), when $\mathbf{p}(t)$, A, and $\mathbf{g}(\mathbf{u})$, are selected as

$$\boldsymbol{p}(t) = \begin{bmatrix} 1\\ \cos(\omega_{d_1}t)\\ \cos(\omega_{d_2}t)\\ \cos^2(\omega_{d_1}t)\\ \cos(\omega_{d_1}t)\cos(\omega_{d_2}t)\\ \cos^2(\omega_{d_2}t) \end{bmatrix}, \quad \boldsymbol{g}(\hat{\boldsymbol{u}}) = \begin{bmatrix} 1\\ D_{u_1}\\ D_{u_2}\\ D_{u_1}^2\\ D_{u_1}\\ D_{u_1}D_{u_2}\\ D_{u_2}^2 \end{bmatrix} Q_J(\hat{\boldsymbol{u}}), \quad (2.8)$$
$$A = diag(1, a_1, a_2, \frac{1}{2a_1^2}, a_1a_2, \frac{1}{2a_2^2}).$$

In classical ES, the cost J is multiplied with a demodulation signal, which is a scaled version of the dither signal, to "extract" content from J(t) at the dither frequency. Observe the multiplication of the measured output $J(t) = Q_J(\boldsymbol{u}(t))$ with the demodulation signal $\boldsymbol{m}(t) \in \mathbb{R}^{n_g \times 1}$:

$$\boldsymbol{m}(t)J(t) = \boldsymbol{m}(t)Q_J(\hat{\boldsymbol{u}} + \boldsymbol{d}(t)) = \boldsymbol{m}(t)\boldsymbol{p}^{\top}(t)A\boldsymbol{g}(\hat{\boldsymbol{u}}) + \boldsymbol{m}(t)R_N.$$
(2.9)

The general expression for the demodulation signal $\boldsymbol{m}(t)$ is

$$\boldsymbol{m}(t) = \begin{bmatrix} 1\\ \cos(\omega_{m_1}t)\\ \vdots\\ \cos(\omega_{m_{(n_g-1)}}t) \end{bmatrix}, \qquad (2.10)$$

where $\omega_{m_j} \in \mathbb{R}_{>0}$, $j = 1, 2, ..., n_g - 1$, are frequencies related to the dither frequencies $\omega_{d_1}, \omega_{d_2}, ..., \omega_{d_n}$, and will be specified later in this section.

In the remainder of this section, it will become clear that for a specific selection of the frequencies in $\mathbf{m}(t)$ and $\mathbf{d}(t)$, there exists a horizon length $T \in \mathbb{R}_{>0}$, such that the following matrix is constant and full rank:

$$K = \int_{t-T}^{t} \boldsymbol{m}(\tau) \boldsymbol{p}^{\top}(\tau) A \mathrm{d}\tau.$$
 (2.11)

Using the matrix K in (2.11), for constant \hat{u} , the integral over the time window [t - T, t] of the quantity in (2.9) is given by

$$\int_{t-T}^{t} \boldsymbol{m}(\tau) J(\tau) d\tau = K \boldsymbol{g}(\hat{\boldsymbol{u}}) + \int_{t-T}^{t} \boldsymbol{m}(\tau) R_N d\tau.$$
(2.12)

Suppose that the matrix K is full rank, then, by neglecting R_N , an estimate $\tilde{g}(t)$ of $g(\hat{u})$ is obtained by the following DE framework:

$$\tilde{\boldsymbol{g}}(t) = K^{-1} \int_{t-T}^{t} \boldsymbol{m}(\tau) J(\tau) \mathrm{d}\tau.$$
(2.13)

The estimation error e_g corresponding to the DE in (2.13) is a function of R_N

$$\boldsymbol{e}_g(t) = -K^{-1} \int_{t-T}^t \boldsymbol{m}(\tau) R_N \mathrm{d}\tau.$$
(2.14)

Remark 2.5. Similar to [Nešić et al., 2010, Proposition 1], the classical result in ES can be derived, which states that the derivative estimation error $e_g(t)$ scales with \bar{a}^2 , where $\bar{a} := \max(a_1, a_2, \ldots, a_n)$.

Remark 2.6. The presented derivation shows that the DE framework in (2.13) is, in essence, a traditional parameter estimation problem, estimating $g(\hat{u})$, see, e.g., Narendra and Annaswamy, 2012. Accordingly, noting that both m and p are a function of the dither signal d, the requirement that the matrix K in (2.11) is full rank, resembles a PE condition on the dither signal d.

The following example illustrates the DE framework in its most basic form, that is, estimating the first-order derivative for a single-input system.

Example 2.7. For n = N = 1, take $\mathbf{p}^{\top}(t) = \begin{bmatrix} 1 & \cos(\omega_{d_1}t) \end{bmatrix}$ accordingly, and select $\omega_{m_1} = \omega_{d_1}$ such that $\mathbf{m}(t) = \begin{bmatrix} 1 & \cos(\omega_{d_1}t) \end{bmatrix}^{\top}$. Then, the matrix K in (2.11) is given by

$$K = \int_{t-T}^{t} \begin{bmatrix} 1 & \cos(\omega_{d_1}\tau) \\ \cos(\omega_{d_1}\tau) & \cos^2(\omega_{d_1}\tau) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix} d\tau.$$

For T equal to the smallest common period time of the harmonic contributions in $\mathbf{p}^{\top}(t)\mathbf{m}(t)$: $T = \frac{2\pi}{\omega_{d_1}}$, the full rank matrix K is

$$K = T \begin{bmatrix} 1 & 0\\ 0 & a_1/2 \end{bmatrix}, \qquad (2.15)$$

where $\int_{t-T}^{t} \cos^2(\omega_{d_1}\tau) d\tau = T/2$ is used. The latter equality follows from

$$\cos(\omega_a t)\cos(\omega_b t) = \frac{1}{2}\cos\left(|\omega_b - \omega_a|t\right) + \frac{1}{2}\cos\left(|\omega_b + \omega_a|t\right), \qquad (2.16)$$

with $\omega_a, \omega_b \in \mathbb{R}$. Equation (2.16) is derived from Euler's result: $\cos(\phi) = 1/2e^{j\phi} + 1/2e^{-j\phi}$, which holds for any $\phi \in \mathbb{R}$.

Substituting K as in (2.15), in the DE in (2.13) yields

$$\tilde{\boldsymbol{g}}(t) = \begin{bmatrix} \tilde{Q}_J(t) \\ \tilde{g}_{u_1}(t) \end{bmatrix} = \frac{1}{T} \int_{t-T}^t \begin{bmatrix} J(\tau) \\ \frac{2}{a_1} \cos(\omega_{d_1} t) J(\tau) \end{bmatrix} d\tau.$$
(2.17)

In (2.17), the first-order derivative estimate with respect to u_1 is denoted using the following notation:

$$\tilde{g}_{u_k}(t) := \tilde{D}_{u_k} Q_J(\hat{\boldsymbol{u}}), \qquad k = 1, 2, \dots, n.$$

Equivalent to classical ES, e.g., in Krstić and Wang, 2000, the second row of the DE in (2.17), that provides the derivative estimate \tilde{g}_{u_1} , contains the product of the measured cost output J(t) with the demodulation signal, $m_2(t) = \cos(\omega_{d_1}t)$, with the dither frequency ω_{d_1} . The scaling $\frac{2}{a_1}$ results in a correct magnitude estimation as is also noted in Nešić et al., 2010. The difference of the proposed method with Krstić and Wang, 2000; Nešić et al., 2010 is the filtering that is applied to the product $m_2(t)J(t)$.

Namely, the integral in the DE in (2.13) and (2.17) acts as a filter, that extracts the average value of $\mathbf{m}(t)J(t)$ over the moving time window [t - T, t]. Such a filter is known as a moving average (MA) filter. Figure 2.3 depicts a schematic representation of the proposed DE framework in (2.17), where the MA filter is separated and where $\mathbf{u}_{MA}(t) = K^{-1}\mathbf{m}(t)J(t)$.

In Krstić and Wang, 2000, a combination of high-pass and a low-pass filtering is applied instead of the MA filter. Figure 2.4 depicts a schematic representation of this approach. For example, the following first-order low-pass filter can be applied to u_{MA} :

$$\mathcal{H}_{LP}(s) = \operatorname{diag}\left(\frac{\omega_{LP}}{s + \omega_{LP}}, \frac{\omega_{LP}}{s + \omega_{LP}}, \dots, \frac{\omega_{LP}}{s + \omega_{LP}}\right),$$

where $s \in \mathbb{C}$ is the complex frequency and $\omega_{LP} \in \mathbb{R}_{>0}$ is the filter pole. The dimension of the filter is $\mathcal{H}_{LP}(s) \in \mathbb{C}^{n_g \times n_g}$. The high-pass filter removes the static contribution $Q_J(\hat{\boldsymbol{u}})$ from J(t), and thereby avoids harmonic contribution



Figure 2.3. The proposed DE schematics using with the MA filter. $\begin{array}{c}
\hat{u} \\
 & \downarrow \\
 & \downarrow$

Figure 2.4. The proposed DE schematics with high-pass and low-pass filters $\mathcal{H}_{HP}(s)$ and $\mathcal{H}_{LP}(s)$.

of the demodulation frequencies in $\tilde{g}(t)$. For example, the following first-order high-pass filter can be applied to J(t):

$$\mathcal{H}_{HP}(s) = \frac{s}{s + \omega_{HP}},\tag{2.18}$$

with frequency $\omega_{HP} \in \mathbb{R}_{>0}$.

The MA filter is proposed for single-input ES in Haring et al., 2013, and applied in Bolder et al., 2012, as a means to improve the accuracy of the derivative estimate. The motivation in Haring et al., 2013; Bolder et al., 2012 is that, opposed to the high-pass and low-pass filters, an MA filter completely removes contributions at the dither frequencies and higher-order harmonics, from a *periodic* input $u_{MA}(t)$. For the presented generalized DE framework for multiple-input systems with estimation of derivatives up to arbitrary order, the MA filter offers the same advantage, for an appropriate T, which is addressed in the remainder of this section.

2.3.2 Dither signal frequency selection

The DE framework in (2.13) requires that the matrix K in (2.11) is full rank, which depends on the selection of the frequencies ω_{d_k} , $k = 1, 2, \ldots, n$, in d(t) and ω_{m_j} , $j = 1, 2, \ldots, n_g$, in m(t) in (2.10). Namely, by definition, see (2.11), the frequencies in p(t) depend on the dither frequencies ω_{d_k} , $k = 1, 2, \ldots, n$. This section provides a sufficient condition on the dither frequency ratio γ , under which the matrix K in (2.11) is full rank. In addition, the optimal ratio γ is defined, which in general enables the fastest possible ES convergence rate, by minimizing the DE time scale.

Without loss of generality, assume that the dither frequencies are arranged over the n inputs of the system, such that

$$\omega_{d_k} < \omega_{d_{k+1}}$$
 for all $k \in \{1, 2, \dots, n-1\},\$

and hence that ω_{d_n} is the highest dither frequency. As such, the following definition of the dither frequencies ω_{d_k} , k = 1, 2, ..., n, can be used:

$$\omega_{d_k} = \gamma_k \omega_{d_n}, \tag{2.19}$$

where $0 < \gamma_k < \gamma_{k+1} \le 1$, k = 1, 2, ..., n-1, and $\gamma_n = 1$, are the elements of a vector $\boldsymbol{\gamma} \in \mathbb{R}^{n \times 1}$.

2.3.2.1 Demodulation signal frequencies

The elements of vector $\mathbf{p}(t)$ can be expressed as a sum of harmonic signals with frequencies that are a combination of the dither frequencies, see (2.16). Consider for instance $\mathbf{p}(t)$ in (2.8), which can be expressed as

$$\boldsymbol{p}(t) = \begin{bmatrix} 1 \\ \cos(\omega_{d_1}t) \\ \cos(\omega_{d_2}t) \\ \cos^2(\omega_{d_1}t) \\ \cos(\omega_{d_1}t)\cos(\omega_{d_2}t) \\ \cos^2(\omega_{d_2}t) \end{bmatrix} = \begin{bmatrix} 1 \\ \cos(\omega_{d_1}t) \\ \frac{1}{2} + \frac{1}{2}\cos(\omega_{d_1}t) \\ \frac{1}{2} + \frac{1}{2}\cos(2\omega_{d_1}t) \\ \frac{1}{2} + \frac{1}{2}\cos(|\omega_{d_1} - \omega_{d_2}|t) + \frac{1}{2}\cos(|\omega_{d_1} + \omega_{d_2}|t) \\ \frac{1}{2} + \frac{1}{2}\cos(2\omega_{d_2}t) \end{bmatrix}. \quad (2.20)$$

The frequencies ω_{m_j} , $j = 1, 2, ..., n_g - 1$, of the demodulation signal $\boldsymbol{m}(t)$ in (2.10), are selected according to the following criterion.

Criterion 2.3.1. The frequency ω_{m_j} of the (j+1)-th element of $\mathbf{m}(t)$ in (2.10), is equal to the highest frequency of the harmonic contributions to the (j+1)-th element of the vector $\mathbf{p}(t)$, $j = 1, 2, ..., n_g - 1$.

Example 2.8. Consider p(t) in (2.8) or (2.20), for which Criterion 2.3.1 yields

$$\{\omega_{m_1}, \omega_{m_2}, \omega_{m_3}, \omega_{m_4}, \omega_{m_5}\} = \{\omega_1, \omega_2, 2\omega_1, (\omega_1 + \omega_2), 2\omega_2\}.$$

2.3.2.2 Dither frequency ratio γ criterion

The elements of the matrix $\boldsymbol{m}(t)\boldsymbol{p}^{\top}(t)$, according to (2.5)-(2.8), (2.10), are a sum of harmonic contributions. The frequency of some of these harmonic contributions is equal to zero by construction, which is explained by (2.16). Observe that, for $\boldsymbol{m}(t)$ according to Criterion 2.3.1, some elements of the matrix $\boldsymbol{m}(t)\boldsymbol{p}^{\top}(t)$ contain a square product $\varepsilon_j \cos^2(\omega_{m_j}t)$, where $0 < \varepsilon_j < 1, j = 1, 2, ..., n_g - 1$. These square products yield positive constant contributions $\varepsilon_j/2$, see (2.16), which are the harmonic contributions with frequency zero, by construction.

Using (2.16), it can be derived that, the frequencies of all *other* harmonic contributions to the matrix $\boldsymbol{m}(t)\boldsymbol{p}^{\top}(t)$, are combinations of the dither frequencies. Observe the following set that can be used to characterize all these combinations, for n inputs up to order N:

$$\boldsymbol{B} = \left\{ \boldsymbol{\beta} \in \mathbb{Z}^{n \times 1} \; \middle| \; 0 < \sum_{k=1}^{n} |\beta_k| \le 2N \; \middle| \; \beta_k \in \{-2N, -(2N-1), \dots, 2N\} \right\},\$$

where $\beta_k \in \mathbb{Z}$, k = 1, 2, ..., n, are the elements of the vector $\boldsymbol{\beta}$. Using the set \boldsymbol{B} , another set can be be derived that contains the frequencies of all other harmonic contributions to the matrix $\boldsymbol{m}(t)\boldsymbol{p}^{\top}(t)$, except those contributions that have frequency zero by construction (due to a square product $\varepsilon_j \cos^2(\omega_{m_j}t)$):

$$\Omega_{mp}(\boldsymbol{\gamma}) = \left\{ |\boldsymbol{\beta}^{\top} \boldsymbol{\gamma}| \omega_{d_n} \in \mathbb{R}_{\geq 0} \ \middle| \ \boldsymbol{\beta} \in \boldsymbol{B} \right\}.$$
(2.21)

The following proposition summarizes the sufficient conditions on the dither frequency ratio γ , under which the matrix K in (2.11) is full rank, for a specific value of T.

Proposition 2.3.1. Consider the DE in (2.13), where p(t) complies with the dither signal d(t) in (2.4) and m(t) in (2.10). Let the dither frequencies ω_{d_k} k = 1, 2, ..., n, be defined as in (2.19). Suppose that the following statements hold:

- (i) The demodulation frequencies ω_{mj}, j = 1, 2, ..., n_g-1, are chosen according to Criterion 2.3.1.
- (ii) The elements γ_k, k = 1, 2, ..., n − 1, of the vector γ are rational numbers, such that all elements of Ω_{mp}(γ) in (2.21) are non-zero, i.e., 0 ∉ Ω_{mp}(γ).

Then, the matrix K in (2.11) is full rank for $T = \frac{2\pi\delta_{\gamma}}{\omega_{d_n}}$, where $\delta_{\gamma} > 0$ is the smallest common divisor of the elements γ_k , k = 1, 2, ..., n, of the vector γ .

Proof. Similar to the set $\Omega_{mp}(\gamma)$ in (2.21), a set $\Omega_p(\gamma)$ can be derived that contains all harmonic contributions to the elements of the vector p(t):

$$\Omega_p(\boldsymbol{\gamma}) = \left\{ |\boldsymbol{\alpha}^\top \boldsymbol{\gamma}| \omega_{d_n} \in \mathbb{R}_{\geq 0} \ \middle| \ \boldsymbol{\alpha} \in \boldsymbol{A} \right\},$$
(2.22)

with A the set of combinations

$$\boldsymbol{A} = \left\{ \boldsymbol{\alpha} \in \mathbb{Z}^{n \times 1} \mid 0 \le \sum_{k=1}^{n} |\alpha_k| \le N \mid \alpha_k \in \{-N, -(N-1), \dots, N\} \right\}, \quad (2.23)$$

where $\alpha_k \in \mathbb{Z}$, k = 1, 2, ..., n, are the elements of the vector $\boldsymbol{\alpha}$. For example, consider $\Omega_p(\boldsymbol{\gamma})$ for n = 2, N = 2, in which case $\boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 & 1 \end{bmatrix}^\top$, then using the set \boldsymbol{A} one can obtain

$$\Omega_p(\boldsymbol{\gamma}) = \{0, \gamma_1 \omega_{d_2}, 2\gamma_1 \omega_{d_2}, \omega_{d_2}, 2\omega_{d_2}, |\gamma_1 - 1|\omega_{d_2}, (\gamma_1 + 1)\omega_{d_2}\},$$
(2.24)

Using $\omega_{d_1} = \gamma_1 \omega_{d_2}$, see (2.19), the frequencies in $\Omega_p(\gamma)$ in (2.24), indeed correspond to the frequencies in p(t) in (2.20).

From (i), it follows that there exist j combinations in the set A in (2.23) that correspond to the demodulation frequencies. The j-th combination is denoted by $\boldsymbol{\alpha}_{m_j}$ and satisfies $|\boldsymbol{\alpha}_{m_j}^{\top}\boldsymbol{\gamma}| = \omega_{m_j}, j = 1, 2, \ldots, n_g - 1$.

It can be verified that the elements $p_{j+1}(t)$, $j = 1, 2, ..., n_g - 1$, of the vector p(t), can be ordered in such a way that:

(a) The harmonic contribution with frequency ω_{m_j} , to the element $p_{j+1}(t)$, corresponding to the combination α_{m_j} , does not appear in the elements $p_1(t), p_2(t), \ldots, p_j(t)$.

Condition (a) is satisfied when the elements of p(t) are ordered by increasing order of the Taylor expansion in (2.5). See for example p(t) in (2.8) and the equivalent expression in (2.20).

As a result of (a), the square products $\varepsilon_j \cos^2(\omega_{m_j}t)$, where $0 < \varepsilon_j < 1$, $j = 1, 2, \ldots, n_g - 1$, cannot appear in the upper diagonal part of the matrix $\boldsymbol{m}(t)\boldsymbol{p}^{\top}(t)$, while necessarily, they do appear on the matrix diagonal. Hence, the diagonal contains positive constant contributions. Moreover, using (ii) in Proposition 2.3.1, it follows that all the other harmonic contributions have a non-zero frequency. As a result, on average over time, the matrix $\boldsymbol{m}(t)\boldsymbol{p}^{\top}(t)$ is a lower diagonal matrix with positive constants on the diagonal elements. To be precise, suppose there exists a smallest common period time T of all frequencies in the set $\Omega_{mp}(\boldsymbol{\gamma})$, then the following matrix is lower diagonal with positive scalars at its diagonal:

$$KA^{-1} = \int_{t-T}^{t} \boldsymbol{m}(\tau) \boldsymbol{p}^{\top}(\tau) \mathrm{d}\tau,$$

which is equivalent to (2.11). While A is constant and full rank by construction, see (2.8), it follows that K is full rank for T the smallest common period time of all frequencies in the set $\Omega_{mp}(\gamma)$

To determine the smallest common period time T, let δ_{γ} denote the smallest common divisor of all (n-1) elements in γ . Such a d_{γ} exists, because all γ_k , $k = 1, 2, \ldots, n-1$, are rational. Then the following result holds

$$|\boldsymbol{\beta}^{\top}\boldsymbol{\gamma}|\omega_{d_n} = \frac{1}{\delta_{\gamma}}|\beta_1 n_{\gamma_1} + \beta_2 n_{\gamma_2} + \ldots + \beta_n \delta_{\gamma}|\omega_{d_n}, \qquad (2.25)$$

where n_{γ_k} are the numerators of the rational γ_k , k = 1, 2, ..., n-1. Using that β_k , k = 1, 2, ..., n, and n_{γ_k} , k = 1, 2, ..., n-1 are integers, it follows that all elements of $\Omega_{mp}(\boldsymbol{\gamma})$ in (2.21) are integer multiples of $\omega_{d_n}/\delta_{\gamma}$. Therefore, $T = 2\pi\delta_{\gamma}/\omega_{d_n}$ is the smallest common period time of all the harmonic contributions to the matrix $\boldsymbol{m}(t)\boldsymbol{p}^{\top}(t)$, which completes the proof.

2.3.2.3 Implication of γ on extremum seeking time scale separation

Consider the following general expression for the optimizer \mathcal{F} :

$$\hat{\boldsymbol{u}}(t) = \omega_{d_n} F(\tilde{\boldsymbol{g}}(J(t), \boldsymbol{m}(t), K)), \qquad (2.26)$$

where F is according to the specific optimization technique, e.g., gradient descent optimization.

A brief sketch of the stability analysis of classical dither-based ES using time scale separation is provided, see, e.g., Nešić et al., 2010, for the single-input case.

The closed loop, of the system \mathcal{H} in (2.1) and the optimizer \mathcal{F} in (2.26), in the time scale $\tau := t\omega_{d_n}$ is given by

$$\omega_{d_n} \frac{d\boldsymbol{x}_{\mathcal{H}}}{d\tau} = f(\boldsymbol{x}_{\mathcal{H}}(\tau), \hat{\boldsymbol{u}}(\tau) + \boldsymbol{d}(\tau))$$
(2.27a)

$$\frac{d\hat{\boldsymbol{u}}}{d\tau} = F(\tilde{\boldsymbol{g}}(J(\tau), \boldsymbol{m}(\tau), K)).$$
(2.27b)

The system in (2.27) is in standard singular perturbation form. As such, from [Khalil, 2000, Theorem 11.1], it follows that the solution $\hat{\boldsymbol{u}}(\tau)$ of (2.27b) is order $O(\omega_{d_n})$ close to the solution $\hat{\boldsymbol{u}}_r(\tau)$ of the reduced system

$$\frac{\hat{\boldsymbol{u}}_r(\tau)}{d\tau} = F(\tilde{\boldsymbol{g}}(Q_J(\hat{\boldsymbol{u}}_r(\tau) + \boldsymbol{d}(\tau)), \boldsymbol{m}(\tau), K)).$$
(2.28)

Verify that, taking $\omega_{d_n} = 0$, "freezes" the system dynamics in (2.27a), such that it holds that $\hat{\boldsymbol{u}}(\tau) = \hat{\boldsymbol{u}}_r(\tau)$.

The MA filter in the DE in (2.13) extracts the average value of the product $\boldsymbol{m}(t)J(t)$ over time. However, a gradient descent optimizer, which essentially integrates the estimated derivatives, does not necessarily need a filter since the optimizer integral acts as an averaging filters itself. In Tan et al., 2010, ES without any filtering to the estimate is referred to as a "minimal ES algorithm". The generalized DE in (2.13), with N = 1, combined with a simple gradient descent optimizer $\dot{\hat{\boldsymbol{u}}}(t) = -c\tilde{\boldsymbol{g}}(t)$, with c > 0, yields the reduced system

$$\frac{\hat{\boldsymbol{u}}_r(\tau)}{d\tau} = -c \left[K^{-1} \boldsymbol{m}(\tau) Q_J(\hat{\boldsymbol{u}}_r(\tau)) \right].$$
(2.29)

For a specific selection of the frequencies in d(t) and m(t), the right-hand side of (2.29) is T periodic. This is the same T that is used in (2.13). As such, [Khalil, 2000, Theorem 10.4] can be used to obtain that the solution $\hat{u}_r(\tau)$ of (2.29) is O(c) close to the averaged solution $\hat{u}_{av}(\tau)$ of

$$\frac{\hat{\boldsymbol{u}}_{av}(\tau)}{d\tau} = -cK^{-1}\int_0^T \boldsymbol{m}(\sigma)Q_J(\hat{\boldsymbol{u}}_{av}(\sigma))\mathrm{d}\sigma.$$
(2.30)

As noted in Remark 2.5, the integral in (2.30), which is the estimated derivative of the DE in (2.13), is $O(\bar{a}^2)$ accurate. Subsequently, the quasi-convexity of $Q_J(\boldsymbol{u})$ in Assumption 2.2 is used to obtain that $\hat{\boldsymbol{u}}_{av} = \boldsymbol{0}$ in (2.30) is globally exponentially stable (GES).

Summarizing the above derivation, the solution $\hat{\boldsymbol{u}}(t)$ is order $O(w_{d_n}, c, \bar{a}^2)$ close to the optimum \boldsymbol{u}^* . The dither frequency selection plays an important

role in the stability analysis through the time scale separation principles: The dither frequency ω_{d_n} is small with respect to the lowest frequency characterizing the system dynamics f, and the optimizer gain c is small with respect to the periodicity of the reduced system, which is equal to ω_{d_1} for N = 1. These observations correspond to the general ES time scale separation principles P1 and P2 in Section 2.2.2.3. In conclusion, the optimal frequency ratio γ is the one that results in the smallest possible value of T.

In Haring et al., 2013, where ES with an MA filter is proposed, it is also noted that a small value of T is desirable, since T is a delay in the estimation.

Example 2.9. Consider the DE in (2.13) with n = 2 inputs for N = 1. In this case, the minimal common period time T of the signals in $\mathbf{m}(t)J(t)$ is obtained for $\boldsymbol{\gamma} = \begin{bmatrix} 1/2 & 1 \end{bmatrix}$, i.e., by selecting $\omega_{d_1} = \frac{1}{2}\omega_{d_2}$.

The observation in Example 2.9, may seem counter intuitive, since the DE frequency band can be decreased by taking $\frac{1}{2}\omega_{d_2} < \omega_{d_1} < \omega_{d_2}$. However, by doing so, the value of T increases and hence the DE time scale is reduced. Finding the optimal ratio γ , for general n and N is not trivial. In the following section, a general expression for the optimal ratio γ is introduced, together with an approach to obtain this optimal ratio.

2.3.2.4 Optimal dither frequency ratio γ

The conditions in Proposition 2.3.1 guarantee that the matrix K is full rank and there exists a T, such that the DE in (2.13) can be applied. However, for arbitrary n and N, it is not trivial to find a γ that satisfies the condition in Proposition 2.3.1. On the other hand, the required γ is not unique. Therefore, this section present a heuristic approach that not only provides a γ that satisfies the condition in Proposition 2.3.1, but in fact enables us to pick the optimal γ , which is defined as the γ that yields the smallest possible value for T, while still satisfying the conditions in Proposition 2.3.1.

Consider the expression for $|\beta^{\dagger}\gamma|\omega_{d_n}$ in (2.25), in the proof of Proposition 2.3.1. In the proof of Proposition 2.3.1, using (2.25), it is derived that all elements in the set $\Omega_{mp}(\gamma)$ in (2.21) are integer multiples of the lowest frequency in the set $\Omega_{mp}(\gamma)$, which is given by $\omega_{d_n}/\delta_{\gamma}$ [rad/s], where δ_{γ} is the smallest common divisor of the elements of γ . Subsequently, the value $T = 2\pi\delta_{\gamma}/\omega_{d_n}$ [s] is derived, being the smallest common period time of all frequencies in the set $\Omega_{mp}(\gamma)$. Hence, minimizing the value of T, requires maximizing the lowest frequency $\omega_{mp_0}(\gamma)$ in the set $\Omega_{mp}(\gamma)$ in (2.21), which is defined as

$$\omega_{mp_0}(\boldsymbol{\gamma}) := \min\left(\omega_{mp} \in \Omega_{mp}(\boldsymbol{\gamma})\right). \tag{2.31}$$

As a result, the optimal value γ^* that minimizes T, is the one that maximizes $\omega_{mp_0}(\gamma)$ in (2.31):

$$\boldsymbol{\gamma}^* = \operatorname*{argmax}_{\boldsymbol{\gamma}} \left(\omega_{mp_0}(\boldsymbol{\gamma}) \right). \tag{2.32}$$

Consider the following two examples in which γ^* is determined by explicitly analyzing all frequencies in the set $\Omega_{mp}(\gamma)$ as a function of γ .

Example 2.10. For n = 2, $\gamma = \gamma_1$ is scalar, and $0 < \gamma_1 < 1$. Figure 2.5 depicts a plot of all frequencies in the set $\Omega_{mp}(\gamma)$ as a function of γ_1 , for the case N = 2. The lowest frequency $\omega_{mp_0}(\gamma)$ in (2.31) is indicated. Using Figure 2.5, the value of γ_1 corresponding to the intersections of the frequencies $\omega_{mp}(\gamma)$ can be determined analytically. Thereby, condition (ii) in Proposition 2.3.1 is satisfied for $\gamma_1 \notin \{1/3, 1/2\}$ and $\gamma^* = \gamma_1^* = 2/3$.

In Table 2.1, values of γ_1^* that satisfy condition (ii) in Proposition 2.3.1, and the corresponding $\omega_{mp_0}^*$ and T, are given for n = 2 for different values of N. Clearly, increasing the estimation order N decreases the lowest frequency $\omega_{mp_0}^*$ and correspondingly increases T.

Table 2.1. γ_1^* , $\omega_{mp_0}^*$, and T, for different values of N for a system with n = 2 inputs.

N	1	2	3	4	5	6
γ_1^*	$^{1/2}$	$^{2}/_{3}$	$^{3/4}$	$^{4/5}$	$^{5}/_{6}$	$^{6}/_{7}$
$\omega_{mp_0}^* \left[\omega_2 \right]$	$^{1/2}$	$^{1}/_{3}$	$^{1}/_{4}$	$^{1}/_{5}$	$^{1}/_{6}$	$^{1}/_{7}$
$T \left[2\pi\omega_2^{-1}\right]$	2	3	4	5	6	7

Example 2.11. For n = 3, $\boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}^{\top}$, and $0 < \gamma_1 < \gamma_2 < 1$. Similar to Example 2.10, $\omega_{mp_0}(\boldsymbol{\gamma})$ is plotted in Figure 2.6, for N = 2. The optimal ratio $\boldsymbol{\gamma}^*$ is numerically approximated by $\boldsymbol{\gamma}^* \approx \begin{bmatrix} 57/100 & 86/100 \end{bmatrix}^{\top}$. Observe Figure 2.6 to note that this is a good approximation, and in any case satisfies condition (ii) in Proposition 2.3.1.

2.4 Frequency-domain analysis of dither-based derivative estimation

In the previous section, a dither-based derivative estimation framework is derived and analyzed in the time domain. This section provides a frequency-domain analysis of dither-based derivative estimation.

Dither-based ES relies on the response of the considered system to periodic inputs. The frequency domain offers a framework to describe a system's frequency response, and as such can be used to illustrate the implication of time scale separation between the system \mathcal{H} and the DE, i.e., the dither frequency, on the estimation accuracy. Moreover, the frequency-domain analysis clarifies the requirements on the system \mathcal{H} , for application of advanced ES approaches, such as ES with phase compensation, see, e.g., Krstić, 2000; Haring et al., 2013, and fast ES Moase and Manzie, 2012; Guay, 2016; Atta and Guay, 2017.



Figure 2.5. Elements of $\Omega_{mp}(\gamma)$, $\omega_{mp_0}(\gamma)$, and γ^* , for n = N = 2.



Figure 2.6. $\omega_{mp_0}(\boldsymbol{\gamma})$, and approximate $\boldsymbol{\gamma}^*$, for n = 3, N = 2.

The frequency-domain analysis shows that, dither-based derivative estimation is a specific form of frequency-domain system identification. Subsequently, a result from system identification literature is adopted, which shows that there exists a lower bound the dither amplitude that scales with the optimizer gain.

For ease of exposition, the frequency-domain derivation is presented considering single-input systems. However, the results apply to multiple-input systems as well, see Remark 2.12. Estimation of higher-order derivatives is not considered, i.e., N = 1 in this section.

2.4.1 Classical dither-based derivative estimation in discrete-time

To connect to frequency-domain system identification results, the analysis considers the DE in discrete time, i.e., in its form that is used for practical implementation. The considered continuous time, first-order single-input DE, with MA filter, is given by

$$\tilde{g}_{u_1}(t) = \frac{1}{T} \int_{t-T}^t \frac{2}{a_{d_1}} \cos(\omega_{d_1} t) J(\tau) \mathrm{d}\tau.$$
(2.33)

The DE in (2.33) is equivalent to the DE in Haring et al., 2013 and to (2.13) for n = 1 and N = 1. The smallest common period time is trivially determined to be $T = \frac{2\pi}{\omega_{d_1}}$ for this simple case with n = 1, N = 1.

Let $T_s \in \mathbb{R}$ be the sampling interval, which is selected such that T is an integer multiple of T_s , i.e.,

$$T = \frac{2\pi}{\omega_{d_1}} = RT_s, \qquad (2.34)$$

for some $R \in \mathbb{Z}_{>0}$. The discrete-time equivalent of (2.33) is

$$\tilde{g}_{u_1}(n_t T_s) = \frac{1}{R} \sum_{n_\tau = n_t - R + 1}^{n_t} \frac{2}{a_{d_1}} \cos(\omega_{d_1} n_\tau T_s) J(n_\tau T_s),$$
(2.35)

where $n_t \in \mathbb{Z}_{\geq 0}$ is the index of the current sampling time instant, and $n_\tau \in \{n_t - R + 1, n_t - R + 2, \ldots, n_t\}$ indicates the MA time window [t - T, t]. In the following section, (2.35) is derived in the frequency domain, demonstrating the equivalence between ES and frequency-domain system identification.

2.4.2 Frequency-domain analysis of classical dither-based derivative estimation

This section provides a frequency-domain analysis that can be used to obtain the aforementioned practical insights into dither frequency selection in ES.Using the discrete Fourier transformation (DFT) of the dither signal and the cost output, dither-based ES is presented as a frequency-domain estimation problem, which connects to existing frequency-domain system identification techniques. The DFT is a spectral representation of the sampled dither signal $d(n_tT_s)$ and sampled cost output $J(n_tT_s)$, where $n_tT_s \in \mathbb{R}_{\geq 0}$ is the current sampling time instant and T_s is the sample interval, can be given by the DFT.

Consider the sampled dither signal

$$d(n_t T_s) = a_{d_1} \cos(\omega_{d_1} n_t T_s) \stackrel{(2.34)}{=} a_{d_1} \cos\left(\frac{2\pi}{R} n_t\right), \qquad (2.36)$$

which is applied in combination with the DE in (2.33), according to Criterion 2.3.1. The k-th element of the DFT of $d(n_tT_s)$ in (2.36), over one moving time period [t - T, t], i.e., over R samples $n_{\tau} \in \{n_t - R + 1, n_t - R + 2, ..., n_t\}$, is given by

$$D(k, n_t) = \frac{1}{R} \sum_{\substack{n_\tau = n_t - R + 1}}^{n_t} d(n_\tau T_s) e^{-j2\pi \frac{n_\tau k}{R}}$$

$$= \frac{1}{R} \sum_{\substack{n_\tau = n_t - R + 1}}^{n_t} d(n_\tau T_s) \cos\left(2\pi \frac{n_\tau k}{R}\right) - j\frac{1}{R} \sum_{\substack{n_\tau = n_t - R + 1}}^{n_t} d(n_\tau T_s) \sin\left(2\pi \frac{n_\tau k}{R}\right),$$
(2.37)

where $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ are used. By taking the DFT in (2.37) over R samples, defined in (2.34), the element k = 1 of (2.37), i.e., the lowest nonzero frequency in the DFT, corresponds to the dither frequency ω_{d_1} . Substituting the sampled dither signal in (2.36) in (2.37) yields, for k = $0, 1, \ldots, R-1$:

$$D(k, n_t) = \begin{cases} \frac{a_{d_1}}{2} & \text{for } k \in \{1, R-1\} \\ 0 & \text{for } k \notin \{1, R-1\}. \end{cases}$$
(2.38)

Equivalent to D(k) in (2.37), the DFT of the cost output J(t), defined by $\mathcal{J}(k, n_t), k = 0, 1, \ldots, R$, is given by

$$\mathcal{J}(k,n_t) = \frac{1}{R} \sum_{\substack{n_\tau = n_t - R + 1 \\ n_\tau = n_t - R + 1}}^{n_t} J(n_\tau T_s) \cos \underbrace{\left(2\pi \frac{n_\tau k}{R}\right)}_{\substack{(2.34)\\ = \omega_{d_1} n_\tau T_s k}} - j \frac{1}{R} \sum_{\substack{n_\tau = n_t - R + 1 \\ n_\tau = n_t - R + 1}}^{n_t} J(n_\tau T_s) \sin \left(2\pi \frac{n_\tau k}{R}\right).$$

(2.39)

Observe that, by using the expressions for D(k) in (2.38), and $\mathcal{J}(k)$ in (2.39), the DE in (2.35) can be expressed as

$$\tilde{g}_{u_1}(n_t T_s) = \operatorname{Re}\left[\mathcal{J}(1, n_t) D^{-1}(1, n_t)\right].$$
(2.40)

Define the optimizer output as "input disturbance" $n_u = \hat{u}$, the dither signal as the "measured input" $u_m = d$, while J remains the measured output. Then, the right-hand side of (2.40) resembles a frequency-domain system identification problem that is known as the best linear approximation (BLA), see Pintelon and Schoukens, 2012. For $k = 0, 1, \ldots, R$, the BLA of the nonlinear system \mathcal{H} is given by

$$\hat{H}(k) = \mathcal{J}(k)U_m^{-1}(k) = \mathcal{J}(k)D^{-1}(k).$$
(2.41)

The BLA is a non-parametric model that provides the best possible linear approximation of a nonlinear system. The BLA is identical to the frequency



Figure 2.7. A schematic representation of the derivative estimator (DE) in (2.35) as a frequency-domain system identification measurement setup. The grey box indicates the dither-based DE_d , which is also depicted in Figure 2.1.

response function (FRF), for the special case that the considered system is LTI. Figure 2.7 schematically depicts the frequency-domain interpretation of the dither-based DE in (2.35). The time scale separation principle P1 in Section 2.2.2.3 requires, for classical dither-based ES, that the dither signal frequency is low with respect to the system time scale, such that the response of the system \mathcal{H} is close to its steady-state behavior. As a result, at the dither frequency which corresponds to k = 1, it holds that $\operatorname{Re}[\hat{H}(1, n_t)] \approx \hat{H}(1, n_t)$.

A difference with ES is that, the BLA assumes periodic signals, which is why $\hat{H}(k)$ in (2.41) is not a function of n_t . In ES, the output J is generally not periodic for time-varying $\hat{u}(t)$. However, the time-domain analysis using a Taylor series approximation, which is presented in Section 2.3.1, also relies on a constant \hat{u} .

Since the BLA is a linear approximation of a nonlinear system, it can only accurately describe a nonlinear system locally in the input. This is the same observation that follows from the time-domain analysis using a Taylor series approximation, which can be used to obtain that the derivative estimate is accurate when the dither amplitude is small, see Remark 2.5.

Summarizing, derivative estimation in dither-based ES is equivalent to a frequency-domain system identification problem at the dither frequency ω_{d_1} , over a moving window of R past time sampling instances, with indexes $n_{\tau} \in \{n_t - R + 1, n_t - R + 2, \ldots, n_t\}$. In the following subsections, this observation is used to analyze the effect of the dither amplitude and the dither frequency ω_{d_n} . This subsection is concluded by the following remarks.

Remark 2.12. The presented frequency-domain analysis of dither-based derivative estimation is trivially extended to multiple input systems, by considering a different identification frequency for each of the n inputs, i.e., n different dither frequencies. For N = 1, when the dither frequencies are selected according to Proposition 2.3.1, the elements of the BLA $\hat{H}(k, n_t)$, $1 \le k \le n$, each correspond to a dither frequency. Identifying a multiple-input system in such a way that the contribution of all inputs can be distinguished in the output, in a single experiment, is referred to as "zippered multisine" identification, see [Pintelon and Schoukens, 2012, Section 2.7.1].

Remark 2.13. In the identification of the BLA, see Pintelon and Schoukens, 2012, it is assumed that the system under consideration is a stable, periodic in, same period out (PISPO) system. When the system \mathcal{H} in (2.1) is uniformly convergent, see Pavlov et al., 2005, this assumption is valid.

Remark 2.14. The BLA, see Pintelon and Schoukens, 2012, can be interpreted as the set of R describing functions [Khalil, 2000, Section 7.2] of the subject nonlinear system, at the frequencies $\omega_{d_1}k$, k = 1, 2, ..., R - 1.

2.4.3 Dither signal amplitude

In the previous section, the equivalence between dither-based derivative estimation and estimation of the BLA of the system \mathcal{H} is presented. This section uses the equivalence to adopt a particular system identification result from Pintelon and Schoukens, 2012, in the context of dither-based ES, which is provided by the following remark.

Remark 2.15. [Pintelon and Schoukens, 2012, Assumption 2.4], states that, unbiased BLA estimation requires that the input disturbance $n_u(t) = \hat{u}(t)$, is independent of the "measured input" $u_m(t) = d(t)$. This requirement is not satisfied in dither-based ES. Namely, the derivative estimate $\tilde{g}(t)$, which is used by the ES optimizer \mathcal{F} , is T-periodic, see Section 2.3.2.3, i.e., $\tilde{g}_{u_1}(t)$ has the same periodicity as the dither signal. Since the optimizer is often LTI, e.g., $\hat{u}(t) = -c\tilde{g}_{u_1}(t)$, with c > 0 the optimizer gain, the optimizer output $\hat{u}(t)$ has a contribution at the dither frequency. As such, the "input disturbance" $n_u(t) = \hat{u}(t)$ is, indeed, dependent on the "measured input" $u_m(t) = d(t)$. In ES, this fundamental problem is counteracted by increasing the magnitude of d with respect to \hat{u} , either by decreasing the optimizer gain c, or increasing the dither amplitude a_{d_1} .

An intuitive explanation for the noted fundamental problem in Remark 2.15 is that the observed correlation between the measured input d and the cost J, which is used for estimation, is partly the result of feedback of J via the DE and the optimizer.

The main observation from Pintelon and Schoukens, 2012, Assumption 2.4, in the context of dither-based ES, is that the dither amplitude has to be sufficiently *large*, with respect to the optimizer gain. We care to stress that this result is obtained, regardless of any actual disturbing noise being present on the input or output of the system \mathcal{H} .

As such, in addition to an upper bound, see Remark 2.5 and e.g., Krstić and Wang, 2000; Nešić et al., 2010, there exists a lower bound on the dither amplitude, that scales with the optimizer gain. This is an important result for tuning of dither-based ES in practice.

2.4.4 Dither signal frequency: Tuning ω_{d_n}

Having addressed the dither signal amplitude, in Remark 2.5 and Section 2.4.3, the ratio γ between different dither frequencies in multiple-input ES in Section 2.3.2.4, the selection of the highest dither frequency ω_{d_n} in classical ditherbased ES, remains an open issue. The ES time scale separation analysis, which is summarized in Section 2.3.2.3, requires ω_{d_n} to be small with respect to the smallest frequency characterizing the system dynamics. However, when ω_{d_n} is lower than necessary, the convergence rate of ES is limited, as a result of an increased value of T.

In this subsection, the frequency-domain analysis, presented in Section 2.4.2, is used to clarify the effect of the choice of ω_{d_n} on the quality of the derivative estimate. At the end of this section, the observations of two examples are summarized as a practical design guideline in Criterion 2.4.1.

Consider the following example, where the DE in (2.35) is demonstrated for a constant optimizer output \hat{u} .

Example 2.16. Consider the single-input-single-output (SISO) nonlinear system \mathcal{H}_{ex} , which is a series connection of an LTI first-order low-pass filter and a quadratic, static, output nonlinearity:

$$\dot{x}_{\mathcal{H}_{ex}}(t) = -\omega_p x_{\mathcal{H}_{ex}}(t) + \omega_p u(t) \tag{2.42a}$$

$$J(t) = x_{\mathcal{H}_{ex}}^2(t), \qquad (2.42b)$$

where $\omega_p = 2\pi$. For constant u, the system \mathcal{H}_{ex} has the equilibrium

$$x_{\mathcal{H}_{ex}} = l(u) = u_{t}$$

such that the steady-state map of the system is

$$Q_J(u) = u^2. (2.43)$$

Consider the DE in (2.35) with sampling interval $T_s = 0.01 \ s$, $a_{d_1} = 0.1$, and $\omega_{d_1} = 0.2\pi \ rad/s$, at $\hat{u} = 1$. As a result, $T = \frac{2\pi}{\omega_{d_1}} = 10 \ s$ and correspondingly R = 1000.

Since \hat{u} is constant, the sampled input $u(n_tT_s)$ is periodic with a period of R samples. Using that \mathcal{H}_{ex} is a stable PISPO system, see Remark 2.13, the output $J(n_tT_s)$ is also periodic over R samples, after an initial transient phase. As a result, the derivative estimate $\tilde{g}_{u_1}(n_tT_s)$ is constant, after a sufficiently large time n_tT_s , because it takes the average over R samples. This initialization time is a result of the initial transient of the dynamics of the system \mathcal{H}_{ex} and the required data, R samples, for the MA filter. For $n_t < R$, taking a correct average over R samples is not possible.

The resulting values of u(t), J(t), and $\tilde{g}_{u_1}(t)$ are plotted in blue lines in Figure 2.8. The bottom plot confirms that the derivative estimate $\tilde{g}_{u_1}(t)$ is not



Figure 2.8. Time-domain plot of example input signals u(t) with different dither frequencies, the corresponding output J(t) of the system \mathcal{H}_{ex} in (2.42), and the resulting derivative estimate $\tilde{g}_{u_1}(t)$ using the derivative-estimator (DE) in (2.35).

exact, for t < T = 10 s, i.e. for $n_t < R$. For t > 10 s, the estimate $\tilde{g}_{u_1}(t)$ converges to the exact value $g_{u_1}(\hat{u}) = 2\hat{u} = 2$. The initial mismatch for t > 10 s, is due to the initial transient of J(t), which can be seen in the middle plot, that affects the average over the first R samples.

Let us now take the BLA $\hat{H}(k)$ in (2.41) of the system \mathcal{H}_{ex} in (2.42), with the same values R = 1000 and $T_s = 0.01$ s as in Example 2.16. Because \hat{u} is constant, the input $u(n_tT_s)$ and the cost $J(n_tT_s)$ are periodic in R samples, after an initial transient phase. As a result the BLA is constant such that the n_t dependency of the DFTs can be dropped. The upper and middle plot in Figure 2.9 depict a Bode plot of the BLA $\hat{H}(k)$, for $k = 1, 2, \ldots, 100$. The BLA clearly describes the low-pass characteristic of the LTI dynamics of the system \mathcal{H}_{ex} in (2.42a), which are given by $\mathcal{H}_{ex,LTI}(j\omega) = (j\omega + \omega_p)^{-1}\omega_p$. The 6 dB difference between $|\mathcal{H}_{ex,LTI}(j\omega)|$ and $|\hat{H}(k)|$ at low frequencies is due to the output scaling of $\mathcal{H}_{ex,LTI}$ in (2.42b); the derivative of $Q_J(u)$ in (2.43), for $\hat{u} = 1$, is $g_{\hat{u}_1}^1 = 2\hat{u}_1 = 2$, which is equal to 6 dB. The bottom plot in Figure 2.9 depicts the real part $\operatorname{Re}[\hat{H}(k)]$.

At the difference ω_{d_1} , which corresponds to k = 1, the derivative $g_{u_1}(\hat{u})$



Figure 2.9. Upper two plots: Bode plot of the best linear approximation (BLA) $\hat{H}(k)$ of the system \mathcal{H}_{ex} in (2.42), and of the LTI part of the system \mathcal{H}_{ex} , denoted by $\mathcal{H}_{ex,LTI}(j\omega)$. Bottom plot: The real part of the BLA $\hat{H}(k)$ which at the dither frequencies is equal to the estimated derivative $\tilde{g}_{u_1}(t)$.

is well approximated by $\tilde{g}_1^1(n_tT_s) = \operatorname{Re}[\hat{H}(1)]$, see the bottom plot in Figure 2.9. The problem in Example 2.16 is repeated with higher dither frequencies: $\omega_{d_1} = \pi$ rad/s and $\omega_{d_1} = 5\pi$ rad/s. The frequency-domain data in Figure 2.9 indicates that the derivative estimate $\tilde{g}_{u_1}(n_tT_s) = \operatorname{Re}[\hat{H}(1)]$ is less accurate for $\omega_{d_1} = \pi$ rad/s and $\omega_{d_1} = 5\pi$ rad/s. This observation is confirmed in the time-domain results in Figure 2.8. Figure 2.8 also confirms that for a higher value of ω_{d_1} , less time is required to obtain a derivative estimate, which on itself is beneficial for the convergence rate of ES as we discussed in Section 2.3.2.3.

For the system \mathcal{H}_{ex} in (2.42), the frequency-domain result in Figure 2.9 does not provide a clear upper limit on ω_{d_n} . In fact, whenever the sign of the estimate $\tilde{g}_{u_1}(n_tT_s) = \operatorname{Re}[\hat{H}(1)]$ is equal to the sign of the true derivative $g_{u_1}(\hat{u}(t))$, a gradient descent type ES does converge to the optimum. However, in ES practice, one suffers from system uncertainty, which in general, increases for higher frequencies. The following example considers a second-order system in which such uncertainty can be illustrated, and a clear upper limit on the dither frequency exists.

Example 2.17. Consider the SISO nonlinear system $\mathcal{H}_{ex,2}$, which is a slight



Figure 2.10. Time-domain plot of example input signals u(t) with different dither frequencies, the corresponding output J(t) of the system $\mathcal{H}_{ex,2}$ in (2.44), and the resulting derivative estimate $\tilde{g}_{u_1}(t)$ using the derivative-estimator (DE) in (2.35).

modification of the system \mathcal{H}_{ex} in (2.42):

$$\dot{\boldsymbol{x}}_{\mathcal{H}_{ex,2}}(t) = \begin{bmatrix} 0 & 1\\ -\omega_p^2 & -2\beta_p\omega_p \end{bmatrix} \boldsymbol{x}_{\mathcal{H}_{ex,2}}(t) + \begin{bmatrix} -1\\ 0 \end{bmatrix} \boldsymbol{u}(t)$$
(2.44a)

$$J(t) = \boldsymbol{x}_{\mathcal{H}_{ex,2}}^{\top}(t) \begin{bmatrix} 0 & 0\\ 0 & 1 \end{bmatrix} \boldsymbol{x}_{\mathcal{H}_{ex,2}}(t), \qquad (2.44b)$$

where $\omega_p = 2\pi$ and $\beta_p = 0.2$. For constant u, the system $\mathcal{H}_{ex,2}$ has the equilibrium

$$\boldsymbol{x}_{\mathcal{H}_{ex}} = l(u) = \begin{bmatrix} \frac{-2\beta_p}{\omega_p} \\ 1 \end{bmatrix} u,$$

such that the steady-state map of the system is equal to that of the system \mathcal{H}_{ex} in (2.42), and is given by

$$Q_J(u) = u^2$$

The same simulations that are explained in Example 2.16 are done with the system $\mathcal{H}_{ex,2}$ in (2.44). Again, the DE in (2.35) is used with sampling interval



Figure 2.11. Upper two plots: Bode plot of the best linear approximation (BLA) $\hat{H}(k)$ of the system $\mathcal{H}_{ex,2}$ in (2.44). Bottom plot: The real part of the BLA $\hat{H}(k)$ which at the dither frequencies is equal to the estimated derivative $\tilde{g}_{u_1}(t)$.

 $T_s = 0.01 \ s, \ a_{d_1} = 0.1, \ and \ \omega_{d_1} = 0.2\pi \ rad/s, \ at \ \hat{u} = 1.$ As a result, $T = 2\pi/\omega_{d_1} = 10 \ s \ and \ correspondingly \ R = 1000.$

Figures 2.10 and 2.11 depict the results in the time and frequency domain, respectively.

The results in Figures 2.10 and 2.11 show that, for the system $\mathcal{H}_{ex,2}$ in (2.44), the derivative $g_{u_1}(\hat{u})$ is well approximated by $\tilde{g}_{u_1}(n_tT_s) = \operatorname{Re}[\hat{H}(1)]$, for $\omega_{d_1} = 0.2\pi$ rad/s. However, for $\omega_{d_1} > \omega_p = 2\pi$ rad/s, the sign of the estimate $\tilde{g}_{u_1}(n_tT_s) = \operatorname{Re}[\hat{H}(1)]$ is opposite to the sign of the true derivative $g_{u_1}(\hat{u}(t))$, see Figure 2.11. This yields an unstable closed-loop ES system, when a gradient descent optimizer is applied.

A general observation from the frequency-domain analysis is that, for sufficiently small ω_{d_1} , the real part of the estimated BLA approximates the steadystate behavior of the system \mathcal{H} , that is described by the map $Q_J(\boldsymbol{u})$. The practical upper bound on ω_{d_n} depends on the system dynamics, to be precise, on the smallest frequency characterizing the system dynamics, which is defined as $\omega_{\mathcal{H}}$. In Examples 2.16 and 2.17, $\omega_{\mathcal{H}} = \omega_p$. The observations obtained from Examples 2.16 and 2.17, are summarized in the following design criterion. **Criterion 2.4.1.** Assuming that, the smallest frequency characterizing the system dynamics $\omega_{\mathcal{H}}$ can be estimated, e.g., from step response measurements at inputs in the typical operating range of the system, the highest dither frequency ω_{d_n} , for classical dither-based ES, is selected as

$$\omega_{d_n} = \frac{1}{2}\omega_{\mathcal{H}}.$$

2.4.5 Frequency-domain perspective on time scale separation in advanced extremum seeking

This section addresses two advanced approaches in the class of dither-based extremum seeking: ES with phase compensated demodulation, and fast ES. The frequency-domain analysis, presented in Section 2.4.2, is used to analyze these approaches. By doing so, the practical implication of the requirements on the system \mathcal{H} , to apply such advanced approaches become clear. Moreover, the frequency-domain description of dither-based ES is used as a framework, that connects classical, phase compensated, and fast, dither-based ES. The following gradient descent optimizer \mathcal{F} is considered for all approaches:

$$\dot{\hat{u}}(t) = -c\tilde{g}_{u_1}(t),$$

with c > 0 the optimizer gain.

2.4.5.1 Phase compensated dither-based extremum seeking

Consider again Example 2.16, in particular the BLA in Figure 2.9. For $\omega_{d_1} = 0.2\pi$ rad/s, we obtain that $\tilde{g}_{u_1}(t) = \operatorname{Re}[\hat{H}(1)] \approx g_{u_1}(\hat{u})$. For $\omega_{d_1} = \pi$ rad/s, the approximation $\tilde{g}_{u_1}(t)$ is smaller than the true derivative $g_{u_1}(\hat{u})$, or in general, $\tilde{g}_{u_1}(t) \neq g_{u_1}(\hat{u})$. For the example system \mathcal{H}_{ex} in (2.42), this mismatch is due to:

- (i) The dynamic part of \mathcal{H}_{ex} in (2.42) suppresses the magnitude of the sinusoidal dither input d(t) in the output J(t) at the dither frequency ω_{d_1} .
- (ii) The dynamic part of \mathcal{H}_{ex} in (2.42) introduces a phase shift in the output J(t) at the dither frequency ω_{d_1} .

To counteract (ii), the phase of the demodulation signal can be compensated such that it matches the phase of the cost output J(t):

$$m(t) = \cos(\omega_{d_1} t - \phi)$$

with $\phi \in [0, 2\pi)$. This approach is proposed in the literature, see, e.g., Krstić, 2000 or Haring et al., 2013. The effect of the system dynamics on the magnitude of the output (i), is not compensated for; however, this is implicitly done by

tuning the optimizer gain c. The following example demonstrates phase compensated dither-based derivative estimation for the first-order system \mathcal{H}_{ex} in (2.42), which is well suited for the presented ES approach.

Example 2.18. Consider the DE in (2.35) with a phase shifted demodulation signal, given by

$$\tilde{g}_{u_1}(n_t T_s) = \frac{1}{R} \sum_{n_\tau = n_t - R + 1}^{n_t} \frac{2}{a_{d_1}} \cos(\omega_{d_1} n_\tau T_s - \phi) J(n_\tau T_s).$$
(2.45)

To analyze the effect of such phase compensation in the frequency-domain, consider the "phase shifted" DFT \mathcal{J}_{ϕ} of the output, given by

$$\mathcal{J}_{\phi}(k, n_{t}) = \frac{1}{R} \sum_{n_{\tau}=n_{t}-R+1}^{n_{t}} J(n_{\tau}T_{s}) \cos \underbrace{\left(2\pi \frac{n_{\tau}k}{R} - \phi\right)}_{\substack{(2.34)\\ \omega_{d_{1}}n_{\tau}T_{s}k}} \dots \\ - j \frac{1}{R} \sum_{n_{\tau}=n_{t}-R+1}^{n_{t}} J(n_{\tau}T_{s}) \sin \left(2\pi \frac{n_{\tau}k}{R} - \phi\right),$$
(2.46)

which is equal to (2.39) when $\phi = 0$. A frequency-domain description of the input-output behavior of the equivalent phase compensated system, with input $d(n_tT_s)$ and output $J_{\phi}(n_tT_s)$ can be obtained equivalent to the BLA in (2.41)

$$\hat{H}_{\phi}(k) = \mathcal{J}_{\phi}(k)U_m^{-1}(k) = \mathcal{J}_{\phi}(k)D^{-1}(k).$$
(2.47)

The resulting $\hat{H}_{\phi}(k)$ for $\phi = \pi/4$ is depicted by Figure 2.12 (in yellow), together with the BLA $\hat{H}(k)$ in (2.41) that is also depicted in Figure 2.9.

Clearly, the phase shift of $\phi = \pi/4$ [rad] is present in $\hat{H}_{\phi}(k)$, see the middle plot in Figure 2.12. The magnitude $|\hat{H}_{\phi}(k)|$ is identical to $|\hat{H}(k)|$, which is to be expected since we only introduced a phase shift, no scaling.

The bottom plot in Figure 2.12 shows the difference between the true derivative $g_{u_1} = 2\hat{u}$, and the estimated derivatives $\operatorname{Re}[\hat{H}_{\phi}(k)]$ and $\operatorname{Re}[\hat{H}(k)]$, with and without phase compensated demodulation. As a result of the phase shift, the estimation error is smaller for frequencies higher than ≈ 0.5 Hz. As such, the phase shift problem, mentioned in (ii) is effective, and the DE with phase shifted demodulation can be applied with a slightly increased dither frequency.

A disadvantage of phase shifted demodulation is that for any choice for the dither frequency, an estimation error remains. For the classical case, $\omega_{d_1} \to 0$ yields $\operatorname{Re}[\hat{H}(1)] = \hat{H}(1)$, see also Figure 2.12, such that the estimation error due to the dynamics of the system \mathcal{H} can be eliminated. The estimation error ror in phase compensated ES, for small ω_{d_1} is however bounded; using basic



Figure 2.12. Upper two plots: Bode plot of the best linear approximation (BLA) $\hat{H}(k)$ of the system \mathcal{H}_{ex} in (2.42). Bottom plot: The real part of the BLA $\hat{H}(k)$ which at the dither frequencies is equal to the estimated derivative $\tilde{g}_{u_1}(t)$. In addition, data corresponding to the frequency-domain characterization $\hat{H}_{\phi}(k)$ of the equivalent phase compensated system with output $J_{\phi}(t)$ is given.

trigonometric math, it can be derived that for $\phi = \pi/4$

$$\lim_{\omega d_1 \to 0} \left(\frac{\operatorname{Re}[\hat{H}_{\phi}(1)]}{\hat{H}_{\phi}(1)} \right) = \frac{\sqrt{2}}{2}.$$

To summarize, in general, phase compensated ES allows to slightly increase the dither frequency towards the system time scale. For example, approximately the same estimation error that is obtained with $\omega_{d_1} = \pi$ [rad/s] for the standard DE, see Figure 2.9, is obtained at a higher dither frequency $\omega_{d_1} = 2\pi$ [rad/s] when a phase compensation of $\phi = \pi/4$ is applied, see Figure 2.12. For the second-order system in Figure $\mathcal{H}_{ex,2}$ in (2.44) however, the phase delay of the system decreases rapidly near the pole frequency ω_p , see Figure 2.11. Hence, a small uncertainty in the pole location ω_p causes a large effect on the phase delay, which makes compensating the phase delay a less robust approach.

2.4.5.2 Fast extremum seeking

Phase compensation in dither-based ES is aimed at predicting the steady-state response of the system, from a measurement at a higher dither frequency ω_{d_1} by accounting for the expected phase loss of the system. However, in the frequency band where the system time scale is observed, the phase loss may change rapidly over the frequency, e.g., near a complex pole pair, see the BLA of $\mathcal{H}_{ex,2}$ in Figure 2.11 near 1 Hz. As such, for a practical setting where the system dynamics are only known approximately, e.g., the complex pole frequency is unknown, it is not possible to determine the required ϕ to push the dither frequency ω_{d_1} towards the system time scale. The robustness for uncertainty in ω_p is limited.

Consider again the BLA of $\mathcal{H}_{ex,2}$ in Figure 2.11. Time scale separation in classical dither-based ES, discussed in Section 2.3.2.3, requires $\omega_{d_1} \ll \omega_p$. As a result, the DE is robust for uncertainty in ω_p .

A different approach is taken in fast ES, see Moase and Manzie, 2012; Guay, 2016; Atta and Guay, 2017. Consider the system $\mathcal{H}_{ex,2}$ in (2.44). When $\omega_{d_1} \gg \omega_p$, the phase loss of the system approaches $-\pi/2$, see the BLA in Figure 2.11. As a result, with $\phi = \pi/2$, the phase compensated DE in (2.45) can be applied. When $\omega_{d_1} \gg \omega_p$, such an approach is robust for uncertainty in ω_p , for the considered example. The key assumption here, is that that the phase loss of $\mathcal{H}_{ex,2}$ for high frequencies is known. This assumption is made in Moase and Manzie, 2012 and Atta and Guay, 2017 by requiring knowledge of the systems relative degree. Opposed to the considered Wiener-Hammerstein systems in Moase and Manzie, 2012; Atta and Guay, 2017, Guay, 2016 considers a nonlinear system with strong relative degree one. When the system's high-frequency dynamics are not exactly known, which is often the case in practice, application of fast ES is not possible.

2.4.5.3 Summary of frequency-domain analysis of time scale separation for dither-based extremum seeking

Having addressed classical, phase compensated, and fast dither-based ES from a frequency-domain perspective, this section is concluded with a unifying view on the relation between the three approaches, in the frequency domain.

A high dither frequency ω_{d_1} increases the DE time scale, and thereby enables fast convergence of ES. Using phase compensated ES and fast ES, it is possible to increase the dither frequency ω_{d_1} . However, approximate knowledge of the system is required: Phase compensation requires that the expected phase delay gradually decreases, while fast ES requires relative degree knowledge and high frequency perturbation.

Figure 2.13 shows a schematic Bode plot of the BLA of a system with secondorder dynamics. The typical dither frequency bands are indicated, for classical, phase compensated, and fast ES. As such, the presented frequency-domain analysis provides a practical analysis tool to assess dynamic properties of the nonlinear system subject to ES, which can be used in deciding which ES approach to apply.

Fast ES is an interesting approach since the convergence rate of ES can be increased significantly (the frequency scale in the Bode plot in Figure 2.13 is logarithmic). However, practical limitations on the application of fast ES exist, which are insightful from the presented frequency-domain analysis, and are summarized in the following remarks.

Remark 2.19. In many practical applications, the dynamics of the system \mathcal{H} at high frequencies are complex, which complicates the application of fast ES, since it is not trivial to determine the relative degree. See, e.g., Hunnekens et al., 2015 where ES is applied to optimize a controller parameter for a motion stage. An additional issue in practice can be that, due to a limited sampling frequency $F_s = T_s^{-1}$ Hz, the required dither frequency cannot be measured when $2\pi\omega_{d_1} > \frac{F_s}{2}$ Hz. Finally, the output signal-to-noise ratio (SNR) is lower for high frequencies, as a result of the reduced response magnitude, given common roll-off properties of the system dynamics.

Remark 2.20. The magnitude of the derivative estimate, obtained with the fast ES approach, is not accurate. To be precise, the estimate contains an unknown scaling. For single-input systems, this is not an issue, since appropriate tuning of the optimizer gain c can account for the unknown scaling. For multiple-input systems however, depending on the optimizer, the ratio between the separate derivatives may be of interest in order to get an accurate gradient estimate. Hence, for those cases, application of fast ES approaches is not always possible.

2.5 Input-based derivative estimation

Sections 2.3 and 2.4 presented several types of dither-based DEs. This section provides an alternative input-based approach, that omits a fundamental drawback of dither-based derivative estimation.

In the analysis of the dither-based DE presented in Section 2.3.1, the optimizer output \hat{u} is constant. This assumption is valid with the averaging theory that is used in the analysis of classical dither-based ES. For small c, the solution $\hat{u}_r(\tau)$ of the reduced system (2.29) is well approximated by the solution $\hat{u}_{av}(\tau)$ of the averaged system, because $\hat{u}_r(\tau)$ varies slowly with respect to the dither signal.

A similar observation is done in Remark 2.15, where, from a system identification perspective, $\hat{\boldsymbol{u}}(t)$ can be treated as input disturbance, that is dependent on the dither signal, see Figure 2.7. When $\hat{\boldsymbol{u}}$ is constant because c = 0, the DFT of $\hat{\boldsymbol{u}}$ has no contribution at the dither frequency and as such, the fundamental problem, explained in Remark 2.15, is absent.

The trivial solution to eliminate the effect of the "input disturbance" $n_u(t) = \hat{u}(t)$ on the "measured input" d(t), is to consider the actual input u(t) as mea-



Figure 2.13. Sketch of the Bode plot of the best linear approximation (BLA) $\hat{H}(k)$ of an example system \mathcal{H} with second-order dynamics. Typical frequency bands for the dither frequency ω_d are indicated for three types of dither-based extremum seeking (ES).

sured input instead. This approach is introduced as input-based ES in Section 2.2.2.2, and is schematically depicted on the right-hand side of Figure 2.1. Potentially, the optimizer gain c can be increased in input-based ES, since the fundamental input disturbance problem in dither-based ES, which in practice is addressed by time scale separation, is omitted. An additional advantage of input-based ES that it can be operated dither-free, when the input u(t) is PE.

In the literature, some examples of input-based ES exist. In Gelbert et al., 2012, an extended Kalman filter is applied. Alternative observer approaches, with ES convergence proofs, are proposed in Guay and Dochain, 2015 and Haring, 2016, Chapter 2.

In Hunnekens et al., 2014, a novel type of ES is presented for single-input systems, which continuously uses a linear least-squares estimate of the derivative of the performance map based on measurements of u(t) and J(t) over a past time window. Here, this approach is generalized to multiple-input systems. Estimation is limited to first-order derivatives, i.e., N = 1. Tuning of the parameters in this least-squares input-based DE is relatively easy. The interesting observation is made, that for constant \hat{u} , it is equivalent to the dither-based DE in (2.13), for N = 1. As such, it enables a clear comparison between dither-based and input-based ES.

2.5.1 Linear least-squares derivative estimation

The least-squares DE approach involves minimizing the following least-squares cost function:

$$W_{LS}(t) = \int_{t-T_u}^{t} \left(J(\tau) - \underbrace{\left[\tilde{Q}_J(t) \quad \tilde{g}_{u_1}(t) \quad \tilde{g}_{u_2}(t) \quad \cdots \quad \tilde{g}_{u_n}(t) \right]}_{\tilde{g}^{\top}(t)} \begin{bmatrix} 1 \\ u(\tau) \end{bmatrix} \right)^2 \mathrm{d}\tau,$$
(2.48)

The product $\tilde{\boldsymbol{g}}^{\top}(t) \begin{bmatrix} 1 & \boldsymbol{u}^{\top}(\tau) \end{bmatrix}^{\top}$ in (2.48) is a linear static model for the cost $J(\tau), \tau \in [t - T_u, t].$

The values of $\tilde{Q}_J(t)$ and $\tilde{g}_{u_k}(t)$, k = 1, 2, ..., n, that minimize W_{LS} in (2.48) satisfy

$$\frac{\partial W_{LS}}{\partial \tilde{Q}_J} = 0, \tag{2.49a}$$

$$\frac{\partial W_{LS}}{\partial \tilde{g}_{u_k}} = 0, \ k = 1, 2, \dots, n.$$
(2.49b)

For the derivatives of W_{LS} in (2.49a) and (2.49b) to be equal to zero, the following should hold:

$$K_{u}(\boldsymbol{u}(t))\tilde{\boldsymbol{g}}(t) = \int_{t-T_{u}}^{t} \begin{bmatrix} 1\\ \boldsymbol{u}(\tau) \end{bmatrix} J(\tau) d\tau,$$

where

$$K_{\boldsymbol{u}}(\boldsymbol{u}(t)) = \int_{t-T_{\boldsymbol{u}}}^{t} \begin{bmatrix} 1\\ \boldsymbol{u}(\tau) \end{bmatrix} \begin{bmatrix} 1\\ \boldsymbol{u}(\tau) \end{bmatrix}^{\top} d\tau.$$
(2.50)

When the matrix $K_u(\boldsymbol{u}(t))$ is invertible, which, in essence, represents a PE condition for the input \boldsymbol{u} , the derivative estimate $\tilde{\boldsymbol{g}}(t)$ is obtained using measurements of $\boldsymbol{u}(t)$ and J(t) as follows:

$$\tilde{\boldsymbol{g}}(t) = K_u^{-1}(\boldsymbol{u}(t)) \int_{t-T_u}^t \begin{bmatrix} 1\\ \boldsymbol{u}(\tau) \end{bmatrix} J(\tau) \mathrm{d}\tau.$$
(2.51)

Remark 2.21. The matrix $K_u(\boldsymbol{u}(t))$ becomes poorly conditioned if the input signal $\boldsymbol{u}(t)$ is not sufficiently exciting the system; this is, for instance, the case if for a single-input system, the input u(t) resides in the same point during time interval T_u , or if $\boldsymbol{u}(t)$ describes a straight line in the input space for a dual-input system. To avoid poor conditioning, a dither signal $\boldsymbol{d}(t)$ can be added to the optimizer output $\hat{\boldsymbol{u}}(t)$ to prevent singularity of $K_u(\boldsymbol{u}(t))$.

At this point, the equivalence of the proposed input-based DE approach, and the dither-based DE approach presented in Section 2.3.1 can be emphasized. Observe that, for N = 1, the vectors $\boldsymbol{m}(t)$ in (2.10) and $\boldsymbol{p}(t)$ in (2.8), that appear in the matrix K in (2.11), can be given as function of the dither signal $\boldsymbol{d}(t)$ as

$$\boldsymbol{m}(t) = \boldsymbol{p}(t) = A^{-1} \begin{bmatrix} 1 \\ \boldsymbol{d}(t) \end{bmatrix}$$

Substituting in (2.11) yields

$$K = A^{-1} \underbrace{\int_{t-T}^{t} \begin{bmatrix} 1 \\ d(\tau) \end{bmatrix} \begin{bmatrix} 1 \\ d(\tau) \end{bmatrix}^{\top} d\tau}_{:=K_d} .$$
(2.52)

Under the conditions in Proposition 2.3.1 there exists a value of T for which the matrix K_d in (2.52) is constant. Using the above expressions and the fact that A is a diagonal matrix, the dither-based DE in (2.13) can be given as:

$$\tilde{\boldsymbol{g}}(t) = AK_d^{-1} \int_{T-t}^t A^{-1} \begin{bmatrix} 1\\ \boldsymbol{d}(\tau) \end{bmatrix} J(\tau) \mathrm{d}\tau = K_d^{-1} \int_{T-t}^t \begin{bmatrix} 1\\ \boldsymbol{d}(\tau) \end{bmatrix} J(\tau) \mathrm{d}\tau.$$
(2.53)

The dither-based DE in (2.53) with K_d in (2.52), has the same structure as the least squares DE in (2.51) with $K_u(\boldsymbol{u}(t))$ in(2.50). As such, the proposed least-squares DE can be interpreted as the input-dependent version of the ditherbased DE.

The full rank requirements on the matrices $K_u(\boldsymbol{u}(t))$ in (2.50) and K_d in (2.52) resemble PE conditions. As noted in Remark 2.21, a dither signal can be added to $\hat{\boldsymbol{u}}(t)$ to achieve PE of $\boldsymbol{u}(t)$, while $\boldsymbol{d}(t)$ is PE when the conditions in Proposition 2.3.1 are satisfied.

2.6 Simulation study

This section presents a simulation study in which the main results of this chapter are illustrated. To be precise, a comparison is made between classical ditherbased ES and the input-based ES presented in Section 2.5.1. Both approaches are schematically depicted in Figure 2.2. The presented example demonstrates the main results from this chapter: (1) The dither frequency tuning guideline in Criterion 2.4.1, and the optimal dither frequency ratio discussed in Section 2.3.2.4 are used, (2) the existence of a lower bound on the dither amplitude, derived in Section 2.4.3, and (3) the advantage of input-based ES, which is fundamentally motivated using system identification results in Section 2.4.3. In addition, the dither-based example demonstrates that application of a high-pass filter, in addition to an MA filter, is beneficial.

Parameter	Value	$a_{d_1} = a_d \ [\text{-}]$	$\{0.1, 0.15\}$
$\omega_{d_1} [\mathrm{rad/s}]$	$^{1/2\pi}$	$a_{d_2} = a_d \ [-]$	$\{0.1, 0.15\}$
$\omega_{d_2} [\mathrm{rad/s}]$	π	T [s]	4
$\omega_{m_1} [\mathrm{rad/s}]$	$^{1}/_{2}\pi$	T_u [s]	4
$\omega_{m_2} [\mathrm{rad/s}]$	π	c [-]	$\{0.035, 0.1\}$
$\omega_{HP} [\mathrm{rad/s}]$	$^{1}/_{4}\pi$	$oldsymbol{u}_0$ [-]	$[1.5 1.5]^{ op}$

Table 2.2. Overview of the extremum seeking controller parameters.

2.6.1 Simulation setup

The considered system has two inputs, which are both passed through the SISO second-order LTI dynamics in (2.44a), in Example 2.17, and subsequently through a two-input-single-output nonlinear map. The system is given by

$$\dot{x}_{\mathcal{H}_{ex,3}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_p^2 & -2\beta_p\omega_p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_p^2 & -2\beta_p\omega_p \end{bmatrix} x_{\mathcal{H}_{ex,3}}(t) - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{u}(t) \quad (2.54a)$$

$$\boldsymbol{y}_{LTI}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_{\mathcal{H}_{ex,3}}(t)$$
(2.54b)

$$J(t) = \boldsymbol{y}_{LTI}^{\top} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \boldsymbol{y}_{LTI}, \qquad (2.54c)$$

with $\omega_p = 2\pi$, and $\beta_p = 0.2$. The map $Q_J(\boldsymbol{u})$, corresponding to (2.54), is equal to (2.54c)

$$Q_J(\boldsymbol{u}) = \boldsymbol{u}^{ op} \begin{bmatrix} 1 & rac{1}{2} \\ rac{1}{2} & 1 \end{bmatrix} \boldsymbol{u}$$

The smallest frequency characterizing the dynamics of (2.54) is $\omega_{\mathcal{H}} = \omega_p$, and hence, Criterion 2.4.1 can be used to select $\omega_{d_2} = \frac{\omega_p}{2} = \pi$. The optimal dither frequency ratio γ^* in (2.32) is $\gamma_1^* = 1/2$, see Example 2.10, such that $\omega_{d_1} = 1/2\pi$. According to Criterion 2.3.1, the demodulation frequencies are $\omega_{m_1} = \omega_{d_1} = 1/2\pi$ and $\omega_{m_2} = \omega_{d_2} = \pi$. The applied dither amplitude for both inputs is $a_d \in \{0.1, 0.15\}$. The ratio γ_1^* satisfies the conditions of Proposition 2.3.1, which provides the value T = 4. To aid a clear comparison, the time window for input-based ES is selected equally, i.e., $T_u = 4$. In general, T_u is not necessarily a function of the dither frequencies. An overview of the ES controller parameters is provided in Table 2.2, including the optimizer gain c, the initial input u_0 , and the pole frequency ω_{HP} for a high-pass filter of the form (2.18).

Both the dither-based DE in (2.13) and the input-based DE in (2.51) require an initialization time for t < T and $t < T_u$, respectively. Moreover, the system $\mathcal{H}_{ex,3}$ in (2.54) has an initial transient. To eliminate both these initialization effects from the example, the optimizer gain c is equal to zero for $-3T \leq t < 0$ to allow initialization of the DE and the system. As such, the applied optimizer \mathcal{F} is described by:

$$\dot{\hat{\boldsymbol{u}}}(t) = \begin{cases} -c \begin{bmatrix} \tilde{g}_{u_1}(t) \\ \tilde{g}_{u_2}(t) \end{bmatrix} & \text{when } t \ge 0\\ 0 & \text{when } -3T \le t < 0 \end{cases}$$

where $c \in \mathbb{R}_{>0}$. The dither and demodulation signals are phase shifted over $\frac{\pi}{2}$, i.e., the sine is used instead of the cosine, to obtain $u(t) = \hat{u}(t)$ at t = 0.

2.6.2 Simulation results

This section provides the results of four simulations, which are depicted by Figure 2.14 and are discussed one at a time:

- (i) Classical dither-based ES baseline: The black line in Figure 2.14 depicts the result of classical dither-based ES with an MA filter, c = 0.035 and $a_d = 0.1$. The value c = 0.035 is selected high on purpose, to illustrate the poor convergence.
- (ii) Classical dither-based ES increased dither amplitude: The gray line in Figure 2.14 depicts the result of classical dither-based ES with an MA filter, c = 0.035 and $a_d = 0.15$. The convergence with the same value c = 0.035 is improved compared to the case with $a_d = 0.1$ (black line).
- (iii) Classical dither-based ES with high-pass filter: The red line in Figure 2.14 depicts the result of classical dither-based ES with a high-pass filter and an MA filter, c = 0.035 and $a_d = 0.1$. The high-pass filter \mathcal{H}_{HP} in (2.18) is applied on the cost J before demodulation. The convergence with the same value c = 0.035 is improved compared to the case with only the MA filter $a_d = 0.1$ (black line).
- (iv) Input-based ES: The blue line in Figure 2.14 depicts the result of input based ES, c = 0.1 and $a_d = 0.1$. The convergence is faster than in the presented dither-based ES examples, as a result of the higher value c = 0.1 that is used, opposed to c = 0.035 for dither-based ES. However, for c = 0.1, all of the presented dither-based ES examples are unstable.

The observed results illustrate two of the main contributions of this chapter. First, the existence of a lower bound on the dither amplitude, introduced in Section 2.4.3, is illustrated by (ii): The transient ES performance is increased for $a_d = 0.15$ compared to $a_d = 0.1$. In fact, for $a_d \leq 0.08$ the presented example with dither-based ES without high-pass filter is unstable. Second, (iv) demonstrates that input-based ES can be operated with a higher optimizer gain c = 0.1, which yields an increased convergence rate. This is a result of omitting the fundamental problem of dither-based ES, which is, that \hat{u} acts as a "disturbance" on the "measured" input d, as introduced in Remark 2.15.

An additional result (iii) is that, in the presented example, the addition of a high-pass filter to classical dither-based ES with an MA filter is beneficial. This can be explained as follows. The MA filter is designed to completely remove contributions at the dither frequencies. However, it can only do so for purely periodic signals with periodicity T. The optimizer output $\hat{u}(t)$ is not periodic, and hence, the MA filter does not completely remove all content at the dither frequencies from the derivative estimates $\tilde{g}_{u_1}(t)$ and $\tilde{g}_{u_1}(t)$, see also the derivative error plots in Figure 2.14. This effect is known as "leakage" in system identification, see Pintelon and Schoukens, 2012. Since the main spectral content of $\hat{u}(t)$ is typically at low frequencies, which corresponds with time scale separation, the high-pass filter reduces the leakage effect.

2.7 Conclusions

An overview is presented of the main classes of continuous derivative-based extremum seeking (ES), by focusing on the derivative estimation of the system's steady-state input-output map.

A dither-based derivative estimation framework is proposed, which generalizes existing approaches by estimating derivatives up to an arbitrary order, for systems with an arbitrary number of inputs.

Explicit tuning guidelines are derived to select the dither frequencies in classical dither-based ES for multiple-input systems. Based on the generalized derivative estimation framework, the optimal ratio between the individual dither frequencies is derived. For general systems, this optimal ratio results in the fastest possible derivative estimation time scale, thereby maximizing the ES convergence rate. By using a frequency-domain description of dither-based ES, a guideline is proposed to select the highest dither frequency in classical dither-based ES.

In addition, the frequency-domain analysis demonstrates the equivalence of dither-based ES and system identification. Using results from system identification theory, the existence of a lower bound on the dither amplitude, and the fundamental advantage of input-based ES are explained. Both these observations are illustrated in a simulation example. The applied input-based ES is the multiple-input extension of an existing "dither-free" ES approach, which is based on a linear least-squares fit of a past time window of input and cost data.

The frequency-domain description applies to classical, phase-shifted, and fast dither-based ES, and as such provides a unifying description for these approaches. Using this result, the system requirements and practical implications of advanced dither-based ES are clarified with respect to classical dither-based ES.



Figure 2.14. Simulation results using three different classical (CL) ditherbased extremum seeking (ES) approaches and one input-based ES approach. The corresponding values of the cost J, the inputs u_1 and u_2 , and the derivative estimation errors $\tilde{g}_{u_k} - g_{u_k}(\hat{u})$ are provided. The dashed black line indicates t = 0 s when the optimizer becomes active.
Chapter 3

Constrained extremum seeking in a diesel engine control system for online fuel efficiency optimization

Abstract – This chapter¹ presents a new method for online fuel efficiency optimization of diesel engines, using constrained extremum seeking (ES). A two-input optimization problem, which is suitable for ES, is integrated into a multivariable tracking control system. As a result, both air-path and fuel-path actuators are used for tracking and ES. A key element of the proposed method is a cost function based on real-time brake specific fuel consumption (BSFC) estimation. Moreover, an existing constrained ES method is extended such that multiple output constraints can be handled. In addition to constraints, related to limitations of the engine, constraints on the actuator position and tracking error are included to maintain tracking performance while optimizing. Experiments on a Euro-VI heavy-duty diesel engine demonstrate the constrained optimization, robustness with respect to real-world disturbances, and the fuel saving potential of the control design. In addition to different stationary engine operating points.

3.1 Introduction

Nowadays, heavy-duty vehicle (HDV) diesel engines are subject to legislated maximum CO_2 emission levels, see Section 1.1.2. This requires maximum efficiency, resulting in a minimal fuel consumption, while at the same time legislated

 $^{^1\}mathrm{This}$ chapter is based on Van der Weijst et al., 2019a.

emission constraints of other pollutants, such as nitrogen oxides (NO_x : a mixture of NO and NO_2), need to be satisfied. This poses a challenging control problem, especially because the emission constraints are subject to real-world evaluation, in addition to existing laboratory test cycles. This means that the control system needs to be robust for real-world disturbances such as varying ambient air conditions, varying fuel composition, production tolerances, component fouling and wear. In addition, physical constraints on the engine need to be satisfied, e.g., an in-cylinder pressure limit, maximum turbocharger rotational speed, and actuator limitations.

A typical diesel powertrain includes the engine itself and an exhaust aftertreatment system (EAS). The EAS is used to reduce the engine-out emission level of the pollutants within the tailpipe-out level prescribed by legislation. However, reducing emission of NO_x in the EAS, requires the addition of urea in the selective catalytic reduction (SCR) system, which is one of the components of the EAS. This contributes to the operational cost of the powertrain. Therefore, a relevant engine control problem, and the problem considered in this chapter, is to deliver the requested power, while optimizing the fuel efficiency for low cost operation, subject to constraints on engine-out NO_x and in-cylinder pressure. Available mechanisms to suppress engine-out NO_x emission are exhaust gas recirculation (EGR) and injection timing. Both are limiting the achievable fuel efficiency. As such, there exists a BSFC-NO_x trade-off, with the brake specific fuel consumption (BSFC) [g/kWh].

The industrial standard in diesel engine control comprises lookup table based feedforward and feedback control. In the literature, different combinations of control inputs and outputs are suggested, see, e.g., Wahlström and Eriksson, 2013; Criens et al., 2015, where air-path (i.e., related to gas flow) control variables are discussed. The potential of feedback control is increased by in-cylinder pressure sensors, which are not yet the industry standard. The in-cylinder pressure can be used to obtain the indicated mean effective pressure (IMEP), which is related to the brake engine torque and hence directly to the brake engine power. In addition, the combustion phasing parameter CA_{50} is obtained, which is the crank angle where half of the total heat per cycle is released. In Luo et al., 2018a, an \mathcal{H}_2 -optimal fuel-path (i.e., fuel injection) controller is presented to control IMEP and CA_{50} , while in Tschanz et al., 2014; Zhao et al., 2014 a combination of fuel-path and air-path control is proposed.

Although feedback control improves the robustness of the engine control system in terms of disturbance rejection, the high-level optimization problem of determining the related reference signals is typically addressed by offline (manual) tuning in an engine test cell. As a result, the obtained performance remains sensitive to the earlier mentioned real-world disturbances.

Online engine performance optimization using extremum seeking (ES) is an interesting research area. ES is an adaptive optimization technique, see, e.g., Tan et al., 2010, which aims to optimize a measured cost, with the advantage that

very little knowledge about the system dynamics and disturbances is required. The main requirement is that, for steady-state operation, the considered system has a (quasi-)convex mapping from the input(s) to be tuned to the performance output. For spark ignition engines, spark timing has been adjusted using ES. see Mohammadi et al., 2014 and Hellström et al., 2013. For diesel engines, in Großbichler et al., 2016 the fuel injection profile is adjusted, while in Lewander et al., 2012 the reference signal for closed-loop controlled CA_{50} is adjusted. In most cases a BSFC equivalent cost function is optimized. Constraints, e.g., on emissions, are not considered in these works. An exception is Ramos et al., 2017 where constrained ES is applied for spark timing tuning, accounting for an NO_x constraint. Explicit optimization of fuel efficiency subject to emission and power constraints by exploiting both fuel-path and air-path actuators is explored in Broomhead et al., 2017. The economic model predictive control (MPC) approach in Broomhead et al., 2017 describes the fuel mass flow with an explicit model of the engine. As such, physical relations and disturbances, e.g., due to real-world disturbances, which are not explicitly incorporated into the model, can lead to a mismatch between the optimum of the MPC cost function and the true optimal fuel efficiency.

The main contribution of this chapter is an ES-based control approach that deals with the engine control problem of delivering power, i.e., torque given the current engine speed, using minimal fuel, while satisfying constraints on engine-out NO_x emission and peak in-cylinder pressure. The NO_x and torque objectives are addressed by a multivariable tracking control system, that utilizes four actuators, both air-path and fuel-path. A key element of the ES application is the cost, which is a BSFC estimate, based on injector opening time, see Kupper et al., 2018. By selecting as ES inputs, the tracking control reference signals for pumping loss dp, and CA_{50} , a convex problem is obtained. The peak in-cylinder pressure is a constraint output. Additional constraint outputs are included to maintain tracking of NO_x . As such, a two-input optimization problem that is suitable for ES, is integrated into a tracking control system, by which the ES affects all four actuators. A second contribution of this chapter is to extend the scalar handling of output constraints in ES presented in Ramos et al., 2017 to multiple output constraints. Finally, online fuel efficiency optimization is demonstrated in experiments on a production type Euro-VI heavy-duty diesel engine, with additional in-cylinder pressure sensors and a high resolution crank angle (CA) encoder. Robustness of the ES is demonstrated by varying the engine operating point, the NO_x reference, and the type of fuel.

This chapter is organized as follows. First, the considered engine is described from a control perspective in Section 3.2. The control objective is summarized in Section 3.3. The low-level tracking control system is introduced in Section 3.4. The ES objective and the ES controller are discussed in Sections 3.5 and 3.6, respectively. In Section 3.7, the experimental results are presented, while Section 3.8 provides the conclusion of this chapter.

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Figure 3.1. Schematic layout of the considered engine.

3.2 Engine description

In this section, the considered type of engine, with the available actuators and sensors is addressed. A schematic outlay of the engine type under consideration is depicted in Figure 3.1.

The engine considered throughout this chapter is a state-of-the-art, heavyduty Euro-VI six-cylinder truck engine. Compared to the production type engine, the test engine is equipped with Kistler 6125 piezoelectric in-cylinder pressure sensors (one in each cylinder), and with an AVL 365 CA encoder, which has a resolution of 0.1 °CA. Data from these sensors is acquired with an AVL Indimodul. The in-cylinder pressure sensors and the CA encoder are indicated in Figure 3.1, so are the engine block, the air-path, the fuel-path, and additional (production type) sensors. This section addresses the sensors, as well as the air-path and fuel-path actuation, and is concluded by defining the considered control inputs and outputs.

3.2.1 Air-path actuation

The air-path system consists of the components related to gas flow: The turbocharger with variable geometry turbine (VGT) and cooler, and the cooled, high pressure EGR system. The compressor compresses the intake air, and thereby enables an increased power output and is beneficial for thermal efficiency. The turbocharger is driven by the VGT, which converts energy from the exhaust gas. The purpose of the EGR system is to reduce NO_x emission, by recirculating a part of the exhaust gas into the intake manifold. By doing so, the intake gas mixture is diluted with inert gas, which reduces the (local) combustion temperature and air-to-fuel ratio, and thereby tempers the formation of NO_x . Application of EGR has, however, a negative effect on fuel efficiency because: 1) To create an EGR flow, a pressure difference over the manifolds is required, see Figure 3.1, which is known as the pumping-loss of the engine, 2) less heat is available for the VGT which reduces the available compressor work of the turbocharger, and 3) EGR can reduce the thermal efficiency of the combustion. Summarizing, suppressing NO_x emission using EGR results in a decreased BSFC, i.e., there exists a BSFC- NO_x trade-off which is influenced by the air-path system. The considered air-path actuators are the VGT and the EGR valve. The considered engine comprises a back pressure valve (BPV), which is, however, not used for control in this study. Other common air-path actuators, such as variable valve actuation, are not present.

3.2.2 Fuel-path actuation

The fuel-path consists of a common rail system with six injectors, see again Figure 3.1, which provides the fuel-path control inputs: Start of injection (SOI) and duration of cylinder-individual injection pulses. These fuel-path inputs are updated once per combustion cycle for each cylinder. This type of control is known as cycle-to-cycle control. The rail pressure input is not used for control, it is kept at a desired reference value.

To discuss the effect of the fuel-path input, consider some of the measured outputs. An important difference between the considered engine, and the current state-of-the-art, is the availability of an in-cylinder pressure sensor in each cylinder and a high resolution CA encoder, see Figure 3.1. Using the CA measurement, which is directly related to the in-cylinder volume V_{cyl} [m³], the incylinder pressure can be given as $p_{cyl}(V_{cyl})$ [bar], which is used to obtain the net IMEP [bar]

$$IMEP_n := \frac{1}{V_d} \oint p_{cyl} \mathrm{d}V_{cyl},\tag{3.1}$$

where the integral covers a complete four-stroke cycle of 720 °CA, see also Heywood, 1988. In (3.1), V_d [m³] is the displacement volume of one piston. The net IMEP is related to the brake engine power P_e [kW] as

$$P_{e} = 10^{-3} \underbrace{\frac{V_{d} n_{cyl}}{4\pi} (IMEP_{n} - FMEP) \cdot 10^{5}}_{M_{e}} n_{e} \frac{2\pi}{60}$$

with the engine speed n_e [rpm], the number of cylinders $n_{cyl} = 6$, the engine torque M_e [Nm], and the friction losses expressed by the friction mean effective pressure (FMEP) [bar]. In addition, using p_{cyl} , the accumulated heat-release of a combustion stroke can be derived as a function of CA. Subsequently CA_{50} [°CA] results, which is the CA relative to top dead centre (TDC), at which half of the total heat is released, see also Heywood, 1988. Both the BSFC and NO_x emission are affected by CA_{50} . The fuel-path input SOI mainly affects CA_{50} , while the injection duration mainly affects $IMEP_n$.

3.2.3 Measured and estimated outputs

As mentioned in Section 3.2.2, the p_{cyl} and CA sensors, see Figure 3.1, can be used to determine CA_{50} and $IMEP_n$. Since each cylinder is equipped with a pressure sensor, CA_{50} and $IMEP_n$ are available for each of the six cylinders. Computationally efficient recursive calculation of CA_{50} and $IMEP_n$ is presented in Wilhelmsson et al., 2006 and applied in Willems et al., 2010.

In addition, the peak in-cylinder pressure p_{peak} [bar] is measured, which is the maximum value of p_{cyl} over one cycle. Due to physical limitations of the engine, p_{peak} should be smaller than a certain value. For the considered engine, at high load operating points, the optimal CA_{50} is constrained from below by the peak in-cylinder pressure p_{peak} upper limit.

The production type NO_x concentration [ppm] sensor is located after the VGT, see Figure 3.1. Combined with an estimate of the mass air flow and the engine power P_e , the NO_x concentration sensor is used to obtain the specific engine-out NO_x mass flow [g/kWh].

Finally, pressure sensors are present to measure the intake and exhaust manifold pressures, p_{in} and p_{ex} [kPa], respectively. These are used to obtain

$$dp := p_{ex} - p_{in} \sim -PMEP. \tag{3.2}$$

The pumping mean effective pressure (PMEP) [bar] is the part of the net IMEP resulting from the intake and exhaust stroke. Typically, PMEP is negative, which is why it is referred to as pumping-loss. As already mentioned, EGR requires dp > 0, i.e., it induces a loss of work, which results in the BSFC-NO_x trade-off mentioned in Section 3.2.1.

3.2.4 Selected control inputs and outputs

To summarize this section from a control perspective, the engine is a system with input $\mathbf{u} \in \mathbb{R}^{14 \times 1}$, given by

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_{dur}^{\top} & \mathbf{u}_{SOI}^{\top} & u_{EGR} & u_{VGT} \end{bmatrix}^{\top}$$
(3.3)

with vectors $\mathbf{u}_{dur}, \mathbf{u}_{SOI} \in \mathbb{R}^{6 \times 1}$, given by

$$\mathbf{u}_{dur} = \begin{bmatrix} u_{dur_1} & u_{dur_2} & \cdots & u_{dur_6} \end{bmatrix}^{\top}$$
(3.4)

$$\mathbf{u}_{SOI} = \begin{bmatrix} u_{SOI_1} & u_{SOI_2} & \cdots & u_{SOI_6} \end{bmatrix}^{\top}$$
(3.5)

containing the cylinder-individual main pulse injection duration signals u_{dur_i} , $i = 1, 2, \ldots, 6$, [ms], and SOI signals u_{SOI_i} , $i = 1, 2, \ldots, 6$, [°CA] relative to TDC, for each of the six cylinders. Signals $u_{EGR}, u_{VGT} \in [0, 100]$ are the EGR valve opening, and the VGT position, in [%], respectively. The output $\mathbf{y} \in \mathbb{R}^{14 \times 1}$ is given by

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_{IMEP_n}^\top & \mathbf{y}_{CA_{50}}^\top & y_{NO_x} & y_{dp} \end{bmatrix}^\top$$
(3.6)

with vectors $\mathbf{y}_{IMEP_n}, \mathbf{y}_{CA_{50}} \in \mathbb{R}^{6 \times 1}$, given by

$$\mathbf{y}_{IMEP_n} = \begin{bmatrix} y_{IMEP_{n_1}} & y_{IMEP_{n_2}} & \cdots & y_{IMEP_{n_6}} \end{bmatrix}^\top \tag{3.7}$$

$$\mathbf{y}_{CA_{50}} = \begin{bmatrix} y_{CA_{50_1}} & y_{CA_{50_2}} & \cdots & y_{CA_{50_6}} \end{bmatrix}^\top$$
(3.8)

containing the cylinder-individual measurements of CA_{50} [°CA] relative to TDC, and $IMEP_n$ [bar]. The units of $y_{NO_x}, y_{dp} \in \mathbb{R}$ are [g/kWh] and [kPa], respectively.

3.3 Control objective and approach

In this section, the control objective is presented. Subsequently, the control approach is introduced, motivated from the control objective.

3.3.1 Control objective

The high-level control objective, in line with the diesel engine control objective introduced in Section 1.2.2, is as follows. The engine should deliver the demanded brake torque $M_{e,dem}$ [Nm], with a minimal BSFC, subject to constrained p_{peak} and specific engine-out NO_x emission, on average over time. Moreover the obtained performance should be robust with respect to real-world disturbances, such as, varying ambient air conditions and fuel composition, production tolerances (e.g., sensor bias), and component fouling and wear.

A more specific control objective, derived from the high-level control objective, is given by

min(BSFC) s.t.
$$p_{peak} \le \bar{p}_{peak}$$
, (3.9a)

$$\boldsymbol{e}_{IMEP_n} = \boldsymbol{r}_{IMEP_n} - \boldsymbol{y}_{IMEP_n} \to \boldsymbol{0},$$
 (3.9b)

$$e_{NO_x} = r_{NO_x} - y_{NO_x} \to 0, \qquad (3.9c)$$

where \bar{p}_{peak} is the safety constraint value of the peak in-cylinder pressure p_{peak} , $r_{IMEP_n}, e_{IMEP_n} \in \mathbb{R}^{6\times 1}$ are vectors containing the net IMEP reference values and respective tracking errors. By tracking of $IMEP_n$, the demanded torque $M_{e,dem}$ can be realized. Equivalently, the specific engine-out NO_x reference and tracking error are denoted by $r_{NO_x}, e_{NO_x} \in \mathbb{R}$. In practice, r_{NO_x} is obtained from a supervisory controller, e.g., the integrated emission management (IEM) strategy proposed in Donkers et al., 2017, see also Section 1.2.2. The notation " \rightarrow 0" in (3.9b) and (3.9c) indicates that both transient and steady-state tracking performance is required. Because the constraint handling approach, introduced in Section 3.6, can only approximately satisfy constraints, the constraint in (3.9a) is a soft constraint in practice.

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Figure 3.2. Schematic overview of the control approach. The system subject to extremum seeking (ES) is indicated by the red dashed box and denoted as Σ , consisting of a dynamic part Σ_{dyn} and a static mapping Σ_{st} .

3.3.2 Control approach

The proposed control approach is schematically depicted in Figure 3.2 and can be subdivided into the following parts:

- (i) A low-level tracking control system, which tracks cylinder-individual CA_{50} reference signals, the net IMEP reference r_{IMEP_n} , the NO_x reference r_{NO_x} , and a dp reference r_{dp} . In Figure 3.2, the closed-loop low-level control system is indicated by Σ_{dyn} , the references are contained by the vector \boldsymbol{r} , and the real-world disturbances by the vector \boldsymbol{w} .
- (ii) A static mapping Σ_{st} , providing constraint outputs, e.g., (3.9a), contained by the vector \boldsymbol{h} , and a BSFC equivalent cost criterion J to be minimized, which uses the available measurements, contained by the vector \boldsymbol{y} , and the actuator positions contained by the vector \boldsymbol{u} , to enable online implementation.
- (iii) The cascaded connection of Σ_{dyn} and Σ_{st} is denoted by Σ and is subject to an ES controller, which online minimizes the cost J, subject to the constraints in h, by adding a delta to the default CA_{50} and dp reference signals, denoted by Δ_r .

The motivation for this structure is as follows. As explained, both CA_{50} and dp have an influence on the trade-off between low BSFC and the considered constraint on p_{peak} and the NO_x tracking objective. As such, they are well suited as optimization inputs. At the same time, the tracking of the net IMEP and NO_x is dealt with by the low-level tracking controller. Being a "model-free" approach, ES does not require explicit knowledge of the real-world disturbances and the dependency of the low-level control system on those disturbances. This is a clear advantage for the application at hand.

As a result of interaction in the low-level control system, all four actuators are affected by adjusting the CA_{50} and dp references. As such, the problem of optimizing fuel efficiency with four actuators, is reduced to a two-input optimization. Moreover, in Section 3.5 it is shown that the resulting two-input problem is "quasi-convex", which is a requirement for ES.



Figure 3.3. Low-level control system schematics. The dashed box indicates Σ_{dyn} : The dynamic part of the system Σ considered for online optimization. In Section 3.5.2, Σ is introduced, see also Figure 3.9.

3.4 Low-level tracking control system

This section introduces the low-level tracking controller, and its main properties. The tracking performance is characterized with Bode plots of the sensitivity function of the closed loop.

The low-level tracking control system is schematically depicted in Figure 3.3. In this figure, \mathcal{P} is a dynamical nonlinear systems which represents the engine. with input \mathbf{u} in (3.3) and output \mathbf{y} in (3.6). The dependency of the engine on the aforementioned real-world disturbances is indicated by the disturbance vector $\mathbf{w} \in \mathbb{R}^{n_w \times 1}, n_w \in \mathbb{Z}_{>0}$. The "cycle-to-cycle calculation" block represents the online derivation of CA_{50} and $IMEP_n$ as in Willems et al., 2010. Feedforward signal $\mathbf{u}_{ff} \in \mathbb{R}^{14 \times 1}$ is obtained from lookup tables (tuned offline), as a function of the engine operating point, and is provided by the default engine control unit (ECU). C_{fb} is a dynamic feedback controller with output $\mathbf{u}_{fb} \in \mathbb{R}^{14 \times 1}$ and as input the tracking error $\mathbf{e} \in \mathbb{R}^{14 \times 1}$, which has the same structure as \mathbf{y} in (3.6)-(3.8). The controller C_{fb} is a decoupled proportional-integral (PI) controller, see, e.g., Skogestad and Postlethwaite, 2005. The corresponding static decoupling matrix is scheduled as a function of engine speed n_e and the $IMEP_n$ reference signal r_{IMEP_n} , to compensate for parameter-varying and nonlinear behavior. The dynamic PI part of \mathcal{C}_{fb} is linear time-invariant (LTI). Finally, the reference signal results from the summation of a nominal reference signal $\mathbf{r} \in \mathbb{R}^{14 \times 1}$, and a delta contribution, $\Delta_r \in \mathbb{R}^{14 \times 1}$, such that we have $\mathbf{e} = \mathbf{r} + \Delta_r - \mathbf{y}$. Signal Δ_r is used to adjust CA_{50} and dp with the ES controller. The reference signal r is based on offline tuned lookup tables, which are parameterized by n_e and r_{IMEP_e} , i.e., the engine operating point. As a result of preserving the default value of rin the ES architecture, the operating point dependency of the system Σ , subject to ES, see Figure 3.2, is reduced. This is a useful step, since application of ES implicitly assumes the operating point dependency of the system Σ to be small.

The control system has cylinder-individual control of CA_{50} and $IMEP_n$,

and, as such, disturbances related to fuel injection are suppressed. However, the reference signals are equal for all cylinders, so r is given by

$$\boldsymbol{r} = \begin{bmatrix} [r_{IMEP_n} & r_{CA_{50}}] \otimes \mathbb{1}^{1 \times 6} & r_{NO_x} & r_{dp} \end{bmatrix}^\top, \qquad (3.10)$$

where \otimes denotes the Kronecker product and $\mathbb{1}^{1\times 6}$ is a one by six vector of ones. Accordingly, signal Δ_r , which is used to adjust the reference signals for CA_{50} and dp, is given by

$$\boldsymbol{\Delta}_{r} = \begin{bmatrix} \begin{bmatrix} 0 & \Delta_{r_{CA_{50}}} \end{bmatrix} \otimes \mathbb{1}^{1 \times 6} & 0 & \Delta_{r_{dp}} \end{bmatrix}^{\top}.$$
(3.11)

Note that, by defining Δ_r as in (3.11), the possibility of cylinder-individual CA_{50} optimization is not exploited. This can be an interesting topic for further research.

For a constant operating point, i.e., constant engine speed and torque, and operating conditions and input \boldsymbol{u} close to nominal conditions, the nonlinear dynamics of the engine \mathcal{P} can be characterized by its frequency response function (FRF). FRFs are non-parametric frequency-domain LTI models, see, e.g., Pintelon and Schoukens, 2012. In Van Keulen et al., 2017, the applied FRF measurement method is discussed, and demonstrated on the engine considered in this chapter. Using these FRFs, the operating point dependent static decoupling matrix is obtained, and the LTI PI part of \mathcal{C}_{fb} is designed using frequency-domain loop-shaping, see, e.g., Skogestad and Postlethwaite, 2005. Using the controller \mathcal{C}_{fb} , which is LTI for a stationary operating point, and the FRF of the engine, a non-parametric model of the sensitivity function $\mathcal{S}(j\omega) \in \mathbb{C}^{4\times 4}$ is obtained. The sensitivity gives an indication of the control system tracking performance and satisfies

$$\begin{bmatrix} \bar{e}_{IMEP_n}(j\omega) \\ \bar{e}_{CA_{50}}(j\omega) \\ e_{NO_x}(j\omega) \\ e_{dp}(j\omega) \end{bmatrix} = \mathcal{S}(j\omega) \begin{bmatrix} r_{IMEP_n}(j\omega) \\ r_{CA_{50}}(j\omega) \\ r_{NO_x}(j\omega) \\ r_{dp}(j\omega) \end{bmatrix}$$
(3.12)

where $j\omega \in \mathbb{C}$ is the complex frequency with $\omega = 2\pi f$, the $(j\omega)$ argument indicates the Fourier transform of the corresponding time-domain signals, and \bar{e}_{IMEP_n} and $\bar{e}_{CA_{50}}$ are the average tracking errors over the six cylinders. Figure 3.4 shows the Bode magnitude plot of an FRF-based estimate of $\mathcal{S}(j\omega)$ for a typical cruise control operating point, referred to as OP-A, see also Figure 3.5. Observe that for f < 0.143 Hz, the magnitude of all elements of $\mathcal{S}(j\omega)$ is smaller than one. To be precise, $|\mathcal{S}_{r_{NO_x} \to e_{NO_x}}(j\omega)| = 1$ at f = 0.143 Hz. This indicates that reference signals up to 0.143 Hz are tracked, and that tracking of NO_x is relatively slow, which is due to the NO_x sensor dynamics.



Figure 3.4. Bode magnitude plot of the sensitivity function $S(j\omega)$ for cruise control operating point OP-A.

3.5 High-level constrained extremum seeking objective

The objective in this chapter is to minimize the BSFC online, during real-world application. Therefore, a BSFC equivalent cost function is required, which uses online available inputs. In addition, the considered constraint outputs are introduced in this subsection, the optimization problem is formalized, and corresponding measured steady-state maps are presented.

3.5.1 Cost and constraint outputs

To obtain the actual BSFC, measurements of the fuel mass flow and the power are required. However, both are not available on the vehicle. To cope with this problem, the implemented cost function is based on estimates instead, which are a function of online available signals. In Kupper et al., 2018 this cost function is treated in detail; the essence is as follows. The engine power estimate \tilde{P}_e is obtained from an estimate of the engine torque M_e , which is based on y_{IMEP_n} , and compensated for the engine speed n_e dependent FMEP. Likewise, the fuel mass flow estimate \tilde{M}_f is primarily a function of the injector opening time u_{dur} , Chapter 3. Constrained extremum seeking in a diesel engine control system for 74 online fuel efficiency optimization

and compensated for the rail pressure p_{rail} , SOI u_{SOI} , and the engine speed n_e . The estimate of the BSFC is given by

$$B\tilde{SFC} = \frac{\tilde{\tilde{m}}_f}{\tilde{P}_e}.$$

In Kupper et al., 2018, the BSFC estimate is demonstrated to be within $\approx 1 \%$ of the actual measured BSFC for different stationary engine operating points. However, the essential property for ES is that the average injector opening time \bar{u}_{dur} is minimized while y_{IMEP_n} is according to its reference. In addition, the absolute value of the BSFC estimate is not important, only the location of its minimum as a function of $\Delta_{r_{CA50}}$, $\Delta_{r_{dp}}$, which is estimated well with respect to the location of the actual minimal BSFC, see Kupper et al., 2018.

To reduce the operating point dependency of the cost function, see challenge C1 in Section 1.3.3, the following normalization is applied:

$$J = \frac{B\tilde{SFC}}{B\tilde{SFC}_{ECU}} \tag{3.13}$$

where $B\tilde{SFC}_{ECU}$ is the BSFC estimate based on the default ECU settings. It holds that $B\tilde{SFC}_{ECU} \approx B\tilde{SFC}$ when $\Delta_r = 0$, and consequently, $J \approx 1$ when $\Delta_r = 0$ for all operating points.

While minimizing cost J, the corresponding $\Delta_{r_{CA_{50}}}$, $\Delta_{r_{dp}}$ can result in a reduced tracking performance of r_{NO_x} . This is due to the corresponding input u deviating significantly from its nominal value, at which the FRF is measured that is used to design the low-level control system. In particular, a reduction of dp requires a higher value of u_{EGR} , i.e., EGR valve opening percentage, to obtain the same amount of EGR. A high value of u_{EGR} , combined with a lower pressure difference dp, leads to a reduced effectiveness of the EGR valve as a control input. Consequently, a negative steady-state tracking error e_{NO_x} can result, which implies that the desired NO_x emission is exceeded, see the upper right plot in Figure 3.6, which will be introduced in Section 3.5.2. Comparing the e_{NO_x} map in Figure 3.6 with the u_{EGR} map, it can be seen that the value of u_{EGR} is easier to anticipate upon than e_{NO_x} . Therefore, constraint outputs are related to both e_{NO_x} and u_{EGR} :

$$h_{NO_x} = -e_{NO_x} - \delta_{NO_x} \tag{3.14}$$

$$h_{EGR} = u_{EGR} - \delta_{EGR} \tag{3.15}$$

which are required to satisfy $h_{NO_x}, h_{EGR} \leq 0$, with $\delta_{NO_x}, \delta_{EGR} \in \mathbb{R}_{>0}$.

In addition, the peak in-cylinder pressure p_{peak} is affected by $\Delta_{r_{CA_{50}}}, \Delta_{r_{dp}}$, thereby possibly violating the physical limitation of the engine. As such, the following constraint output is included

$$h_{p_{peak}} = p_{peak} - \delta_{p_{peak}} \tag{3.16}$$

which is required to satisfy $h_{p_{peak}} \leq 0$, with $\delta_{p_{peak}} \in \mathbb{R}_{>0}$.



Figure 3.5. Indication of the engine operating points in relation to the maximum engine speed $n_{e,max}$ and torque $M_{e,max} = n_e \in (0, n_{e,max}]$ (M_e). The transition from OP-A to OP-A^{*} is used as a transient test in Section 3.7.2.

3.5.2 Optimization problem

The system Σ , subject to ES, see Figure 3.2, consists of the cascaded connection of the closed-loop low-level tracking control system Σ_{dyn} , see Figure 3.3, and the static mapping Σ_{st} , which yields J, h_{NO_x} , h_{EGR} , and $h_{p_{peak}}$ according to (3.13)-(3.16), respectively. Define the input $\mathbf{\Delta} = [\Delta_{r_{CA_{50}}} \ \Delta_{r_{dp}}]^{\top}$, then the considered optimization problem is summarized by:

$$\min_{\mathbf{\Delta}}(J) \quad \text{s.t.} \quad h_{NO_x} \le 0, \ h_{EGR} \le 0, h_{p_{peak}} \le 0, \tag{3.17a}$$

$$\Sigma: (\boldsymbol{\Delta}, \boldsymbol{w}, \boldsymbol{u}_{ff}, \boldsymbol{r}) \to (J, h_{NO_x}, h_{EGR}, h_{p_{peak}}).$$
(3.17b)

It is noted that, with the applied constrained ES approach, which is introduced in Section 3.6, the constraints in (3.17a) are dealt with as soft constraints, see Remark 3.3. To be more precise, the applied approach provides convergence to a region around the constrained minimum. Note that, (3.13)-(3.16) are static, and hence the dynamics of the system Σ in (3.17b) are in Σ_{dyn} , indicated by the dashed box in Figure 3.3.

Figures 3.6, 3.7, and 3.8 depict steady-state maps corresponding to the system Σ in (3.17b). The maps are obtained by measurements on a grid of constant $\Delta_{r_{CA_{50}}}, \Delta_{r_{dp}}$, for four stationary engine operating points, indicated in Figure 3.5, for which $\boldsymbol{w}, \boldsymbol{u}_{ff}$, and \boldsymbol{r} are constant. The subscript $_{st}$ indicates that the depicted values approximate the steady state. The applied constraint limits are $\delta_{NO_x} = 0.3$ g/kWh, $\delta_{EGR} = 20$ %, and $\delta_{p_{peak}} = \bar{p}_{peak}$ bar. The value \bar{p}_{peak} and the exact specification of the operating points are omitted due to confidentiality. Evidently, the grid points where p_{peak} is (presumably) above the constraint limit are not measured, as can be seen in Figure 3.7.

The measurements of Σ show that, in steady state, the optimization problem (3.17) is quasi-convex for all operating points. In accordance with Section 3.2,

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Figure 3.6. Measured steady-state values J_{st} , $h_{NO_x,st}$, $h_{p_{peak},st}$ and $h_{EGR,st}$ (black circles) of J, h_{NO_x} , $h_{p_{peak}}$, and h_{EGR} in (3.13)-(3.16), respectively, for cruise control operating point OP-A. The surfaces are obtained using a heuristic fitting algorithm, the transparent planes indicate $h_{NO_x,st}$, $h_{EGR,st} = 0$.

reducing r_{dp} results in a decrease of J_{st} and an increase of $u_{EGR,st}$, which ultimately yield a steady-state NO_x tracking error $e_{NO_x,st}$. For a large part of the $(\Delta_{r_{CA_{50}}}, \Delta_{r_{dp}})$ grid, $h_{NO_x,st} \approx -\delta_{NO_x} = -0.3$, i.e. $e_{NO_x} \approx 0$, indicating that the low-level control system is able to track r_{NO_x} . Observe that $h_{EGR,st} \approx 0$ only occurs in the vicinity of $h_{EGR,st} = 0$, while $h_{NO_x,st}$ is close to zero (depending on the value of δ_{NO_x}) in a large area where there is no violation of the NO_x constraint. The small difference in the value of h_{NO_x} between violation and satisfaction of the h_{NO_x} constraint complicates anticipating violation. Therefore, the EGR constraint is in place, while motivated from the high-level diesel engine control objective, only NO_x and p_{peak} are the signals to constraint.

Consider Figures 3.6 and 3.7, and observe that the influence of the ES input Δ on p_{peak} , mentioned in Section 3.5.1, mainly concerns the $\Delta_{r_{CA_{50}}}$ -direction. Moreover, note that for the presented operating points, p_{peak} is only an active constraint for the high-load operating point OP-B.



Figure 3.7. Measured steady-state values J_{st} , $h_{NO_x,st}$, $h_{p_{peak},st}$ and $h_{EGR,st}$ (black circles) of J, h_{NO_x} , $h_{p_{peak}}$, and h_{EGR} in (3.13)-(3.16), respectively, for high-load operating point OP-B. The surfaces are obtained using a heuristic fitting algorithm, the transparent planes indicate $h_{NO_x,st}$, $h_{p_{peak},st}$, $h_{EGR,st} = 0$.

3.6 Constrained extremum seeking controller

The applied ES method employs classical dither-based derivative estimation, see Nešić et al., 2010, and Van der Weijst et al., 2017 for the multivariable case. For the constraint handling, the single constraint approach proposed in Ramos et al., 2017 is extended such that it is applicable for multiple constraints. After presenting the ES approach, the tuning of the ES controller parameters is addressed.

3.6.1 Constrained extremum seeking controller structure

The ES system schematics are depicted in Figure 3.9. Inputs $\Delta_{r_{CA50}}$ and $\Delta_{r_{dp}}$ are perturbed by the dither signals

$$d_i(t) = a_{d_i} \cos(\omega_{d_i} t) \tag{3.18}$$

with $a_{d_i}, \omega_{d_i} \in \mathbb{R}_{>0}$, i = 1, 2, the dither amplitude and dither frequency, respectively. The dither signals are not restricted to be sinusoidal. In fact, according

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Figure 3.8. Fitted surfaces based on measured steady-state data J_{st} of cost J for four different operating points. Constraint information is included: $h_{p_{peak},st} = 0$ (dotted black line), $h_{EGR,st} = 0$ (dashed black line), and $h_{NO_x,st} = 0$ (solid black line).

to Tan et al., 2008, any periodic, zero mean, persistently exciting signal with bounded magnitude can be used. The motivation to use a sinusoidal dither signal is to avoid excitation of high-frequency dynamics.

The derivative estimator (DE) blocks estimate the derivatives of the cost output J and the constraint outputs $h_{NO_x}, h_{EGR}, h_{p_{peak}}$, with respect to the inputs $\Delta_{r_{CA50}}$ and $\Delta_{r_{dp}}$. The derivative estimates are denoted by $\tilde{\boldsymbol{g}}_{J}, \tilde{\boldsymbol{g}}_{h_{NO_x}}, \tilde{\boldsymbol{g}}_{h_{EGR}}, \tilde{\boldsymbol{g}}_{h_{p_{peak}}} \in \mathbb{R}^{2\times 1}$, respectively. Consider for example an input signal $q(t) \in \mathbb{R}$, then its derivative estimate $\tilde{\boldsymbol{g}}_q$ is obtained as follows:

$$\mathbf{f}\dot{x}_{HP}(t) = -\omega_{HP}x_{HP}(t) + \omega_{HP}q(t) \tag{3.19a}$$

DE:
$$\begin{cases} y_{HP}(t) = -x_{HP}(t) + q(t) \end{cases}$$
 (3.19b)

$$\tilde{\boldsymbol{g}}_{q}(t) = \frac{1}{T_{MA}} \int_{t-T_{MA}}^{t} \begin{bmatrix} 2/a_{d_{1}} \cos(\omega_{d_{1}}\tau) y_{HP}(\tau) \\ 2/a_{d_{2}} \cos(\omega_{d_{2}}\tau) y_{HP}(\tau) \end{bmatrix} \mathrm{d}\tau$$
(3.19c)

where $x_{HP}, y_{HP} \in \mathbb{R}$, $\omega_{HP} \in \mathbb{R}_{>0}$, are the first-order high-pass filter state, output, and pole frequency [rad/s], respectively.

The high-pass filter in (3.19a),(3.19b) removes the DC component from the DE input q(t), and is common in ES, see Tan et al., 2010. The integral in (3.19c)



Figure 3.9. Extremum seeking (ES) system schematics. The thick arrow indicate signals in $\mathbb{R}^{2\times 1}$.

is a moving average (MA)-filter, which has a low-pass characteristic. For T_{MA} equal to the smallest common period time of the dither signals d_1, d_2 in (3.18), the MA-filter removes all content in the derivative estimate at integer multiples of the dither frequencies $\omega_{d_1}, \omega_{d_2}$.

Estimating the derivatives with respect to $\Delta_{r_{CA_{50}}}$ and $\Delta_{r_{dp}}$ independently, with the DE in (3.19) and the dither signals d_1, d_2 in (3.18), requires $\omega_{d_1} \neq \omega_{d_2}$. As proposed in Van der Weijst et al., 2017, we select $\omega_{d_1} = 2\omega_{d_2}$ resulting in $T_{MA} = \frac{2\pi}{\omega_{d_2}}$ which is the smallest possible value for T_{MA} . As a result, the estimation delay is minimized.

Consider Figure 3.4 and observe that $|\mathcal{S}_{r_{CA_{50}} \to \bar{e}_{CA_{50}}}| < |\mathcal{S}_{r_{dp} \to e_{dp}}|$, which implies that the tracking control for $r_{CA_{50}}$ is faster than for r_{dp} . Therefore, $\omega_{d_1} = 2\omega_{d_2}$ is selected instead of $\omega_{d_2} = 2\omega_{d_1}$.

Remark 3.1. The MA-filter was first proposed in the context of ES in Haring et al., 2013 and is favored over the usually applied low-pass filter (see Tan et al., 2010) because: 1) It completely removes contributions at integer multiples of the dither frequencies, thereby improving the quality of the derivative estimate, and 2) it offers a more transparent tuning as T_{MA} is directly related to the dither frequencies.

The constraint handling, adopted from Ramos et al., 2017, is achieved by combining all individual derivative estimates $\tilde{\boldsymbol{g}}_J, \tilde{\boldsymbol{g}}_{h_{NO_x}}, \tilde{\boldsymbol{g}}_{h_{EGR}}, \tilde{\boldsymbol{g}}_{h_{p_{peak}}}$ into a weighted combination $\tilde{\boldsymbol{g}} \in \mathbb{R}^{2 \times 1} = [\tilde{g}^{CA_{50}} \ \tilde{g}^{dp}]^{\top}$. The weighting is a function of the values of the corresponding constraint outputs $h_{NO_x}, h_{EGR}, h_{p_{peak}}$. The intuition is that, whenever a constraint violation is detected, the corresponding gradient is used to decrease the constraint, thereby counteracting the constraint violation. The extension of the approach in Ramos et al., 2017 towards multiple

constraints requires a more complex weighting function to deal with simultaneous violation of multiple constraints. The weighting is done using the scheduling function

$$f_{con}: \tilde{\boldsymbol{g}} = \begin{bmatrix} \tilde{\boldsymbol{g}}_{J}^{\top} \\ \gamma_{NO_x} \tilde{\boldsymbol{g}}_{h_{NOx}}^{\top} \\ \gamma_{EGR} \tilde{\boldsymbol{g}}_{h_{EGR}}^{\top} \\ \gamma_{p_{peak}} \tilde{\boldsymbol{g}}_{h_{ppeak}}^{\top} \end{bmatrix}^{\top} \begin{bmatrix} (1 - \alpha_{NO_x})(1 - \alpha_{EGR})(1 - \alpha_{p_{peak}}) \\ \alpha_{NO_x}(1 - \alpha_{EGR})(1 - \alpha_{p_{peak}}) \\ \alpha_{EGR}(1 - \alpha_{p_{peak}}) \\ \alpha_{p_{peak}} \end{bmatrix}$$
(3.20)

with scaling parameters $\gamma_{NO_x}, \gamma_{EGR}, \gamma_{p_{peak}} \in \mathbb{R}_{>0}$, and smooth scheduling functions $\alpha_{NO_x}, \alpha_{EGR}, \alpha_{p_{peak}} \in (0, 1)$, as a function of h_{NO_x}, h_{EGR} , and $h_{p_{peak}}$, respectively. For example $\alpha_{NO_x}(h_{NO_x})$ is given by

$$\alpha_{NO_x}(h_{NO_x}) = \frac{1}{1 + \exp\left(-\frac{h_{NO_x}}{\kappa_{NO_x}}\right)}$$
(3.21)

with $\kappa_{NO_x} \in \mathbb{R}_{>0}$ a constant that determines the smoothness. Observe that for $\kappa_{NO_x} \to 0$, $\alpha_{NO_x}(h_{NO_x})$ approaches a discontinuous switch. Functions $\alpha_{EGR}(h_{EGR})$ and $\alpha_{p_{peak}}(h_{p_{peak}})$ are defined as (3.21) with κ_{EGR} and $\kappa_{p_{peak}}$, respectively.

Since system Σ is dynamic, the order in which the elements of the vectors in f_{con} appear does influence the solution $\Delta(t)$. To be precise, violation of $h_{p_{peak}}$ is given priority over violation of h_{EGR} , and, in turn, over h_{NO_x} .

The output $\tilde{\boldsymbol{g}}$ of the function f_{con} in (3.20) is used in the optimizer, indicated in Figure 3.9, to obtain the optimizer output $\hat{\boldsymbol{\Delta}} = \begin{bmatrix} \hat{\Delta}_{CA_{50}} & \hat{\Delta}_{dp} \end{bmatrix}^{\top}$, with $c_1, c_2 \in \mathbb{R}_{>0}$ the optimizer gains. As such, $\tilde{\boldsymbol{g}} = 0$ corresponds to the, possibly constrained, optimum resulting from ES.

Using the function f_{con} in (3.20), the value of $\tilde{\boldsymbol{g}}$ in the vicinity of a constraint, e.g., $h_{NO_x} \approx 0$ where $\alpha_{NO_x}(h_{NO_x}) \approx 0.5$, depends on the ratio between the values of the individual derivative estimates, $\tilde{\boldsymbol{g}}_J$ and $\tilde{\boldsymbol{g}}_{h_{NO_x}}$ in case of NO_x. As a result, the ES optimum, where $\tilde{\boldsymbol{g}} = 0$, may deviate from the actual constrained optimum where $h_{NO_x} = 0$. To better balance the ratio between the individual derivative estimates, the scaling constants $\gamma_{NO_x}, \gamma_{EGR}, \gamma_{p_{peak}}$ are included in f_{con} in (3.20).

In Ramos et al., 2017 an integrator state as a function of the constraint output is added to the argument of the scheduling function in (3.21). By doing so, the balancing of the individual derivative estimates is achieved in the scheduling parameters $\alpha_{NO_x}, \alpha_{EGR}$, and $\alpha_{p_{peak}}$, thereby omitting the necessity for $\gamma_{NO_x}, \gamma_{EGR}, \gamma_{p_{peak}}$. A disadvantage of this approach is windup of the integrator state, when the corresponding constraint is not active. To avoid the necessity of anti-windup measures, the scaling parameters $\gamma_{NO_x}, \gamma_{EGR}, \gamma_{p_{peak}}$ are used instead.

As noted in Ramos et al., 2017, the ES approach also converges without such an additional integrator state. Observe in (3.20) that the possibly resulting

deviation of the ES optimum, where $\tilde{g} = 0$, from the actual constrained optimum, depends on the balance of the individual derivative estimates, and the smoothness of the scheduling parameters $\alpha_{NO_x}, \alpha_{EGR}, \alpha_{p_{peak}}$.

Remark 3.2. The applied DE approach is the classical ES method. Note that alternative ES methods exist. For (3.17), with corresponding steady-state mappings in Figure 3.8, the advantage of such methods can for instance be faster convergence, see, e.g., Moase and Manzie, 2012 or Guay, 2016. However, these methods require (among others) high amplitude fast dither excitation. This is not possible due to the cycle-to-cycle control induced discrete-time behavior of the system, and physical limitations on the actuators. Moreover, as a result of the constraint-based scheduling between the individual gradients, see Figure 3.9 and (3.20),(3.21), for a constrained optimum, the transient performance of the constrained ES is mainly limited by the dynamics of system Σ , not the DE. Application of input-based approaches, see, e.g., Hunnekens et al., 2014; Guay and Dochain, 2015, or Section 2.5, is an interesting topic for further research, as such methods generally require less dither excitation. As a result, dither-induced constraint violation can be reduced, which allows the optimizer output $\hat{\Delta}$ to be closer to the constraint, thereby reducing the cost at a constrained optimum.

Remark 3.3. The constraint handling of the ES approach relies on feedback of the constraint outputs h_{NO_x} , h_{EGR} , and $h_{p_{peak}}$. Because the system Σ in (3.17b) is dynamic, the constraint outputs are prone to overshoot their limit in transient operation of the ES. Moreover, the tuning of the scaling constants γ_{NO_x} , γ_{EGR} , and $\gamma_{p_{peak}}$, may induce a difference between the optimal ES input, i.e., the input Δ for which $\tilde{g} = 0$, and the actual constraint limit. Finally, in steady state, the convergence of ES is limited to practical convergence to a neighborhood of the (constrained) optimum. Due to these aspects, constraints are dealt with as soft constraints.

3.6.2 Extremum seeking controller parameter tuning

The motivation to apply ES, is to provide robustness in achieving optimal fuel efficiency in the presence of real-world disturbances. Therefore, the objective in tuning the parameters of the ES controller is fast convergence of $\Delta(t)$, towards a reasonably small neighborhood of the optimal input, denoted by Δ^* . This section provides systematic guidelines for tuning of the parameters. The tuning of the parameters, see Table 3.1, is used for all measurements that are presented in Section 3.7.

The convergence analysis of standard ES, with the applied DE with MA filter, see Haring et al., 2013, relies on the following time scale separation principles:

(i) The dither frequencies are sufficiently low with respect to the lowest frequency characterizing the dynamics of the system Σ.

Parameter	Value	$\delta_{NO_x} \left[{ m g/kWh} ight]$	0.3
$\omega_{d_1} [\mathrm{rad/s}]$	0.16π	δ_{EGR} [%]	20
$\omega_{d_2} [\mathrm{rad/s}]$	0.08π	$\delta_{p_{peak}}$ [bar]	\bar{p}_{peak}
a_{d_1} [°CA]	0.4	$\hat{\gamma}_{NO_x}$ [-]	0.005
a_{d_2} [kPa]	0.8	γ_{EGR} [-]	0.0008
T_{MA} [s]	25	$\gamma_{p_{peak}}$ [-]	0.001
$\omega_{HP} [rad/s]$	0.072π	κ_{NO_x} [-]	0.01
c_1 [-]	12	κ_{EGR} [-]	1
c_2 [-]	120	$\kappa_{p_{peak}}$ [-]	1

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Table 3.1. Constrained extremum seeking (ES) controller parameters.

(ii) The optimizer, see Figure 3.9, adapts Δ sufficiently slow with respect to the dither frequency, to not disturb the dither-based derivative estimation.

The functionality of the presented constrained ES relies on the same time scale separation principles, which motivates the applied parameter tuning.

Haring et al., 2013 note that, the time window T_{MA} in the MA filter, see (3.19c), is a delay in the derivative estimation. This delay should be minimal to enable fast ES convergence. Hence, high dither frequencies are required, which is a conflicting requirement given the aforementioned time scale separation principle (i). The Bode plots in Figure 3.4 are used to select $\omega_{d_1} = 0.08$ Hz and $\omega_{d_2} = 0.04$ Hz, see Table 3.1. These are relatively high frequencies, for which all elements of $|\mathcal{S}(j\omega)|$ are well below 0 dB, which implies that the dither signals are tracked by the low-level control system. Hence, the dynamics of Σ_{dyn} , see Figure 3.3, are faster than the dither perturbation. Since all dynamics of the system Σ are contained by Σ_{dyn} , the time scale separation requirement (i) is satisfied for the selected dither frequencies. The value of T_{MA} follows directly from the selected dither frequencies.

The high-pass filter frequency ω_{HP} is selected sufficiently low to maintain the dither contribution in J, h_{NO_x}, h_{EGR} , and $h_{p_{peak}}$. Note that reducing the value of ω_{HP} , reduces the effectiveness of the filter in suppressing the contribution to the outputs, that is induced by time-varying behavior of the optimizer output $\hat{\Delta}(t)$. Therefore, a relatively high value is selected for ω_{HP} , see Table 3.1.

The dither amplitudes affect the size of the neighborhood of Δ^* , to which $\Delta(t)$ converges, see Krstić and Wang, 2000 and Haring et al., 2013. As such, small dither amplitudes are desired. However, from a system identification perspective, the dither amplitudes should be large enough to generate a sufficiently high signal-to-noise ratio (SNR), at the dither frequencies, in the outputs $J, h_{NO_x}, h_{EGR}, h_{p_{peak}}$ of the system Σ , and the input Δ , see Section 2.4.3. The selected dither amplitudes, see Table 3.1, are obtained by tuning on the engine setup.

Having the dither frequencies and amplitudes, and the high-pass filter frequency, the derivative estimates \tilde{g}_J , $\tilde{g}_{h_{NO_x}}$, $\tilde{g}_{h_{EGR}}$, and $\tilde{g}_{h_{p_{peak}}}$, can be ob-

tained, on the setup, for constant values of $\hat{\Delta}$ near the constraint limits. Subsequently, the scaling parameters $\gamma_{NO_x}, \gamma_{EGR}, \gamma_{p_{peak}}$ can be determined, such that $|\tilde{g}_J| \approx \gamma_{NO_x} |\tilde{g}_{h_{NO_x}}| \approx \gamma_{EGR} |\tilde{g}_{h_{EGR}}| \approx \gamma_{p_{peak}} |\tilde{g}_{h_{p_{peak}}}|$, see Table 3.1 for the values.

The values of δ_{NO_x} , δ_{EGR} , and $\delta_{p_{peak}}$, have been introduced in Section 3.5.2 and are provided in Table 3.1. The values for κ_{NO_x} , κ_{EGR} , and $\kappa_{p_{peak}}$ are obtained as follows. Selecting $\kappa_{EGR} = \kappa_{p_{peak}} = 1$ results in a relatively smooth scheduling between the derivatives $\tilde{g}_{h_{EGR}}$, $\tilde{g}_{h_{p_{peak}}}$ and \tilde{g}_J , which appears to be beneficial during the experiments: Small κ_{EGR} and $\kappa_{p_{peak}}$ result in a larger amplitude sawtooth-like oscillation around constrained Δ^* . Contrary to $h_{p_{peak},st}$ and $h_{EGR,st}$, which are close to zero only in the vicinity of the constraints, $h_{NO_x,st}$ is close to zero for $\Delta \in D \in \mathbb{R}^2$ where the constraint is satisfied, see Figures 3.6 and 3.7, and observe the set D. To be precise, $e_{NO_x} \approx 0$ due to the integral action in the low-level control system, which results in $h_{NO_x} \approx -\delta_{NO_x} = -0.3$. With $\kappa_{NO_x} = 1$, for $\alpha_{NO_x}(h_{NO_x})$ in (3.21), this yields $\alpha_{NO_x}(-0.3) \approx 0.43$ for $\Delta \in D$, i.e., the NO_x constraint remains active in the ES while it is satisfied. As such, the smoothness of α_{NO_x} has to be limited, by taking a small value for κ_{NO_x} , e.g., $\kappa_{NO_x} = 0.01$ as selected.

Finally, the optimizer gains c_1 and c_2 are determined by tuning on the setup, aiming to maintain $\Delta(t)$ in a reasonable small (approximately the dither amplitude) neighborhood of Δ^* , after convergence.

3.7 Heavy-duty Euro-VI engine experiments

This section discusses experimental results obtained on the engine, which is introduced in Section 3.2. For experiments, the engine is connected to a dynamometer, which dissipates the delivered engine power. First, the controller implementation is addressed, followed by the tuning of the ES controller parameters, after which the experimental results are presented.

The initial condition for $\Delta(t)$ is denoted by $\Delta_0 = \Delta(0)$. Throughout all measurements, the ambient air conditions are controlled at nominal lab conditions. Moreover, as mentioned in Section 3.6.2, the same ES controller is used for all measurements, with the same parameter tuning.

3.7.1 Controller implementation

The standard ECU is bypassed with a Speedgoat real-time target machine, which has an Intel Core i3 3.3 GHz dual-core CPU, and a programmable Xilinx FPGA (field-programmable gate array). The FPGA is used for cycle-to-cycle calculation of y_{IMEP_n} and $y_{CA_{50}}$ in (3.7), (3.8), according to Wilhelmsson et al., 2006, based on the data obtained with the in-cylinder pressure sensors and the crank angle encoder. The CPU is used to implement the low-level and ES controller, which are presented in Sections 3.4 and 3.6, respectively, at a sampling frequency of 100 Hz. Matlab/Simulink[®] is used for programming, automatic code generation, and deployment.

3.7.2 Measurement results

The results of three measurements are discussed: Two at stationary engine operating points and one time-varying operating point. An overview is given in Table 3.2. The stationary points are OP-A and a high load point OP-E similar to OP-B, see Figure 3.5. Measured steady-state maps are depicted in Figures 3.6 and 3.7, respectively. To demonstrate the constraint handling, $r_{NO_x} \in \{4, 6\}$ g/kWh is varied during the OP-A measurement, and $\delta_{p_{peak}} \in \{\bar{p}_{peak} - 20, \bar{p}_{peak}\}$ bar during the OP-E measurement. The steady-state measurements of Σ in Figure 3.6 are obtained with $r_{NO_x} = 5$ g/kWh. The results of the measurement in OP-A are depicted in Figures 3.10, 3.11, 3.12, and 3.13, and for the measurement in OP-E in Figures 3.14, 3.15, 3.16, and 3.17. In the actuator plots, the average duration and SOI over the six cylinders is given, i.e.,

$$\begin{bmatrix} \bar{u}_{dur} & \bar{u}_{SOI} \end{bmatrix} = \frac{1}{6} \mathbb{1}^{1 \times 6} \begin{bmatrix} \boldsymbol{u}_{dur} & \boldsymbol{u}_{SOI} \end{bmatrix}$$

with u_{dur} and u_{SOI} in (3.4), (3.5).

The time-varying operating point consists of a linear transition from OP-A to OP-A^{*}, see Figure 3.5. Compared to OP-A, for OP-A^{*}, M_e is ≈ 33 % lower and n_e is ≈ 21 % higher. Note that, a different fuel is used during the time-varying operating point measurement, being hydrotreated vegetable oil (HVO) which is a renewable diesel fuel. The results are in Figure 3.19.

MA filtering is applied to better visualize noisy signals, e.g., for J(t)

$$J_{MA(T)}(t) = \frac{1}{T} \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} J(\tau) \mathrm{d}\tau,$$

where T = 25 s for all evaluated signals. As a result of the MA filter in the DE in (3.19), the DE blocks require an initialization time of $T_{MA} = 25$ s. Therefore, the optimizer, see Figure 3.9, is activated after 25 s, i.e., $c_1 = c_2 = 0$ for t < 25 s. Observe that the oscillatory behavior of $\Delta(t)$ (and all other signals) is due to the dither excitation and the scheduling between different derivatives in f_{con} in (3.20). These oscillations are necessary for any type of ES which requires a persistence-of-excitation (PE) condition. This type of convergence, i.e., convergence of $\Delta(t)$ to a neighborhood of Δ^* , is known as practical stability, see also Krstić and Wang, 2000.

3.7.2.1 Extremum seeking convergence and constraint handling

Consider Figures 3.10 and 3.14. For both measurements $\Delta(t)$ converges to a constrained Δ^* , see also Figures 3.11 and 3.15. For OP-A, $\Delta(t)$ is mainly

Test	Case 1	Case 2	Case 3
Operating point	OP-A	OP-E	$[OP-A, OP-A^*]$
r_{NO_x} [g/kWh]	$\{4, 6\}$	4.8	$\{4.8, 5.2\}$
$\delta_{p_{peak}}$ [bar]	\bar{p}_{peak}	$[\bar{p}_{peak}, \bar{p}_{peak} - 20]$	$ar{p}_{peak}$
$\mathbf{\Delta}_{0}^{\top}$ [°CA], [kPa]	[4 5]	$[0 \ 0]$	[-2.3 - 3.6]
Fuel	diesel	diesel	HVO

Table 3.2. Specification of the test cases.

constrained in the $\Delta_{r_{cA_{50}}}$ -direction (Figure 3.10), while for OP-E it is mainly constrained in the $\Delta_{r_{CA_{50}}}$ -direction (Figure 3.14). Cost J(t) decreases accordingly. The corresponding values of $h_{NO_x}(t), h_{EGR}(t)$, respectively $h_{p_{peak}}(t)$, are depicted in Figures 3.11 and 3.15.

Figures 3.14 and 3.15 show the functionality of the constraint handling on p_{peak} : A different Δ^* is obtained for $\delta_{p_{peak}} = \bar{p}_{peak} - 20$ bar, such that $h_{p_{peak}}(t)$ practically converges to zero again. Figure 3.14 shows that changing CA_{50} clearly affects the BSFC.

For OP-A, see Figures 3.10 and 3.11, both the NO_x and EGR constraint are active. To illustrate the functionality of the scheduling function f_{con} in (3.20), the values of the scheduling parameters $\alpha_{NO_x}(t), \alpha_{EGR}(t)$ and $\alpha_{p_{peak}}(t)$ and the MA of the, scaled, individual derivative estimates in the Δ_{rdp} -direction are depicted in Figure 3.11. Observe that $\alpha_{p_{peak}} \approx 0$, due to the fact that $h_{p_{peak}} << 0$ for OP-A, see Figure 3.6. Therefore, the corresponding derivative estimate $\tilde{g}_{p_{peak}}^{dp}$ is not depicted in Figure 3.11. For OP-A, u_{EGR} and e_{NO_x} are mainly constrained in the Δ_{rdp} -direction, see Figure 3.6. Correspondingly, \tilde{g}^{dp} is, on average, close to zero for $t \gtrsim 130$ s, see the second plot in Figure 3.11. The MA of $y_{NO_x}(t)$, see Figure 3.11, shows that, on average, the engine-out NO_x level is according to the reference $r_{NO_x}(t)$.

In Figure 3.10 the significant reduction of $\Delta_{BSFC}(t)$ and J(t), for the higher value of $r_{NO_x}(t)$, confirms the BSFC-NO_x trade-off discussed in Section 3.2. Correspondingly, a reduced amount of EGR is required, such that $\Delta_{r_{d_p}}$ at Δ^* can be lower, see again Figure 3.10.

The ES convergence to different constrained Δ^* is clear in Figures 3.12 and 3.16, where $\Delta_{r_{d_p}}(t)$ is plotted against $\Delta_{r_{CA_{50}}}(t)$.

In Figure 3.12, the corresponding measured steady-state map depicted in the upper left plot in Figure 3.8 is plotted. The optimum Δ^* obtained by ES does not exactly match the expectation from the steady-state map. This can be due to the fact that the measurements are not performed at the same time instance (as the measurements of the steady-state mapping) and the location of the optimum may have shifted, and as a result of interpolation between the measured steady-state grid points. Nevertheless, it can be observed that the ES result matches the contours of J_{st} , and the shape of the constraints $h_{NO_T,st}$, $h_{EGR,st} = 0$.

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Figure 3.10. Measurement in OP-A. Extremum seeking (ES) inputs $\Delta_{r_{CA_{50}}}(t), \Delta_{r_{dp}}(t)$, measured cost J(t), and the difference in actual BSFC $\Delta_{BSFC}(t)$. The yellow surfaces indicate the time window where $r_{NO_x} = 6$ g/kWh, $r_{NO_x} = 4$ g/kWh outside this time window.

3.7.2.2 Robustness of the extremum seeking optimization

Conducting experiments under all significant real-world disturbances is not possible, due to limited measurement time and limitations of the test cell. However, observations can be obtained from the presented measurement results, which demonstrate robustness of the proposed ES-based approach.

The presented results show that $\Delta(t)$ converges to Δ^* , with the same ES parameter tuning, for very different operating points (see Figure 3.5). In addition, the controller copes with time-varying $r_{NO_x}(t)$ and $\delta_{p_{peak}}(t)$, and a different fuel. Although the variation of $r_{NO_x}(t)$ is dealt with by the tracking control system, the ES obtains a different Δ^* , see Figure 3.12, which indicates a correlation between r_{NO_x} and the cost and constraint outputs. As such, the variation in r_{NO_x} affects the system Σ in a similar way as a real-world disturbance, and thereby demonstrates disturbance rejection of the ES-based control approach.

Besides the robustness with respect to real-world disturbances, for practical application of the proposed approach, the ability of ES to deal with a timevarying operating point should be considered. Despite the applied operating point-dependent normalization (3.13) on the BSFC estimate in the cost J, and using the nominal operating point-dependent reference signals r in (3.10) as a



Figure 3.11. Measurement in OP-A. Upper plot: Constraint outputs $h_{NO_x}(t), h_{EGR}(t)$. Second plot: MA-filtered (γ -scaled) gradients in the $\Delta_{r_{dp}}$ -direction. Third plot: Scheduling parameters $\alpha_{NO_x}, \alpha_{EGR}, \alpha_{p_{peak}}$ in (3.20). Lower plot: Tracking result of r_{NO_x} . The yellow surfaces indicate the time window where $r_{NO_x} = 6$ g/kWh, $r_{NO_x} = 4$ g/kWh outside this time window.

basis for the ES input, the resulting optimal ES input Δ^* is not completely independent of the engine operating point, see Figure 3.8 and the results in this section. The required convergence rate to deal with a time-varying operating point depends on the rate of change of Δ^* , which is a result of the rate of change of the operating point, in combination with the corresponding steady-state maps in Figure 3.8. For a relatively slow time-varying operating point in Figure 3.19, $\Delta(t)$ converges accordingly. Being constrained in the $\Delta_{r_{dp}}$ -direction, a large NO_x tracking error occurs during the transition between OP-A and OP-A^{*} as a result of $h_{EGR} > 0$ violation.

Observe that during OP-A ($t \leq 380$ s) in the time-varying operating point test, the cost J(t) for $\Delta(t) \approx \Delta^*$ is higher than, and Δ^* is different from, the results of the stationary OP-A measurement, compare Figures 3.10 and 3.19, respectively. Although a slightly different value $r_{NO_x}(t) = 5.2$ g/kWh is used (instead of $r_{NO_x}(t) \in \{4, 6\}$ g/kWh), the difference is mainly due to the

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Figure 3.12. Extremum seeking (ES) input $\Delta(t)$ for the measurement in OP-A, plotted over level curves of the corresponding interpolated steady-state measured map from Figure 3.8.

different fuel, HVO instead of diesel. This adds to the motivation for an adaptive optimizing control approach in general, such as ES.

3.7.2.3 Obtained fuel efficiency optimum

The cost output J(t) relates well to the measured difference in BSFC $\Delta_{BSFC}(t)$, see Figures 3.10 and 3.14. As such, it is concluded that Δ^* according to J(t), is a good estimate of the value for Δ which would yield optimal fuel efficiency in practice. The average duration $\bar{u}_{dur}(t)$ varies accordingly, i.e., $\bar{u}_{dur}(t) \sim \Delta_{BSFC}(t)$, see Figures 3.13 and 3.17.

For the measurement in OP-A, see Figure 3.10, an offset initial point $\Delta_0^{\top} = [4 \ 5]$ is used. This is done to accentuate the ES functionality. Due to this offset, the corresponding reduction of Δ_{BSFC} is not realistic. However, the ES consistently results in $\Delta_{r_{dp}}^* \neq 0$, which implies, as a result of gradient descent optimization in the ES, that cost J has a (local) minimum at $\Delta^{*\top} \neq [0 \ 0]$. Observe that both for the OP-A and OP-E measurements, the obtained Δ^* is close to the optimum in the steady-state measurements in Figures 3.6 and 3.7. Furthermore, the ability to find the optimum Δ^* might lead to a larger reduction of Δ_{BSFC} in a disturbed real-world case.

As already discussed in Section 3.7.2.1, the increased value for $r_{NO_x}(t)$ results



Figure 3.13. Actuator positions u(t) for the measurement in OP-A. The yellow surfaces indicate the time window where $r_{NO_x} = 6$ g/kWh, $r_{NO_x} = 4$ g/kWh outside this time window.

in a significant reduction in Δ_{BSFC} . This is however only partly the result of ES, as the BSFC-NO_x trade-off plays a role.

In the OP-E measurement, $\Delta_0^{\dagger} = [0 \ 0]$, i.e., initially the default tuning is used. As such, the observed decrease in Δ_{BSFC} of ≈ 1 g/kWh is realistic. This decrease is mainly the result of the incorporation of the constraint on the maximum in-cylinder pressure p_{peak} . By default, the tuning of $r_{CA_{50}}$ needs to be conservative, to avoid violation of the p_{peak} constraint.

Remark 3.4. The default tuning of the ES input, i.e., $r_{CA_{50}}$ and r_{dp} , is aimed at maximizing the fuel efficiency of the considered engine under nominal test conditions. The ES measurements in OP-A and OP-E, see Figures 3.10 and 3.14, respectively, are conducted under the same nominal test conditions. Hence, a significantly increase of fuel efficiency by ES cannot be expected. Instead, the functionality of ES is of interest, as it indicates its ability to optimize fuel efficiency in a non-nominal situation, e.g., for different ambient conditions.

The oscillation in $\Delta(t)$ around Δ^* for OP-E is seen to affect both the constraint output $h_{p_{peak}}(t)$ in Figure 3.15 and in $\Delta_{BSFC}(t)$ in Figure 3.14. The soft limit $\delta_{p_{peak}}$ needs to be reduced proportionally to the oscillation in $h_{p_{peak}}(t)$ to avoid violation of the physical maximum of p_{peak} . Effectively, this increases the

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Figure 3.14. Measurement in OP-E. Extremum seeking (ES) inputs $\Delta_{r_{CA_{50}}}(t), \Delta_{r_{dp}}(t)$, measured cost J(t), and the difference in actual BSFC $\Delta_{BSFC}(t)$. The yellow surfaces indicate the time window where $\delta_{p_{peak}} = \bar{p}_{peak} - 20$ bar, $\delta_{p_{peak}} = \bar{p}_{peak}$ bar outside this time window.

average optimal cost J at Δ^* and thus reduces the increase in fuel efficiency. As noted in Section 3.6.2, the oscillations around Δ^* scale with the dither amplitude. As such, it is important to select a_{d_1}, a_{d_2} as small as possible.

For OP-A, the effect of oscillations in $\Delta(t)$ on $\Delta_{BSFC}(t)$, see Figure 3.10, is not so apparent. This is because in OP-A, the constrained optimum Δ^* is close to the actual minimum of J, where the derivative of J is much smaller than in the constrained optimum for OP-E.

3.7.2.4 Low-level control system performance

The effect of the applied ES controller on the low-level control system is best evaluated by considering the actuator positions, depicted in Figures 3.13 and 3.17. The key observation is that all four actuators respond, while the optimization is done using only two variables in $\Delta(t)$. The actuators and hence the low-level control system are clearly affected by the dither excitation and the constraint scheduling. Note that, the oscillation of $u_{EGR}(t)$ is stronger for OP-E (Figure 3.17) than for OP-A (Figure 3.13). This is due to the larger value of $u_{EGR}(t)$, at which the EGR valve is less sensitive, as explained in Section 3.5.1.

The tracking of $r_{NO_x}(t)$ suffers from ES induced oscillations, specifically for



Figure 3.15. Constraint output $h_{p_{peak}}(t)$ for the measurement in OP-E. The yellow surface indicates the time window where $\delta_{p_{peak}} = \bar{p}_{peak} - 20$ bar, $\delta_{p_{peak}} = \bar{p}_{peak}$ bar outside this time window.

large values of $u_{EGR}(t)$, see also Figure 3.13. This is essentially why the value of δ_{EGR} in (3.15) is selected lower than the actual saturation of the EGR valve. The MA of $y_{NO_x}(t)$ shows however that, on average, there is no violation of the reference engine-out NO_x level, which is in accordance with the high-level diesel engine control objective stated in Section 3.3.1. Regarding the demanded torque $M_{e.dem}$, the cylinder-average tracking error \bar{e}_{IMEP_n} is of interest. Figure 3.18 depicts the cumulative power spectral densitys (PSDs) of $\bar{e}_{IMEP_n}(t)$ for OP-A and OP-E, with and without ES active. For both operating points, ES induces no significant difference in the PSD of $\bar{e}_{IMEP_n}(t)$. In particular for $f < \omega_{d_1}$, which is the frequency band in which ES is active, there is no difference observed.

In conclusion, ES does affect the low-level control system performance, however, the influence is limited and not problematic with respect to the high-level control objective stated in Section 3.3.1.

3.7.2.5 Discussion: Practical applicability

The obtained results show that $\Delta^* \neq [0 \ 0]^\top$, and that the related cost J(t) corresponds well to the actual BSFC. Hence, there is a potential to increase the fuel efficiency of the considered engine. Therefore, applying online adaptive optimization, such as ES, is useful. Especially in non-nominal real-world conditions, where the default tuning does not necessarily yield optimal fuel efficiency, see also the effect of a different fuel: HVO instead of diesel.

The applied ES controller is capable of robustly finding the (constrained) optimum Δ^* . However, the functionality of the ES-based approach for a timevarying operating point is limited. This is due to the operating point dependency of the optimal input Δ^* , as discussed in Section 3.7.2.2, combined with the inherently limited convergence rate of the considered gradient descent-based ES. As such, straightforward application of the ES-based approach in a transient test cycle is not possible. The transient performance may be improved by using a NO_x sensor with less delay, which enables an increased bandwidth of the low-level tracking control system, and therefore allows for an increased dither frequency and ES convergence rate. Alternatively, the approaches in Marinkov et al., 2014;

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Figure 3.16. Extremum seeking (ES) input $\Delta(t)$ for the measurement in OP-E.

Sharafi et al., 2016 may be used to obtain an operating point-dependent ES.

In case g_J is relatively high at a constrained optimum, Δ_{BSFC} could be further reduced by application of input-based ES methods because such methods require less (or no) dither excitation, see Remark 3.2. As discussed in Section 3.7.2.4, the performance, in terms of average engine-out NO_x and *IMEP_n* tracking, is however not affected by the dither perturbation, and ES in general.

3.8 Conclusion

A two-input (quasi-)convex constrained optimization problem is proposed, which connects to the diesel engine control goal of delivering power, using a minimal amount of fuel, subject to constraints on specific engine-out NO_x emission, incylinder pressure, and exhaust gas recirculation (EGR) valve position. The two inputs are reference signals of a low-level feedback control system, related to pumping losses and combustion phasing. As a result of interaction in the multivariable low-level control system, all four air-path and fuel-path actuators respond to the optimization of only two inputs. A constrained extremum seeking (ES) approach is applied to obtain optimal fuel efficiency, potentially also under real-world disturbances, as no model and disturbance information is required.



Figure 3.17. Actuator positions u(t) for the measurement in OP-E. The yellow surfaces indicate the time window where $\delta_{p_{peak}} = \bar{p}_{peak} - 20$ bar, $\delta_{p_{peak}} = \bar{p}_{peak}$ bar outside this time window.

Experiments are conducted on an advanced setup, based on a Euro-VI heavyduty truck engine, with additional in-cylinder pressure sensors. The experiments demonstrate the functionality and robustness of the proposed ES-based control approach, the equivalence of the cost function to the actual brake specific fuel consumption (BSFC), and a reduction of BSFC up to ≈ 1 g/kWh under nominal operating conditions. Application of the ES-based control approach in a transient test cycle requires further research, to deal with the remaining operating point dependency in the ES cost.



Figure 3.18. Cumulative power spectral density (PSD) of cylinder-average $IMEP_n$ tracking errors \bar{e}_{IMEP_n} for OP-A and OP-E with(out) extremum seeking (ES) based on three windows of length 81.92 s. The bottom plot shows the same results as in the top plot, zoomed in at $f \leq 0.25$ Hz. Without ES corresponds to $c_1 = c_2 = a_{d_1} = a_{d_2} = 0$ and $\Delta_0 = \Delta = [0 \ 0]^{\top}$. For ES active, $\bar{e}_{IMEP_n}(t)$ corresponding to the presented time-domain results are used, for OP-A with $t \in [206.66, 452.41]$ s and for OP-E with $t \in [31.34, 277.09]$ s.



Figure 3.19. Signals $\Delta(t)$, J(t), $J_{MA(25)}(t)$, and NO_x, with HVO fuel, for a time-varying operating point: OP-A for $t \leq 380$ s and OP-A^{*} for $t \gtrsim 400$ s, see also Figure 3.5.

Chapter 4

Extremum seeking in over-actuated tracking control systems

Abstract – This chapter¹ considers the problem of simultaneous reference tracking control and cost optimization using extremum seeking (ES), for overactuated systems. An extremum seeking tracking control (ESTC) design is proposed, in which the interaction between the tracking control and the ES objectives is accounted for by an adaptive decoupling mechanism. As a result, there is no need for time scale separation between the tracking dynamics and the ES, which enables an increased ES convergence rate. The proposed control design is demonstrated in a case study on air-path control of a diesel engine.

4.1 Introduction

Many control problems allow cost optimization, besides tracking of other system outputs. The optimization could for instance aim for minimal energy consumption, or maximizing a chemical reaction. Examples can be found in automotive control Wahlström et al., 2010; Van der Meulen et al., 2014; Gelso and Dahl, 2016; Broomhead et al., 2017; Luo et al., 2018a and process control Skogestad, 2000. The case study in this chapter considers combined tracking of the exhaust gas recirculation (EGR) fraction in a diesel engine, correlated to emission of the pollutant NO_x , and minimization of the engine pumping-loss.

In over-actuated systems, the number of control degrees-of-freedom (DOFs) is larger than the number of tracking outputs, i.e., outputs that are required to

 $^{^1\}mathrm{This}$ chapter is partly based on Van der Weijst et al., 2019b.

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follow a reference signal. This chapter considers zero steady-state error tracking control, for over-actuated systems with a measurable cost output. The control input that satisfies the tracking objective is non-unique for over-actuated systems, i.e., the tracking objective is satisfied for a range of input combinations. The corresponding value of the cost output for input combinations in this range, is generally not equal. As a result, an optimal input combination exists, which yields the lowest cost output for the given tracking objective. The problem considered in this chapter is to find this optimal input, with robustness to system uncertainty. Note that, the cost optimum is considered to be a minimum, which is without loss of generality.

Providing robustness to system uncertainty in cost optimization is more challenging than in tracking control. Since the tracking outputs and references are known, the tracking error is always known. Contrarily, the optimal cost value, which is affected by system uncertainty and the tracking objective, is unknown.

Economic model predictive control (MPC), see , e.g., Angeli et al., 2012; Müller and Allgöwer, 2017, provides combined tracking and cost optimization, by minimizing the sum of a tracking control cost and an economic cost over a prediction horizon, using a parametric model of the system dynamics. Input and state constraints are effectively handled in economic MPC. However, the obtained optimal cost depends on the model.

For the diesel engine case study in this chapter, economic MPC has been applied in Wahlström and Eriksson, 2013. Wahlström et al., 2010; Criens et al., 2015 propose heursitic approaches to operate the system in such a way that the cost is near optimal during tracking control. In Wahlström et al., 2010, the tracking control allocation is scheduled as a function of measured disturbances which are known to affect the location of the optimum. In Criens et al., 2015, in addition to the tracking output, an inferred cost parameter is tracked to a predefined, feasible, reference value that is assumed to be close to the optimum.

An open issue exists in obtaining the optimal cost in the presences of system uncertainty. This can be the case for highly complex system dynamics, or due to uncertain parameters or time-varying behavior of the system, possibly depending on parameters that are not measured. Extremum seeking (ES) is a control approach that can be used to optimize the cost output of a system in steady state, while only limited knowledge of the system dynamics is required, see, e.g., Tan et al., 2010. As such, by combining ES with a robust tracking controller, potentially a control design can be obtained that provides combined tracking and optimization, with robustness to system uncertainty.

The input corresponding to the cost minimum does generally not yield the demanded reference values of the tracking outputs, i.e., the optimization and tracking objectives are conflicting. As a result, taking the sum of independent ES and tracking controllers as input to the over-actuated system, generally yields a sub-optimal cost and induces a nonzero steady-state tracking error. Reducing the convergence rate of ES with respect to the tracking dynamics, reduces
the steady-state tracking error. However, prioritizing tracking control by such a time scale separation approach is undesirable in applications where fast ES convergence is desired. This chapter proposes a novel control architecture, which enables tracking control and ES in the same time scale, thereby increasing the ES convergence rate. The key element is a projection of the estimated cost gradient on the orthogonal component of the estimated tracking gradient. The projection acts as an adaptive decoupling mechanism, that decouples the ES adaptation from the tracking control objective. As a result, the steady-state tracking performance is unaffected by ES. Similar modified gradient ES approaches are used for constrained ES, see, e.g., Poveda and Quijano, 2012; Ramos et al., 2017, and multi-objective ES, see, e.g., Atta et al., 2018.

The projection requires derivative estimates of the system's steady-state map from the input to both the tracking and cost output. Classical dither-based ES derivative estimation, see Chapter 2, relies on external excitation of the system input by a periodic dither signal. The tracking controller is designed to counteract such input disturbances. To enable dither-based derivative estimation, in Van der Meulen et al., 2014; Van der Weijst et al., 2019b, the tracking control effort is suppressed with notch filters at the dither frequencies. However, suppressing the control effort reduces the tracking control performance. Moreover, a narrow, lightly damped notch filter has a slow transient response, which is problematic in practice. In this chapter, an alternative solution is found by application of input-based derivative estimation, which uses the actual system input instead of the external dither signal. See, e.g., Guay and Dochain, 2015, Chapter 2 in Haring, 2016, or Chapter 2 in this thesis.

Summarizing, this chapter presents an extremum seeking tracking control (ESTC) design, which optimizes the steady-state ES cost in the same time scale as tracking, within the range of inputs for which the steady-state tracking objective is satisfied. The approach is presented for systems with two inputs, with a proportional-integral (PI) type tracking controller. Preliminary steps for a stability analysis of the static closed-loop system are presented. The proposed ESTC is applied for online fuel efficiency optimization of diesel engines and is demonstrated using the physics-based simulation model presented in Wahlström and Eriksson, 2011.

This chapter consists of two parts. The first part considers the general ESTC problem, controller design, and analysis, which are presented in Sections 4.2, 4.3, and 4.4, respectively. The second part presents the diesel engine case study, which consists of a system description provided in Section 4.5, a modified version of the ESTC design presented in Section 4.6, and finally simulation results which are provided in Section 4.7. The conclusions are summarized in Section 4.8.

4.2 System and problem description

4.2.1 Class of systems

The considered class of systems is given by

$$\mathcal{H}: \begin{cases} \dot{\boldsymbol{x}}_{\mathcal{H}}(t) &= f(\boldsymbol{x}_{\mathcal{H}}(t), \boldsymbol{u}(t)) \\ y(t) &= g(\boldsymbol{x}_{\mathcal{H}}(t), \boldsymbol{u}(t)) \\ J(t) &= h(\boldsymbol{x}_{\mathcal{H}}(t), \boldsymbol{u}(t)), \end{cases}$$
(4.1)

where $\boldsymbol{x}_{\mathcal{H}} \in \mathbb{R}^{n_{\mathcal{H}} \times 1}$ is a state vector with $n_{\mathcal{H}}$ states, $y \in \mathbb{R}$ is the tracking output, $J \in \mathbb{R}$ is the cost output, $\boldsymbol{u} = [u_1 \ u_2]^{\top} \in \mathbb{R}^{2 \times 1}$ is the input, $f : \mathbb{R}^{n_{\mathcal{H}}} \times \mathbb{R}^2 \to \mathbb{R}^{n_{\mathcal{H}}}$, and $g : \mathbb{R}^{n_{\mathcal{H}}} \times \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R}^{n_{\mathcal{H}}} \times \mathbb{R}^2 \to \mathbb{R}$. The system \mathcal{H} in (4.1) satisfies the following assumptions.

Assumption 4.1. For constant inputs u, the system \mathcal{H} in (4.1) possesses a unique asymptotically stable equilibrium $x_{\mathcal{H}} = l(u)$, such that its outputs are described by the steady-state maps $Q_y : u \to y$ and $Q_J : u \to J$, given by:

$$Q_y(\boldsymbol{u}) = g(l(\boldsymbol{u}), \boldsymbol{u}), \tag{4.2a}$$

$$Q_J(\boldsymbol{u}) = h(l(\boldsymbol{u}), \boldsymbol{u}). \tag{4.2b}$$

Assumption 4.2. There exists a positive constant ε , and a control allocation vector $\mathbf{m} \in \mathbb{R}^{2\times 1}$, such that for all $\mathbf{u} \in \mathbb{R}^2$ the gradient $\nabla_y(\mathbf{u})$ of the map $Q_y(\mathbf{u})$ in (4.2a) satisfies

$$\nabla_y(\boldsymbol{u})\boldsymbol{m} \ge \varepsilon > 0. \tag{4.3}$$

Assumption 4.2 states that when m is used as control allocation vector, e.g., the input to the system is $u = my_{\mathcal{C}}$, where $y_{\mathcal{C}} \in \mathbb{R}$ is the output of a singleinput-single-output (SISO) tracking controller, the steady-state gain from $y_{\mathcal{C}}$ to y does not contain sign reversals. This is required to apply a linear tracking control design with a constant control allocation, and is used in the convergence analysis in Section 4.4.

4.2.2 Control objective

The objective of the tracking controller is to provide tracking of the output y of a reference signal r. The tracking error is defined as e = r - y. To aid the presentation and analysis of the ESTC, the following assumption is made.

Assumption 4.3. The reference signal r is constant.

Although the reference signal is constant, to reject time-varying disturbances, the tracking controller provides *transient*, as well as steady-state performance.

Since ES optimizes steady-state performance, for the interaction between tracking and ES, the steady-state tracking performance is considered. As such, the ESTC objective is defined considering the steady-state behavior of the system, using the steady-state maps $Q_{y}(\boldsymbol{u})$ and $Q_{J}(\boldsymbol{u})$ in (4.2), and is summarized by

$$\min_{\boldsymbol{u}} Q_J(\boldsymbol{u}), \tag{4.4a}$$

s.t.
$$Q_y(u) - r = 0.$$
 (4.4b)

The minimization objective for the steady-state cost in (4.4a), is subject to the zero steady-state tracking error requirement in (4.4b).

The ESTC objective in (4.4) is an equality-constrained optimization problem. As such, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$, for which the optimal input u^* satisfies the Lagrange necessary conditions for a stationary point of (4.4), which are given by

$$\nabla_J(\boldsymbol{u}) + \lambda \nabla_y(\boldsymbol{u}) = \boldsymbol{0} \tag{4.5a}$$

$$Q_y(\boldsymbol{u}) - r = 0, \tag{4.5b}$$

where

$$\nabla_J(\boldsymbol{u}) = \begin{bmatrix} \frac{\partial Q_J(\boldsymbol{u})}{\partial u_1} & \frac{\partial Q_J(\boldsymbol{u})}{\partial u_2} \end{bmatrix}, \qquad (4.6a)$$

$$\nabla_{y}(\boldsymbol{u}) = \begin{bmatrix} \frac{\partial Q_{y}(\boldsymbol{u})}{\partial u_{1}} & \frac{\partial Q_{y}(\boldsymbol{u})}{\partial u_{2}} \end{bmatrix}.$$
(4.6b)

Assumption 4.4. The optimization problem in (4.4) has only one stationary point, which is a minimum. Hence, the necessary conditions in (4.5) are sufficient conditions for $\mathbf{u}^* = \mathbf{u}$, with \mathbf{u}^* satisfying (4.5), to be the optimal input.

Using Assumption 4.4, the optimum u^* , which satisfies the conditions in (4.5), can be defined as

$$\boldsymbol{u}^* = \boldsymbol{u}^*_{\mathcal{C}} \cap \boldsymbol{u}^*_{ES} \tag{4.7}$$

with \boldsymbol{u}_{ES}^* the set of inputs for which (4.5a) is satisfied, given by

$$\boldsymbol{u}_{ES}^{*} = \left\{ \boldsymbol{u} \in \mathbb{R}^{2} \mid \frac{\partial Q_{J}(\boldsymbol{u})}{\partial u_{1}} \frac{\partial Q_{y}(\boldsymbol{u})}{\partial u_{2}} = \frac{\partial Q_{J}(\boldsymbol{u})}{\partial u_{2}} \frac{\partial Q_{y}(\boldsymbol{u})}{\partial u_{1}} \right\},$$
(4.8)

and $u_{\mathcal{C}}^*$ the set of inputs for which (4.5b) is satisfied, given by

$$\boldsymbol{u}_{\mathcal{C}}^* = \left\{ \boldsymbol{u} \in \mathbb{R}^2 \mid Q_y(\boldsymbol{u}) = r \right\}.$$
(4.9)

Example 4.5. Consider a system of the form (4.1) that has the following steadystate maps:

$$Q_J(\boldsymbol{u}) = rac{1}{2} \left(u_1^2 + u_2^2
ight)$$

 $Q_y(\boldsymbol{u}) = u_1 + u_2.$

For these maps and a constant reference signal r, the set $\mathbf{u}_{\mathcal{C}}^*$ in (4.9) is described by $u_2 = r - u_1$, while the set \mathbf{u}_{ES}^* in (4.8) is described by $u_2 = u_1$. Figure 4.1 depicts level sets of the map $Q_J(\mathbf{u})$, the sets $\mathbf{u}_{\mathcal{C}}^*$, \mathbf{u}_{ES}^* , and the optimum \mathbf{u}^* defined in (4.7). Observe that the actual minimum of $Q_J(\mathbf{u})$ is obtained for \mathbf{u} in the origin, and not for $\mathbf{u} \in \mathbf{u}_{\mathcal{C}}^*$ for the reference r that is selected in this example, i.e., the tracking and optimization objectives are conflicting in this example.



Figure 4.1. Example of the steady-state problem for $Q_y(\mathbf{u}) = u_1 + u_2$ and $Q_J(\mathbf{u}) = \frac{1}{2}(u_1^2 + u_2^2)$. The sets \mathbf{u}_C^* , \mathbf{u}_{ES}^* in (4.8), (4.9), respectively, the optimum \mathbf{u}^* in (4.7), and level sets of $Q_J(\mathbf{u})$ are indicated.

4.3 Extremum seeking tracking control design

This section introduces the ESTC approach, schematically depicted in Figure 4.2. The ESTC scheme consists of a tracking controller, which aims to steer the input \boldsymbol{u} to the set $\boldsymbol{u}_{\mathcal{C}}^*$ in (4.9), and the ES with adaptive decoupling, which aims to steer the input \boldsymbol{u} to the set \boldsymbol{u}_{ES}^* in (4.8).

The ES with adaptive decoupling uses estimated gradients of the maps $Q_y(\boldsymbol{u})$ and $Q_J(\boldsymbol{u})$, which are obtained by using the input-based derivative estimator (DE) that is introduced in Section 2.5. However, the ES with adaptive decoupling can be applied with any type of input-based DE.

The focus of this chapter is on the integration of ES in a tracking control system. Therefore, a relatively simple PI controller is considered with a constant control allocation.



Figure 4.2. The closed-loop extremum seeking tracking control (ESTC) system. The integrators are shared between the tracking controller and extremum seeking (ES).

4.3.1 Tracking controller

The tracking controller is essentially a standard PI controller described by the transfer function

$$\mathcal{C}(s) = \boldsymbol{m}\left(k_P + k_I \frac{1}{s}\right),$$

where $k_P, k_I \in \mathbb{R}_{\geq 0}$, and $\boldsymbol{m} \in \mathbb{R}^{2 \times 1}$ is a constant control allocation vector. To accommodate the integration with ES, two additional inputs, $\boldsymbol{w}, \boldsymbol{d} \in \mathbb{R}^{2 \times 1}$ are included in the controller. As such, the input \boldsymbol{u} to the system is provided by the controller which is given by

$$\dot{\boldsymbol{x}}_{\mathcal{C}}(t) = \boldsymbol{m}k_{I}\boldsymbol{e}(t) + \boldsymbol{w}(t) \tag{4.10a}$$

$$\hat{\boldsymbol{u}}(t) = \boldsymbol{x}_{\mathcal{C}}(t) + \boldsymbol{m}k_{P}\boldsymbol{e}(t) \tag{4.10b}$$

$$\boldsymbol{u}(t) = \hat{\boldsymbol{u}}(t) + \boldsymbol{d}(t), \qquad (4.10c)$$

where $\boldsymbol{x}_{\mathcal{C}} \in \mathbb{R}^{2 \times 1}$ is the controller state and e(t) = r - y(t) is the tracking error.

4.3.2 Input-based derivative estimation

To ensure that the input $\boldsymbol{u}(t)$ to the system \mathcal{H} satisfies a persistence-of-excitation (PE) condition, a dither signal $\boldsymbol{d}(t)$ is added to $\hat{\boldsymbol{u}}(t)$ in (4.10c). The dither signal is given by

$$\boldsymbol{d}(t) = \begin{bmatrix} a_{d_1} \cos(\omega_{d_1} t) \\ a_{d_2} \cos(\omega_{d_2} t) \end{bmatrix},\tag{4.11}$$

where $a_{d_1}, a_{d_2} \in \mathbb{R}_{\geq 0}$ are the dither amplitudes and $\omega_{d_1} \neq \omega_{d_2} \in \mathbb{R}_{\geq 0}$ are the dither frequencies.

The applied input-based DE provides estimates $\tilde{g}_y(t)$ and $\tilde{g}_J(t)$ of the derivative vectors

$$egin{aligned} oldsymbol{g}_y(oldsymbol{u}) &= egin{bmatrix} Q_y(oldsymbol{u}) & rac{\partial Q_y(oldsymbol{u})}{\partialoldsymbol{u}} & rac{\partial Q_y(oldsymbol{u})}{\partialoldsymbol{u}} \end{bmatrix}^ op, \ oldsymbol{g}_J(oldsymbol{u}) &= egin{bmatrix} Q_J(oldsymbol{u}) & rac{\partial Q_J(oldsymbol{u})}{\partialoldsymbol{u}} & rac{\partial Q_y(oldsymbol{u})}{\partialoldsymbol{u}} \end{bmatrix}^ op, \end{aligned}$$

respectively. Note that, the estimates of $Q_y(\mathbf{u})$ and $Q_J(\mathbf{u})$ are not used for control; their presence in the estimate vectors $\mathbf{g}_y(\mathbf{u})$ and $\mathbf{g}_J(\mathbf{u})$, results from the affine estimation models that are used in the least-squares optimization, which is underlying the input-based derivative estimation approach. As introduced in Section 2.5, the input-based DE is given by

$$\tilde{\boldsymbol{g}}_{\boldsymbol{y}}(t) = K_{\boldsymbol{u}}^{-1}(\boldsymbol{u}(t)) \int_{t-T_{\boldsymbol{u}}}^{t} \begin{bmatrix} 1\\ \boldsymbol{u}(\tau) \end{bmatrix} \boldsymbol{y}(\tau) \mathrm{d}\tau, \qquad (4.12)$$

where $T_u \in \mathbb{R}_{>0}$ and

$$K_{\boldsymbol{u}}(\boldsymbol{u}(t)) = \int_{t-T_{\boldsymbol{u}}}^{t} \begin{bmatrix} 1\\ \boldsymbol{u}(\tau) \end{bmatrix} \begin{bmatrix} 1\\ \boldsymbol{u}(\tau) \end{bmatrix}^{\top} d\tau.$$
(4.13)

The estimate $\tilde{g}_J(t)$ is obtained by substituting the measured cost J, for y, in (4.12). When the input u(t) is PE, the inverse of the matrix $K_u(u(t))$ exists. Using the derivative estimates in $\tilde{g}_y(t)$ and $\tilde{g}_J(t)$, estimates $\tilde{\nabla}_y(u)$ and $\tilde{\nabla}_J(u)$ of the gradients in (4.6) are obtained.

4.3.3 Extremum seeking with adaptive decoupling

As already mentioned, and illustrated in Example 4.5, the minimal cost value $Q_J(\boldsymbol{u})$, in the considered class of problems (4.4), is generally not obtained for inputs \boldsymbol{u} that satisfy the tracking objective (4.4b). As such, the tracking and optimization objectives are generally conflicting. The main contribution of this chapter is the adaptive decoupling of ES from the tracking objective, such that zero steady-state error tracking is possible without reducing the ES convergence rate.

The reasoning behind the decoupling is that, locally in \boldsymbol{u} , the tracking output y is not affected by ES, when the ES contribution to the input \boldsymbol{u} induced by \boldsymbol{w} in (4.10a), is orthogonal to the gradient $\nabla_y(\boldsymbol{u})$. This orthogonal direction is obtained from the estimated gradient as

$$\tilde{\nabla}_{y^{\perp}}(t) = \tilde{\nabla}_{y}(t) \begin{bmatrix} 0 & -1\\ 1 & 0 \end{bmatrix}.$$
(4.14)

The adaptive decoupling is obtained by projection of the cost output gradient estimate on $\tilde{\nabla}_{y^{\perp}}$. The resulting modified gradient is defined as $\tilde{g}_{ES} \in \mathbb{R}^{2 \times 1}$ and

is described by

$$\tilde{\boldsymbol{g}}_{ES}^{\top}(t) = \operatorname{proj}_{\tilde{\nabla}_{y}^{\perp}(t)}(\tilde{\nabla}_{J}(t)) = \frac{\tilde{\nabla}_{J}(t)\tilde{\nabla}_{y}^{\top}(t)}{\tilde{\nabla}_{y}^{\perp}(t)\tilde{\nabla}_{y}^{\top}(t)}\tilde{\nabla}_{y}^{\perp}(t).$$
(4.15)

Subsequently, the ES adaptation of the input \boldsymbol{u} to the system consists of

$$\boldsymbol{w}(t) = -c\tilde{\boldsymbol{g}}_{ES}(t), \qquad (4.16)$$

with $c \in \mathbb{R}_{>0}$ the optimizer gain, and the integral action of the controller (4.10).

The projection in (4.15) provides the decoupling of \boldsymbol{w} from the estimated tracking gradient $\tilde{\nabla}_y(t)$. Since the optimization direction $\tilde{\boldsymbol{g}}_{ES}$ uses continuous estimates of the gradients of $Q_y(\boldsymbol{u})$ and $Q_J(\boldsymbol{u})$, the decoupling is adaptive.

Observe that, when the true gradients $\nabla_J(\boldsymbol{u})$ and $\nabla_{y^{\perp}}(\boldsymbol{u})$ are substituted in (4.15), $\boldsymbol{g}_{ES}(t) = \boldsymbol{0}$ when $\boldsymbol{u}(t) \in \boldsymbol{u}_{ES}^*$. This observation can be clarified by observing the definition of the set \boldsymbol{u}_{ES}^* in (4.8). The condition in (4.8) is equivalent to $\nabla_J(\boldsymbol{u})\nabla_{\boldsymbol{u}^{\perp}}^{\top}(\boldsymbol{u}) = 0$, which occurs in the denominator of (4.15).

The complete ESTC design is given by (4.10), with $\boldsymbol{d}(t)$ in (4.11), $\boldsymbol{w}(t)$ in (4.16), with $\tilde{\boldsymbol{g}}_{ES}(t)$ in (4.15) and the DE in (4.12).

Remark 4.6. When the derivative estimates that are used in the projection in (4.15) are not exact, the ES is not fully decoupled from the tracking objective, which induces a residual steady-state tracking error. By sharing the integrator states $\boldsymbol{x}_{\mathcal{C}}$ in (4.10a) between ES and tracking, internal instability is avoided in case of such a residual steady-state tracking error. Essentially, by sharing the integrator states the ESTC is a minimal realization.

4.4 Convergence analysis

The stability analysis of the closed-loop ESTC system consists of several steps. Following the approach in Haring, 2016, Chapter 2, where an observer-based, input-based ES approach is considered, a high-level overview of these steps is provided as follows:

- (i) A convergence analysis of the static system, described by $Q_y(\boldsymbol{u})$ and $Q_J(\boldsymbol{u})$ combined with the tracking controller and optimizer, without dither perturbation, using the true derivatives of $Q_y(\boldsymbol{u})$ and $Q_J(\boldsymbol{u})$.
- (ii) Robustness margins in the convergence analysis in (i), to deal with bounded disturbances on the estimated derivatives and the dither excitation.
- (iii) Conditions to provide a bound on the difference between the estimated derivatives of the maps $Q_y(\boldsymbol{u})$ and $Q_J(\boldsymbol{u})$, and the true derivatives.

(iv) Conditions to provide a bound on the difference between the state $\boldsymbol{x}_{\mathcal{H}}(t)$ and the steady-state equilibrium state $l(\boldsymbol{u}(t))$ defined in Assumption 4.1.

Combined in a small gain analysis, these steps imply that the input u converges to the optimum u^* defined in (4.7). The presented analysis is limited to step (i). As such, the analysis provided here serves as a stepping stone towards a complete stability analysis in the future. Moreover, it illustrates the intuition behind the ESTC design. Moreover, while steps (i) and (ii) are specific for the ESTC design, steps (iii) and (iv) are expected to be similar to those steps in existing ES analyses.

4.4.1 Simplified closed-loop system description

The static version of the system \mathcal{H} in (4.1) is given by

$$\boldsymbol{u} \quad (4.17a)$$

$$J(t) = Q_J(\boldsymbol{u}(t)), \qquad (4.17b)$$

with $Q_y(\boldsymbol{u})$ and $Q_J(\boldsymbol{u})$ in (4.2). The closed-loop tracking control system with the static system \mathcal{H}_{st} in (4.17), without dither signal, is given by

$$\dot{\boldsymbol{x}}_{\mathcal{C}}(t) = \boldsymbol{m}k_{I}\left[r - Q_{y}(\boldsymbol{u}(t))\right] + \boldsymbol{w}(t)$$
(4.18a)

$$\boldsymbol{u}(t) = \boldsymbol{m} k_P \left[r - Q_y(\boldsymbol{u}(t)) \right] + \boldsymbol{x}_{\mathcal{C}}(t), \qquad (4.18b)$$

with r a constant reference signal, $\boldsymbol{w}(t)$ as in (4.16) with $\tilde{\boldsymbol{g}}_{ES}(t)$ as in (4.15). Considering step (i), the gradient estimates are exact, i.e., $\tilde{\nabla}_y(t) = \nabla_y(\boldsymbol{u}(t))$ and $\tilde{\nabla}_J(t) = \nabla_J(\boldsymbol{u}(t))$, and the corresponding exact modified gradient is denoted by $\boldsymbol{g}_{ES}(t)$.

4.4.2 Convergence analysis

The convergence analysis consists of two steps. First, convergence of the input u to the set $u_{\mathcal{C}}^*$ defined in (4.9) is shown, regardless of the ES. Second, for any $u \in u_{\mathcal{C}}^*$, converges to the optimum u^* defined in (4.7) is established.

Convergence of u to $u_{\mathcal{C}}^*$

The convergence analysis in this section consists of two parts. First, it is demonstrated that under Assumptions 4.2 and 4.3 the tracking error dynamics for the system in (4.18) are globally exponentially stable (GES). Second, an upper bound is derived on the distance between $\boldsymbol{u} \in \mathbb{R}^2$ and a specific input in the set $\boldsymbol{u}_{\mathcal{C}}^*$, as a function of the absolute value of the tracking error e. As such, the GES property of e implies convergence of \boldsymbol{u} to the set $\boldsymbol{u}_{\mathcal{C}}^*$.

Dropping the $(\boldsymbol{u}(t))$ arguments in the gradients for clarity of presentation, and using Assumption 4.3, the dynamics of the tracking error $e(t) = r - Q_y(\boldsymbol{u}(t))$ are derived as

$$\dot{e}(t) = -\nabla_{y} \dot{\boldsymbol{u}}(t) = -\nabla_{y} \left(\boldsymbol{m} k_{P} \dot{e}(t) + \boldsymbol{m} k_{I} e(t) + \boldsymbol{w}(t)\right),$$

$$\dot{e}(t) = \frac{-1}{1 + \nabla_{y} \boldsymbol{m} k_{P}} \nabla_{y} \left(\boldsymbol{m} k_{I} e(t) + \boldsymbol{w}(t)\right)$$
(4.19a)

$$=\frac{-1}{1+\nabla_{y}\boldsymbol{m}k_{P}}\left(\nabla_{y}\boldsymbol{m}k_{I}\boldsymbol{e}(t)-c\nabla_{y}\underbrace{\nabla_{y^{\perp}}^{\top}\frac{\nabla_{J}\nabla_{y}^{\top}}{\nabla_{y^{\perp}}\nabla_{y^{\perp}}}}_{\boldsymbol{g}_{ES}}\right)$$
(4.19b)

$$=\frac{-1}{1+\nabla_y \boldsymbol{m} k_P} \nabla_y \boldsymbol{m} k_I \boldsymbol{e}(t), \qquad (4.19c)$$

where $\boldsymbol{w} = -c\boldsymbol{g}_{ES}$ is substituted in (4.19b) and we use that $\nabla_y \nabla_{y^{\perp}}^{\top} = 0$. As such, the error dynamics in (4.19) have the equilibrium e = 0, which is not affected by the presence of \boldsymbol{w} . This is the key property of the ESTC design; the optimizing term \boldsymbol{w} is a scaled version of the orthogonal component $\nabla_{y^{\perp}}$, see (4.16), (4.15). As a result, the product $\nabla_y \nabla_{y^{\perp}}^{\top} = 0$ appears in the mapping from \boldsymbol{w} to y, such that the influence of \boldsymbol{w} on y is eliminated, i.e., the ES optimization is decoupled from the tracking objective.

To analyze stability of e = 0 for the dynamics in (4.19), consider the Lyapunov function candidate $V_e(e) = 1/2e^2$. The time derivative of $V_e(e(t))$ along trajectories of (4.19) is given by

$$\dot{V}_e(t) = e(t)\dot{e}(t) = -e^2(t)\frac{\nabla_y \boldsymbol{m}k_I}{1 + \nabla_y \boldsymbol{m}k_P},$$
(4.20)

Using the property $\nabla_y \boldsymbol{m} \geq \varepsilon > 0$ in (4.3), and $k_P, k_I > 0$, there exists a constant $\delta_1 > 0$ such that

$$\dot{V}_e(t) \le -\delta_1 e^2(t) = -2\delta_1 V_e(t),$$

which implies that e = 0 is GES. Note that this fact holds in the presence of the optimizing term \boldsymbol{w} in the closed-loop dynamics in (4.18).

Remark 4.7. The GES property of e = 0 relies on Assumption 4.3, which states that the reference r is constant, i.e., $\dot{r} = 0$. However, the stability of e = 0 can trivially be extended to input-to-state stability (ISS) with respect to \dot{r} in case ris time-varying. For $\dot{r} \neq 0$, the time derivative of V_e in (4.20) becomes

$$\begin{split} \dot{V_e}(t) &= e(t)\dot{e}(t) = e(t)\left(-e(t)\frac{\nabla_y \boldsymbol{m}k_I}{1+\nabla_y \boldsymbol{m}k_P} - \frac{1}{1+\nabla_y \boldsymbol{m}k_P}\dot{r}(t)\right) \\ &\leq -e^2(t)\frac{\nabla_y \boldsymbol{m}k_I}{1+\nabla_y \boldsymbol{m}k_P} + \frac{1}{1+\nabla_y \boldsymbol{m}k_P}\left(\frac{\zeta_1}{2}e^2(t) + \frac{1}{2\zeta_1}\dot{r}^2(t)\right), \end{split}$$

for any $\zeta_1 > 0$, using Young's inequality. Subsequently, using again the property $\nabla_y \mathbf{m} \geq \varepsilon > 0$ in (4.3), and $k_P, k_I > 0$, there must exist positive constants $\zeta_2, \zeta_3 > 0$, such that

$$\dot{V}_e(t) \le \zeta_2 V_e(t) + \zeta_3 |\dot{r}(t)|^2,$$

which implies that e = 0 is ISS with respect to $\dot{r}(t)$, see Sontag, 1995. As such, the tracking controller can deal with time-varying references. Note that, for $\dot{r}(t) = 0$, the GES property of e = 0 is recovered.

In the remainder of the analysis, on convergence of the input \boldsymbol{u} to the optimum \boldsymbol{u}^* , we use that e = 0 is GES. Therefore, the ISS property of e = 0 is stated as a remark.

Next, a bound on the distance between \boldsymbol{u} and the set $\boldsymbol{u}_{\mathcal{C}}^*$ is derived as a function of e. This bound is used to connect the GES property of e to convergence of \boldsymbol{u} to the set $\boldsymbol{u}_{\mathcal{C}}^*$. For any input $\boldsymbol{u} \in \mathbb{R}^2$, we can define

$$\boldsymbol{u}_{\rho} := \boldsymbol{u} + \boldsymbol{m}\rho \tag{4.21}$$

for some $\rho \in \mathbb{R}$ and with the constant control allocation vector \boldsymbol{m} . The value of $Q_y(\boldsymbol{u}_{\rho})$ can be evaluated as a function of ρ ; consider the derivative

$$\frac{\partial Q_y(\boldsymbol{u} + \boldsymbol{m}\rho)}{\partial \rho} = \nabla_y(\boldsymbol{u}_\rho)\boldsymbol{m} \ge \varepsilon > 0, \qquad (4.22)$$

where (4.3) in Assumption 4.2 is used. The inequality (4.22) implies that, for any \boldsymbol{u} and ρ ,

$$\frac{Q_y(\boldsymbol{u} + \boldsymbol{m}\rho) - Q_y(\boldsymbol{u})}{\rho} \ge \varepsilon.$$
(4.23)

Using (4.23), it can be verified that given an input \boldsymbol{u} for which the value of $Q_y(\boldsymbol{u})$ is bounded, there always exists a finite ρ such that $Q_y(\boldsymbol{u} + \boldsymbol{m}\rho) = Q_y(\boldsymbol{u}_{\rho})$ has any arbitrary bounded real value. As such, for a constant reference r, there always exists a $\rho_r \in \mathbb{R}$ such that

$$Q_y(\boldsymbol{u} + \boldsymbol{m}\rho_r) = Q_y(\boldsymbol{u}_{\rho_r}) = r, \qquad (4.24)$$

where $\boldsymbol{u}_{\rho_r} := \boldsymbol{u} + \boldsymbol{m}\rho_r$ consequently belongs to the tracking objective set $\boldsymbol{u}_{\mathcal{C}}^*$ defined in (4.9). Consider next the tracking error at the input \boldsymbol{u}

$$e(\boldsymbol{u}) = r - Q_y(\boldsymbol{u}) = Q_y(\boldsymbol{u} + \boldsymbol{m}\rho_r) - Q_y(\boldsymbol{u}), \qquad (4.25)$$

which is now a function of the "distance along the vector \boldsymbol{m} " given by ρ_r . The partial derivative of e with respect to this distance is

$$\frac{\partial e(\boldsymbol{u})}{\partial \rho_r} = \nabla_y(\boldsymbol{u})\boldsymbol{m} \ge \varepsilon > 0, \qquad (4.26)$$

where again Assumption 4.2 is employed. The inequality (4.26) implies that

$$|e(\boldsymbol{u})| \ge \varepsilon |\rho_r|. \tag{4.27}$$

As a result, the distance between an arbitrary input $\boldsymbol{u} \in \mathbb{R}^2$ and the input $\boldsymbol{u}_{\rho_r} := \boldsymbol{u} + \boldsymbol{m} \rho_r \in \boldsymbol{u}_{\mathcal{C}}^*$ satisfies the following inequality

$$\|\boldsymbol{u}_{\rho_r} - \boldsymbol{u}\| = \|\boldsymbol{m}\rho_r\| \le \|\boldsymbol{m}\| \cdot |\rho_r| \stackrel{(4.27)}{\le} \frac{\|\boldsymbol{m}\|}{\varepsilon} |\boldsymbol{e}(\boldsymbol{u})|.$$
(4.28)

Using that e = 0 is GES, it follows that the any input u converges to the set $u_{\mathcal{C}}^*$, given the fact that e converges to zero as shown before, in the presence of the optimizing term w in the controller.

Convergence of u in the set u_c^* to the optimum u^*

In this section the input dynamics in the set $u_{\mathcal{C}}^*$ are considered. The input u^* corresponding to the minimal cost $Q_J(u)$ is ISS with respect to the tracking error e and its time derivative \dot{e} . Combined with the GES property of e, convergence to u^* for any input $u \in \mathbb{R}^2$ is concluded.

The analysis in this section assumes that the map $Q_y(\boldsymbol{u})$ is linear, such that

$$Q_y(\boldsymbol{u}) = \nabla_y \boldsymbol{u} = \nabla_{y_1} u_1 + \nabla_{y_2} u_2,$$

where $\nabla_{y_1}, \nabla_{y_2} \in \mathbb{R}$ are constant. For $u \in u_{\mathcal{C}}^*$, the following relation holds:

$$u_2 = f_{u_2}(u_1, r) = \frac{r}{\nabla_{y_2}} - \frac{\nabla_{y_1}}{\nabla_{y_2}} u_1.$$
(4.29)

Note that, (4.29) requires $\nabla_{y_2} \neq 0$. For a linear map $Q_y(\boldsymbol{u})$, where ∇_{y_2} is constant and equal for all $\boldsymbol{u} \in \mathbb{R}^2$, there is no over-actuation when $\nabla_{y_2} = 0$. Since this chapter considers over-actuated systems, it is non-restrictive to assume that $\nabla_{y_2} \neq 0$.

Using (4.29), the cost map $Q_J(\boldsymbol{u})$ for $\boldsymbol{u} \in \boldsymbol{u}_{\mathcal{C}}^*$ can be expressed as a function of u_1 (and r). Observe its derivative with respect to u_1 , given by

$$\frac{dQ_J(u_1, f_{u_2}(u_1, r))}{du_1} = \nabla_J(u_1, f_{u_2}(u_1, r)) \left[\frac{1}{\frac{df_{u_2}(u_1, r)}{du_1}}\right]$$
(4.30a)

$$= \nabla_J(u_1, f_{u_2}(u_1, r)) \begin{bmatrix} 1\\ -\frac{\nabla_{y_1}}{\nabla_{y_2}} \end{bmatrix}.$$
(4.30b)

Consider the modified gradient \boldsymbol{g}_{ES} given by substituting $\tilde{\nabla}_J(t) = \nabla_J(\boldsymbol{u}(t))$ and $\tilde{\nabla}_y(t) = [\nabla_{y_1} \nabla_{y_2}]$ in (4.15). Denoting the first element of \boldsymbol{g}_{ES} as g_{ES_1} , then

 g_{ES_1} can be rewritten as

$$g_{ES_1} = \nabla_J(u_1, f_{u_2}(u_1, r)) \nabla_{y^{\perp}}^{\top} \frac{\nabla_{y_2}}{\nabla_{y_1}^2 + \nabla_{y_2}^2} = \nabla_J(u_1, f_{u_2}(u_1, r)) \begin{bmatrix} 1\\ -\frac{\nabla_{y_1}}{\nabla_{y_2}} \end{bmatrix} \frac{\nabla_{y_2}^2}{\nabla_{y_1}^2 + \nabla_{y_2}^2} \stackrel{(4.30)}{=} \frac{dQ_J(u_1, f_{u_2}(u_1, r))}{du_1} \delta_2,$$

$$(4.31)$$

for some constant $\delta_2 > 0$, using that over-actuation implies $\nabla_{y_2} \neq 0$. Let us now adopt the following assumption.

Assumption 4.8. The map $Q_J(\mathbf{u})$ is strictly convex for all $\mathbf{u} \in \mathbb{R}^2$.

Using the fact that $Q_y(\boldsymbol{u})$ is considered to be linear in this analysis, the set $\boldsymbol{u}_{\mathcal{C}}^*$ is a linear combination of the input \boldsymbol{u} . As a result, Assumption 4.8 implies that the cost map on the set $\boldsymbol{u}_{\mathcal{C}}^*$, expressed as $Q_J(u_1, f_{u_2}(u_1, r))$, is strictly convex in u_1 . The input corresponding to the minimum of $Q_J(u_1, f_{u_2}(u_1, r))$ is denoted by u_1^* , which consequently is the first element of the optimum \boldsymbol{u}^* defined in (4.7).

The map $Q_J(u_1, f_{u_2}(u_1, r))$ as a function of u_1 , is strictly convex if and only if its derivative with respect to u_1 is strictly increasing. Given the fact that the derivative at u_1^* is equal to zero, the following relation must hold:

$$\frac{dQ_J(u_1, f_{u_2}(u_1, r))}{du_1}\bar{u}_1 > \epsilon_2\bar{u}_1, \tag{4.32}$$

where $\bar{u}_1 := u_1 - u_1^*$ and $\epsilon_2 > 0$ is a constant.

Using that $\boldsymbol{w} = -c\boldsymbol{g}_{ES}$ and the expression for g_{ES_1} in (4.31), the closed-loop dynamics in (4.18b) in the u_1 -direction, in the coordinate \bar{u}_1 , are given by

$$\dot{\bar{u}}_1 = -c\delta_2 \frac{dQ_J(u_1, f_{u_2}(u_1, r))}{du_1} + m_1 k_P \dot{e} + m_1 k_I e, \qquad (4.33)$$

where m_1 denotes the first element of m.

Assumption 4.8 implies that the equilibrium of (4.33) is $\bar{u}_1 = 0$ if $\dot{e} = e = 0$. Consider the radially unbounded positive definite function $V_{u_1}(\bar{u}_1) = 1/2\bar{u}_1^2$, for which the time derivative along trajectories of (4.33) is given by

$$\dot{V}_{u_{1}} = \dot{\bar{u}}_{1}\bar{\bar{u}}_{1} = -c\delta_{2}\frac{dQ_{J}(u_{1}, f_{u_{2}}(u_{1}, r))}{du_{1}}\bar{\bar{u}}_{1} + m_{1}k_{P}\dot{e}\bar{\bar{u}}_{1} + m_{1}k_{I}e\bar{\bar{u}}_{1}
\stackrel{(4.32)}{\leq} -c\delta_{2}\epsilon_{2}\bar{\bar{u}}_{1}^{2} + m_{1}k_{P}\dot{e}\bar{\bar{u}}_{1} + m_{1}k_{I}e\bar{\bar{u}}_{1}
\leq -\left(c\delta_{2}\epsilon_{2} - \frac{1}{2\delta_{3}}m_{1}k_{P} - \frac{1}{2\delta_{3}}m_{1}k_{I}\right)\bar{\bar{u}}_{1}^{2} + \frac{\delta_{3}}{2}m_{1}k_{P}\dot{e}^{2} + \frac{\delta_{3}}{2}m_{1}k_{I}e^{2},$$
(4.34)

where Young's inequality is used for some constant $\delta_3 > 0$. Consider the error dynamics in (4.19c) and observe that the product $\nabla_y \boldsymbol{m}$ occurs in the numerator

and the denominator. As a result, using that \boldsymbol{m} , k_P , and k_I are bounded, there exists a constant $\delta_4 > 0$ such that

$$|\dot{e}| \le \delta_4 |e|.$$

Using this bound, a δ_3 can be selected such that there exist class \mathcal{K}_{∞} functions α_1 and α_2 for which the time derivative of V_{u_1} in (4.34) satisfies

$$V_{u_1} \leq -\alpha_1 (|\bar{u}_1|) + \alpha_2 (|e|).$$

As such, V_{u_1} is an ISS Lyapunov function for (4.33), which implies according to Sontag, 1995 that for $t \ge 0$ the input $\bar{u}_1(t)$ satisfies

$$|\bar{u}_1(t)| \le \beta(|\bar{u}_1(0)|, t) + \gamma\left(\sup_{0 \le \tau \le t} |e(\tau)|\right)$$

$$(4.35)$$

with a class \mathcal{KL} function β and a class \mathcal{K} function γ .

Using that e = 0 is GES, a cascaded argument can be used to show that the equilibrium $\bar{u}_1^* = 0$ of (4.33) is globally asymptotically stable (GAS). The intuitive interpretation is as follows. Using the ISS property we know that \bar{u}_1 remains bounded in the transient of e converging to zero, i.e., before $u \in u_c^*$. Subsequently, when $u \in u_c^*$, and hence e = 0, $\bar{u}_1(t)$ converges to zero which implies that the input u converges to the optimum u^* .

The presented analysis provides us with step (i) of the analysis of the ESTC for dynamic systems, introduced at the beginning of this section. However, note that, at this point the second part of the analysis on convergence of any $\boldsymbol{u} \in \boldsymbol{u}_{\mathcal{C}}^*$ to the optimum \boldsymbol{u}^* , assumes that the map $Q_y(\boldsymbol{u})$ is linear.

4.5 Diesel engine control example: System description

This section introduces an example of an over-actuated tracking control system in a diesel engine, which allows cost optimization. Diesel engines are subject to constraints on the emission of pollutants, while at the same time, a minimal brake specific fuel consumption (BSFC) [g/kWh] is desired. The emission of NO_x, a mixture of NO and NO₂, can be reduced by EGR. However, this increases the BSFC, among others, because a pumping-loss dp [kPa] is induced as the exhaust gas needs to be pumped back into the intake manifold. As such, a NO_x-BSFC trade-off exists.

4.5.1 Engine system description

Actuation of the considered type of engine, schematically depicted in Fig. 4.3, can be divided into the so called "fuel-path" and the "air-path".



Figure 4.3. Schematic layout of a diesel engine equipped with high pressure exhaust gas recirculation (EGR), a variable geometry turbine (VGT), and a compressor. The controlled parameters are indicated by the magenta blocks, where p_{in} and p_{ex} are used to provide $dp = p_{ex} - p_{in}$.

The fuel-path consists of the common rail fuel injection system with an injector in each of the cylinders. In the considered engine, it is not possible to measure the effect of fuel-path actuation on the combustion process. Therefore, control of the fuel-path is open-loop, i.e., without feedback.

The air-path consists of the components related to gas flow in the engine. These are a cooled, high-pressure, EGR system, and a variable geometry turbine (VGT) turbocharger with charge air cooler, see Figure 4.3. The control input consists of the air-path actuators

$$\boldsymbol{u} = [u_{egr} \ u_{vgt}]^{\top} \in U_{sat} \subset \mathbb{R}^2_{\geq 0}, \tag{4.36}$$

with $u_{egr}, u_{vgt} \in [0, 100]$ % the EGR valve opening percentage and the VGT position, and U_{sat} a set that consists of actuator constraints. Note that $u_{vgt} = 100$ % corresponds to the smallest possible pressure drop over the VGT.

Adjusting the EGR valve and VGT position affects the intake and exhaust manifold pressures p_{in} and p_{ex} [kPa], respectively, the air/fuel equivalence ratio λ [-], and the relative amount of exhaust gas in the intake manifold, i.e., the EGR fraction X_{eqr} [%].

By diluting the intake air with exhaust gas using EGR, the combustion temperature is reduced, which can result in a significant reduction of NO_x formation. A higher value of X_{egr} corresponds to a decrease of NO_x formation. A disadvantage of EGR is the aforementioned pumping-loss $dp = p_{ex} - p_{in}$, which is directly related to a loss of engine power.

For $\lambda = 1$, the combustion is stoichiometric, $\lambda > 1$ corresponds to an excess of air. Typically, $\lambda > 1.4$ is required to prevent high emission of particulate matter (PM) (soot). Reduced λ values can decrease the thermal efficiency as well.

To illustrate the controller design, tuning, and functionality, the engine model presented in Wahlström and Eriksson, 2011 is used². The tracking objective considers the emission of NO_x, see, e.g., Criens et al., 2015. However, the model does not provide a NO_x measurement. A commonly applied solution, see, e.g., Wahlström et al., 2010, is to use the EGR fraction X_{egr} as inferred parameter, motivated from the correlation between X_{egr} and NO_x formation. The pumping loss dp is commonly available as measurement. As such, the control problem is summarized as follows: A reference value of X_{egr} is tracked, while the pumping-loss dp is minimized, subject to an inequality constraint on the air/fuel equivalence ratio λ . Hence, the following outputs are considered

$$y = X_{egr},\tag{4.37a}$$

$$I = dp, \tag{4.37b}$$

as well as the constraint

$$\lambda \ge \underline{\lambda}.\tag{4.38}$$

The output constraint handling, which is introduced in Section 4.6, is based on the measured value of λ . Whenever $\lambda < \underline{\lambda}$, the optimization uses the λ gradient estimate $\tilde{\nabla}_{\lambda}$ instead of the cost gradient estimate $\tilde{\nabla}_{J}$, to increase λ . As a result, the constraint (4.38) is dealt with as a soft constraint.

4.5.2 System properties

This section discusses some of the properties of the physics-based engine model from Wahlström and Eriksson, 2011. The static behavior demonstrates the equivalence of the model and the general problem description provided in Section 4.2. The dynamic characterization of the model is used for tuning of the ESTC in Section 4.7. The engine operating point, i.e., the engine speed $n_e = 1250$ rpm and fuel injection duration $u_{dur} = 115$ ms, is constant throughout this chapter.

4.5.2.1 Static system properties

By simulating the engine model from Wahlström and Eriksson, 2011, at a grid of constant input values $\boldsymbol{u} \in U_{sat}$, the steady-state values of X_{egr} , dp, and λ in (4.37) are obtained, which are denoted by $X_{egr_{st}}$, dp_{st} , and λ_{st} , respectively. The values $X_{egr_{st}}$ and dp_{st} describe the maps $Q_y(\boldsymbol{u})$ and $Q_J(\boldsymbol{u})$ in (4.2), respectively, and are depicted in Figure 4.4. In addition, Figure 4.4 depicts the map Q_{λ} : $\boldsymbol{u} \to \lambda$ based on λ_{st} . The level sets $Q_y(\boldsymbol{u}) = \{2.5, 5, 10\}$ % describe the set $\boldsymbol{u}_{\mathcal{C}}^*$ in (4.9) for $r = \{2.5, 5, 10\}$. The lower right plot in Figure 4.4 shows that the value of Q_J within the depicted sets $\boldsymbol{u}_{\mathcal{C}}^*$ is not equal and hence (ES-based) optimization on the tracking constraint manifold is needed.

²Open-source available at www.vehicular.isy.liu.se/Software/TCDI_EGR_VGT.



Figure 4.4. Steady-state outputs $X_{egr_{st}}$, dp_{st} , and λ_{st} , obtained at a grid of constant $\boldsymbol{u} \in U_{sat}$, characterizing the maps $Q_y(\boldsymbol{u})$ and $Q_J(\boldsymbol{u})$ in (4.2), and aditionally the steady-state constraint map $Q_{\lambda} : \boldsymbol{u} \to \lambda$. The results are obtained using the model from Wahlström and Eriksson, 2011 with fuel injection duration $u_{dur} = 115$ ms, and engine speed $n_e = 1250$ rpm. Level sets are provided which describe the set $\boldsymbol{u}_{\mathcal{C}}^*$ in (4.9) for $r \in \{2.5, 5, 10\}$ %.

4.5.2.2 Dynamic system properties

In an experimental study on a heavy-duty diesel engine presented in Criens et al., 2016, it is shown that locally in \boldsymbol{u} , the dynamics of the engine are well described by a linear model. As such, for small amplitude time-varying $\boldsymbol{u}(t)$, the system is characterized by the best linear approximation (BLA). The BLA, see Pintelon and Schoukens, 2012, is a frequency-domain non-parametric model for nonlinear systems. For an linear time-invariant (LTI) system, the BLA is the system's frequency response function (FRF). For each of the outputs y, J, and

 λ , a two-input-single-output BLA is obtained, such that

$$\begin{bmatrix} Y(j\omega) \\ \mathcal{J}(j\omega) \\ \Lambda(j\omega) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{P}_{X_{egr}}(j\omega) \\ \mathcal{P}_{dp}(j\omega) \\ \mathcal{P}_{\lambda}(j\omega) \end{bmatrix}}_{\mathcal{P}(j\omega)} U(j\omega), \tag{4.39}$$

where $j\omega \in \mathbb{C}$ is the complex frequency, $\omega \in \mathbb{R}_{>0}$, $Y(j\omega), \mathcal{J}(j\omega), \Lambda(j\omega)$, and $U(j\omega)$, are the Fourier transforms of $y(t), J(t), \lambda(t)$, and u(t), respectively, and $\mathcal{P}_{X_{egr}}(j\omega), \mathcal{P}_{dp}(j\omega), \mathcal{P}_{\lambda}(j\omega) \in \mathbb{C}^{1\times 2}$ are the BLAs. Figure 4.5 depicts Bode plots of these BLAs for $u_{egr} = u_{vgt} = 45$ %.

Remark 4.9. For frequencies smaller than $\approx 1/2$ Hz, the BLAs $\mathcal{P}_{X_{egr}}(j\omega)$ and $\mathcal{P}_{dp}(j\omega)$, see Figure 4.5, show a slope of zero in the magnitude plot, and correspondingly a constant phase of $\{0, 180\}$ deg. This indicates that for frequencies smaller than $\approx 1/2$ Hz, the systems steady-state behavior is obtained in the outputs $y = X_{egr}$ and J = dp. The low-frequency phase and magnitude correspond to the gradients of the maps $Q_y(\mathbf{u})$ and $Q_J(\mathbf{u})$ in Figure 4.4, for $u_{egr} = u_{vgt} = 45$ %. When compared to $\mathcal{P}_{X_{egr}}(j\omega)$ and $\mathcal{P}_{dp}(j\omega)$, see Figure 4.5, the phase delay of $\mathcal{P}_{\lambda}(j\omega)$ is significantly larger at low frequencies. This indicates that the steady-state λ dynamics are observed at a much lower time scale (≈ 10 times) than the steady-state dynamics of $y = X_{egr}$ and J = dp.

4.6 Constrained extremum seeking tracking control design

The diesel engine example, introduced in the previous section, is subject to input and output constraints. This section presents an extension of the ESTC scheme proposed in Section 4.3, to incorporate these constraints. The input constraints are dealt with by saturation and anti-windup, based on Section 2.4.1 in Glattfelder and Schaufelberger, 2003. The output constraints are handled using the approach from Ramos et al., 2017. The extended ESTC scheme is depicted in Figure 4.6.

4.6.1 Input constraint handling

A saturation mapping $\phi: \boldsymbol{\nu} \to \hat{\boldsymbol{u}} \in U_{sat}$ is applied to the unconstrained input to the system, which is denoted by $\boldsymbol{\nu} = [\nu_{egr} \ \nu_{vgt}]^{\top}$. The saturation ensures that $\underline{u}_{egr} \leq u_{egr} \leq \overline{u}_{egr}$ and $\underline{u}_{vgt} \leq u_{vgt} \leq \overline{u}_{vgt}$, where $\underline{u}_{egr}, \underline{u}_{vgt}$ and $\overline{u}_{egr}, \overline{u}_{vgt}$ are lower and upper constraints, respectively. The saturation is described by

$$\hat{\boldsymbol{u}} = \phi(\boldsymbol{\nu}) = \begin{bmatrix} \min\left(\max\left(\nu_{egr}, \underline{u}_{egr}\right), \bar{u}_{egr}\right) \\ \min\left(\max\left(\nu_{vgt}, \underline{u}_{vgt}\right), \bar{u}_{vgt}\right) \end{bmatrix}.$$
(4.40)

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Figure 4.5. Bode plots of the BLAs $\mathcal{P}(j\omega)$ defined in (4.39), around $u_{egr} = u_{vgt} = 45 \%$, for $n_e = 1250$ rpm and $u_{dur} = 115$ ms, with phase wrapping to [-180, 180] deg. The dashed lines indicated the ESTC dither frequencies $f_{d_{egr}} = \frac{1}{2\pi}\omega_{d_1}$ Hz and $f_{d_{vgt}} = \frac{1}{2\pi}\omega_{d_2}$ Hz, which are introduced in Section 4.7.

Anti-windup measures are required to avoid integrator windup in the controller (4.10). Using the difference over the saturation map $\phi(\boldsymbol{\nu})$ in (4.40), the



Figure 4.6. The closed-loop extremum seeking tracking control (ESTC) system with input and output constraint handling.

controller with anti-windup is given by

$$\dot{\boldsymbol{x}}_{\mathcal{C}}(t) = \boldsymbol{m} k_I \boldsymbol{e}(t) - k_I \left[\boldsymbol{\nu}(t) - \hat{\boldsymbol{u}}(t) \right] + \boldsymbol{w}(t)$$
(4.41a)

$$\boldsymbol{\nu}(t) = \boldsymbol{x}_{\mathcal{C}}(t) + \boldsymbol{m}k_{P}\boldsymbol{e}(t) \tag{4.41b}$$

$$\boldsymbol{u}(t) = \phi(\boldsymbol{\nu}) + \boldsymbol{d}(t). \tag{4.41c}$$

Note that, the dither signal d(t) is added after the saturation element, to ensure that u(t) remains PE in case both inputs are saturated. As a result, in practice the saturation levels in $\phi(\nu)$ should account for the dither amplitude.

4.6.2 Output constraint handling

Ramos et al., 2017 proposes an output constrained ES approach, where the gradient used for optimization, is a weighted sum of the gradients of the cost output and the constraint output. The weighting is achieved by a smooth scheduling function, in such a way that the cost gradient is used in when the constraint is satisfied, while the constraint gradient is used when the constraint is violated.

Adopting the approach from Ramos et al., 2017, for the proposed ESTC scheme, the weighted combination is denoted by $\tilde{\nabla}_{J\lambda} \in \mathbb{R}^{1 \times 2}$, and is given by

$$\tilde{\nabla}_{J\lambda} = (1 - \alpha(\lambda))\tilde{\nabla}_J - \alpha(\lambda)\tilde{\nabla}_\lambda, \qquad (4.42)$$

where $\tilde{\nabla}_{\lambda}$ is the estimate of the λ -gradient. The smooth scheduling function $\alpha : \lambda \to (0, 1)$ is given by

$$\alpha(\lambda) = \frac{1}{1 + \exp\left(\frac{1}{\kappa}\left(\lambda - \underline{\lambda}\right)\right)},\tag{4.43}$$

where $\kappa > 0$ is a constant which affects the smoothness of $\alpha(\lambda)$. In Figure 4.6 the gradient scheduling (4.42) is indicated by the function $f_{J\lambda}(\lambda)$.

As noted in Remark 4.9, the steady-state behavior of the λ dynamics is only observed for relatively low frequencies compared to the tracking and cost dynamics. As a result, estimating the constraint gradient $\tilde{\nabla}_{\lambda}$ of the steady-state map $Q_{\lambda}(\boldsymbol{u})$, is slow in comparison to estimation of the tracking and cost gradients. To avoid a reduction of the ES convergence rate as a result of slow estimation of $\tilde{\nabla}_{\lambda}$, for the application at hand, the constraint gradient is approximated by $\tilde{\nabla}_{\lambda} = \tilde{\nabla}_{dp}$. Subsequently, the weighted combination $\tilde{\nabla}_{J\lambda}$ is obtained as

$$\tilde{\nabla}_{J\lambda} = (1 - 2\alpha(\lambda))\,\tilde{\nabla}_J \tag{4.44}$$

with $\alpha(y_{\lambda})$ in (4.43). Note that, $\alpha(\underline{\lambda}) = 1/2$, and hence that $\tilde{\nabla}_{J\lambda} = 0$ when $\lambda = \underline{\lambda}$.

Remark 4.10. The approximation $\tilde{\nabla}_{\lambda} = \tilde{\nabla}_{dp}$ is not exact for all inputs, compare the maps $Q_J(\mathbf{u})$ and $Q_{\lambda}(\mathbf{u})$ in Figure 4.4. However, since the input \mathbf{u} is constrained to the tracking manifold $\mathbf{u}_{\mathcal{C}}^*$, the proposed approximation counteracts violation of the λ -constraint as a result of cost optimization, assuming an initial condition without constraint violation.

Remark 4.11. For the proposed ESTC scheme, an alternative approach to adopt the constraint handling from Ramos et al., 2017 is to take the weighted sum of the modified gradient \tilde{g}_{ES} and $\tilde{\nabla}_{\lambda}$, instead of (4.42). Thereby, handling the constraint on λ is given priority over the optimization and over the tracking objective. Note that, the approximation (4.44) cannot be used in that case.

4.7 Diesel engine control example: Simulation results

This section presents a simulation study in which the proposed ESTC approach, with the constraint handling extensions presented in Section 4.6, is applied on the diesel engine model from Wahlström and Eriksson, 2011. The example demonstrates that the ESTC design obtains the solution of the problem in (4.4). Moreover, the possibility of operating tracking control and ES in the same time scale is illustrates by selecting different values for the optimizer gain c.

The analysis of the ESTC in Section 4.4 uses Assumption 4.3, which requires that the reference r is constant. However, as shown in Remark 4.7, the tracking dynamics on itself remain stable for a time-varying reference. Therefore, in the example we opt to apply a time-varying reference signal r(t), consisting of different constant values, connected by ramps. Such a reference signal is encountered in practice, when changing between different emission modes.

4.7.1 Parameter tuning

An overview of the applied ESTC parameters is provided in Table 4.1. The selected tuning is discussed in this section.

 Table 4.1. Overview of the extremum seeking tracking control (ESTC) parameters.

Parameter	Value	κ	0.01
$\omega_{d_1} = 2\pi f_{d_{eqr}} \text{ [rad/s]}$	$^{4/3\pi}$	$\underline{\lambda}$	1.9
$\omega_{d_2} = 2\pi f_{d_{vat}} \text{ [rad/s]}$	$^{2}/_{3}\pi$	U_{sat} [%]	$u_{egr} \in [10, 65], u_{vgt} \in [30, 80]$
$a_{d_1} = a_{d_{eqr}} [\%]$	0.5	$oldsymbol{u}(0)$ [%]	$[15.4 67.3]^ op$
$a_{d_2} = a_{d_{vqt}} [\%]$	0.125	m	$[0.1485 \ -0.9889]^{ op}$
T_u [s]	3	k_p	80
c	$\{10, 30, 90\}$	k_i	$2k_p2\pi$

The control allocation \boldsymbol{m} is selected as the normalized gain of the BLA $\mathcal{P}_{X_{egr}}(j\omega)$, see Figure 4.5, at the lowest measured frequency. As a result, the tracking output $y = X_{egr}$ is affected with minimal actuator effort, locally around $u_{egr} = u_{vgt} = 45 \%$. The value of \boldsymbol{m} is provided in Table 4.1. Using the provided value for \boldsymbol{m} , the PI parameters k_I and k_P are obtained using frequency-domain loop-shaping techniques with the BLA $\mathcal{P}(j\omega)$ in (4.39), see, e.g., Franklin et al., 2015. For $\boldsymbol{\nu} \in U_{sat}$, i.e., without actuator saturation, and using the BLA $\mathcal{P}_{X_{egr}}(j\omega)$, a bandwidth of 1.2 Hz is derived for the tracking control closed-loop system. The bandwidth is defined as the frequency at which the open-loop transfer function from the tracking error e to the output y, has an amplification equal to one. Loosely speaking, a feedback control system enables tracking and disturbance rejection of signals with a frequency content below the bandwidth.

Increasing the dither frequencies generally results in faster derivative estimation. In addition, a ratio 1/2 between frequencies is selected because it yields the smallest common period time of the two dither signals, which is used as the time window T_u in the DE in (4.12). High values for the dither frequencies ω_{d_1} and ω_{d_2} are selected, provided in Table 4.1, for which the systems steady-state behavior can still be observed in the outputs y and J. To do so, the Bode plots of $\mathcal{P}_{X_{egr}}(j\omega)$ and $\mathcal{P}_{dp}(j\omega)$ in Figure 4.5 are used. To be precise, the highest frequencies are selected, where the Bode plots of $\mathcal{P}_{X_{egr}}(j\omega)$ and $\mathcal{P}_{dp}(j\omega)$ still have a small, less than ≈ 30 deg, phase lag. Figure 4.5 shows that $\mathcal{P}_{X_{egr}}(j\omega)$ has slightly less phase lag than $\mathcal{P}_{dp}(j\omega)$ for low frequencies, therefore $\omega_{d_1} = 2\omega_{d_2}$ instead of $\omega_{d_2} = 2\omega_{d_1}$.

For the presented engine operating point the value of λ is large, see the map $Q_{\lambda}(\boldsymbol{u})$ in Figure 4.4. Therefore, a relatively high constraint level $\underline{\lambda} = 1.9$ is applied for demonstration of the constraint handling. The value $\kappa = 0.01$ results in a relatively smooth scheduling, which is required to prevent large overshoot of the λ -constraint, as a result of the slow λ dynamics.



Figure 4.7. Input u(t) obtained in a simulation of the proposed extremum seeking tracking control (ESTC) system with the model from Wahlström and Eriksson, 2011, for $n_e = 1250$ rpm, and $u_{dur} = 115$ ms, for different optimizer gain values $c \in \{10, 30, 90\}$. The underlying contour plot is the map $Q_J(u)$, which is also depicted in Figure 4.4.

The saturation limits of the model require $u_{egr} \in [0, 100]$ and $u_{vgt} \in [20, 100]$. For the simulation, lower values are used, see Table 4.1, to demonstrate the input constraint handling.

4.7.2 Simulation results

The simulation results are depicted in Figures 4.7 and 4.8. The reference signal r(t) consists of constant parts, connected by ramps at t = 3 s and t = 53 s, with a duration of respectively 2.5 s and 5 s. At t = 3 s, the ES becomes active, i.e., for t < 3 s, the optimizer gain c = 0. The same simulation is performed with three different values for the optimizer gain c. The same initial input u(0) is used, provided in Table 4.1, for which the tracking objective (4.4b) is satisfied, while the corresponding cost is larger than the minimum of (4.4).

From the top plot in Figure 4.8, the steady-state tracking error is concluded to be approximately zero for the constant parts of r(t), while the tracking error remains small during the ramps. Correspondingly, Figure 4.7 shows that the input u(t) converges to the set $u_{\mathcal{C}}^*$, which is equal to the depicted level sets of y. The cost J is minimized for $u(t) \in u_{\mathcal{C}}^*$. As such, the ESTC objective in (4.5) is satisfied in a practical sense, i.e., input u(t) converges to a neighborhood of the optimum u^* in (4.7). The soft constraint $\lambda \geq \underline{\lambda}$, as well as the saturation



Figure 4.8. Signals r(t), $y(t) = X_{egr}(t)$, u(t), J(t) = dp(t), and $\lambda(t)$, corresponding to the result in Figure 4.7.

 $\phi: \boldsymbol{\nu} \to \boldsymbol{u} \in U_{sat}$, are active in the example.

For all different values for $c \in \{10, 30, 90\}$, the tracking performance is similar, see the top plot in Figure 4.8, as a result of the adaptive decoupling of the ES adaptation from the tracking objective. Clearly, increasing the optimizer gain c yields an increased convergence rate. In Figure 4.7, the interaction between tracking and ES can be observed with $c \in \{30, 90\}$, before $\mathbf{u}(t)$ has converged to \mathbf{u}^* . For c = 10, a clear time scale separation is observed, as initially $\mathbf{u}(t) \to \mathbf{u}^*_{\mathcal{C}}$, after which $\mathbf{u}(t) \to \mathbf{u}^*$ while $\mathbf{u}(t)$ remains approximately in the set $\mathbf{u}^*_{\mathcal{C}}$. For $c \in \{30, 90\}$, the time scale separation is less strong; tracking and ES affect the input $\mathbf{u}(t)$ simultaneously in the same time scale. As such, the presented example clearly demonstrates the value of the adaptive decoupling mechanism, based on gradient projection.

4.8 Conclusions

A extremum seeking tracking control (ESTC) design is proposed, which can be used for steady-state cost optimization in over-actuated tracking control systems. Being based on extremum seeking (ES), the optimization is robust with respect to system uncertainty. The ESTC enables the tracking and ES to operate simultaneously in the same time scale, which increases the achievable ES convergence rate. To this extent, the key elements in the ESTC design are: (1) An adaptive decoupling mechanism based on projection of estimated gradients, which decouples the ES adaptation from the tracking objective, (2) application of input-based ES, and (3) shared integrator states for ES and tracking control. A case study is presented which considers air-path control of a diesel engine. The corresponding objective is simultaneous tracking of a reference exhaust gas recirculation (EGR) fraction, and minimization of the pumping-loss. To apply the proposed ESTC design, existing input and output constraint handling approaches are adopted and incorporated in an extended version of the ESTC design. A simulation study of the proposed ESTC design applied to a physicsbased diesel engine model demonstrates constrained cost optimization with zero steady-state tracking error, in the same time scale as the tracking dynamics.

Chapter 5

Conclusions and recommendations

The societal demand for clean and efficient road transport, enforced by increasingly stringent emission legislation, motivates the development of fuel efficient diesel engine control strategies. From a practical perspective, there is a need for control systems that provide robust optimal performance with respect to real-world disturbances, with low implementation, calibration, and modeling effort. To this extent, the presented research in this thesis explores possibilities to apply data-based online fuel efficiency optimization using extremum seeking (ES) in diesel engine control systems, and addresses corresponding challenges. An overview of the main conclusions and recommendations for future research is provided in this chapter.

5.1 Conclusions

The conclusions are presented in three sections as follows:

- (i) ES application guidelines for fast and accurate convergence.
- (ii) ES cost optimization in over-actuated tracking control systems.
- (iii) Application of ES for fuel-efficient control of diesel engines.

If applicable, corresponding challenges **C1-C6** introduced in Chapter 1, listed below, are mentioned in between brackets.

C1 Fuel efficiency equivalent cost output design

C2 Input selection for fuel efficiency optimization

C3 Extremum seeking approach evaluation

C4 Extremum seeking parameter tuning

C5 Multiple-output constraint handling

C6 Combined extremum seeking and tracking control

For a more detailed introduction of the challenges and motivation, the reader is referred to Sections 1.3.3 and 1.4.3.

(i) Extremum seeking application guidelines for fast and accurate convergence

Chapter 2 presents an overview of the main classes of continuous derivative-based ES for multiple-input systems. The corresponding performance and system requirements are clarified (C3), and tuning guidelines are provided that lead to fast and accurate convergence to the unknown optimum (C4).

In Chapter 2, a classical time scale separation analysis demonstrates that the achievable convergence rate of dither-based ES depends on the dither frequencies. In this perspective, an optimal ratio between the individual dither frequencies is proposed, based on a generalized dither-based derivative estimation framework. This framework unifies and extends existing dither-based derivative estimators by estimating derivatives up to an arbitrary order for systems with an arbitrary number of inputs. For general unknown systems, this optimal ratio results in the fastest possible derivative estimation time scale, thereby maximizing the ES convergence rate. The dither frequency tuning guideline (C4) is completed by a practical guideline to select the highest dither frequency, based on a frequency-domain analysis of dither-based ES and an approximation of the lowest frequency characterizing the system dynamics.

The frequency-domain analysis demonstrates that dither-based ES is equivalent to frequency-domain system identification. Using this equivalence, results from system identification theory are adopted in the ES context, to derive the existence of a lower bound on the dither amplitude. This bound exists regardless of disturbing noise being present, which is an important observation concerning ES parameter tuning (C4), in particular, considering the existing ES convergence analyses, which only derive a practical upper bound on the dither amplitude.

In addition to classical dither-based ES, Chapter 2 discusses phase-shifted, and fast dither-based ES. The aforementioned frequency-domain description applies to these advanced approaches as well, and as such provides a unifying description for different classes of dither-based ES. Using this result, the system requirements and practical implications of advanced dither-based ES are clarified with respect to classical dither-based ES (C3).

Considering the class of input-based ES approaches, the aforementioned system identification equivalence is used to provide a fundamental analysis of the advantage of input-based ES with respect to dither-based ES (C3).

(ii) Extremum seeking cost optimization in over-actuated tracking control systems

The diesel engine control problem, introduced in Section 1.2.2, is an example of an over-actuated tracking control problem, where the over-actuation can be allocated such that the cost, in this case fuel efficiency, is optimal. However, as summarized in challenge **C6**, the tracking objective is generally not satisfied for the input corresponding to the minimal cost. This conflict causes interaction between the ES optimization and the tracking objective.

The first solution is presented in Chapter 3, where ES is applied to a lowlevel multivariable engine control system. The objective interaction is dealt with by including constraints on the low-level tracking errors in the high-level ES optimization problem. As a result, the optimal ES inputs are found, for which tracking performance of specific engine-out NO_x and the net indicated mean effective pressure (IMEP) is preserved. To deal with the created constraints, an extension for handling of multiple output constraints is proposed to an existing constrained ES approach in which a single output constraint is considered (C5).

The second solution is presented in Chapter 4, where an extremum seeking tracking control (ESTC) design is proposed, which directly considers the objective interaction. The key element of the proposed control design is an adaptive decoupling mechanism, based on a projection of estimated gradients of the system's steady-state map. By explicitly accounting for the interaction and making use of input-based derivative estimation, tracking and ES can operate in the same time scale. Thereby, the convergence rate of ES can be increased without affecting the tracking performance. A simulation study on air-path control in a physics-based diesel engine model demonstrates the functionality of the proposed ESTC design. The optimal cost is found for which the tracking objective remains satisfied. By increasing the ES optimizer gain, the convergence rate occurs in the same time scale as the tracking dynamics.

(iii) Application of extremum seeking for fuel-efficient control of diesel engines

In Chapter 3, an experimental study is presented considering a state-of-the-art heavy-duty EURO-VI diesel engine with in-cylinder pressure sensors. A twoinput quasi-convex constrained optimization problem is proposed (C2), which connects to the diesel engine control goal of delivering power, using a minimal amount of fuel, while tracking a specific engine-out NO_x level [g/kWh] and satisfying safety constraints. The ES inputs are two reference signals, related to pumping losses and combustion phasing, applied to a low-level feedback control system. As a result of interaction in the multivariable low-level control system, all four air-path and fuel-path actuators respond to the optimization of the two ES inputs. Besides tracking error constraints mentioned in the previous section, a safety constraint on the peak in-cylinder pressure is included in the constrained ES (C5). In the presented case study, in-cylinder pressure measurements are used for closed-loop fuel-path control, measurement of the constrained peak in-cylinder pressure, and accurate brake specific fuel consumption (BSFC) estimation, which is used as basis for the ES cost (C1).

The conducted experiments in Chapter 3 demonstrate convergence to constrained optima, for different engine operating points, types of fuel, peak pressure limits, and NO_x reference values. The cost output proves to be an accurate estimate of the actual BSFC, and a BSFC reduction up to $\approx 1 \text{ g/kWh}$ is obtained under nominal operating conditions. Given that the baseline control approach in the comparison is optimized for nominal conditions, the fuel saving of the ES-based controller is potentially larger for non-nominal, real-world, conditions.

In Chapter 4, a simulation study using a physics-based diesel engine model is presented considering air-path control using the exhaust gas recirculation (EGR) valve and the variable geometry turbine (VGT) as control inputs (**C2**). The corresponding objective is simultaneous tracking of a reference EGR fraction, and minimization of the pumping-loss (**C1**). The proposed ESTC design is applied and extended with existing input and output constraint handling approaches, to deal with actuator saturation and a constraint on the air-fuel equivalence ratio λ . As such tracking of the EGR fraction is possible using a low-complexity controller, based on frequency-domain loop-shaping and easyto-obtain non-parametric system models. Compared to economic model predictive control (MPC), the online cost optimization in the ESTC design is limited to steady-state optimization and output constraints are dealt with as soft constraint. However, contrary to MPC, the ESTC design potentially offers robustness to real-world disturbances, without requiring parametric models, while being implementable on a standard engine control unit (ECU).

As introduced in Section 1.3.3, the main motivations for ES-based diesel engine control are real-world robustness of the optimization, while the implementation, calibration, and modeling efforts are low. The application examples in Chapters 3 and 4 demonstrate these advantages: No parametric models are being used, ES improves upon the default calibration, and different optima result for disturbed conditions. Moreover in Chapter 3, the ES-based approach deals with a different type of fuel, various constraint levels, and adapts to a time-varying NO_x reference and engine operating point. Contrary, an inherent drawback is the requirement for a BSFC equivalent cost function. For dynamic systems, the proposed ES-based approaches do not offer strict output constraint handling, being based on feedback of the measured constraint output. Output constraints are potentially handled more effectively in MPC. In practice the optimal input corresponding to the minimal cost shows a strong correlation with the engine operating point, despite the applied normalization of the cost output. Moreover, although the focus throughout this thesis is on fast ES convergence, the convergence rate remains limited. Combining these two aspects, the optimal input is not obtained under fast operating point transients. As such, for fast transients the pursued robust performance is not obtained, as is the case for stationary operating points. Therefore, industrial application of the presented ES-based control approaches requires further research, for which suggestions are provided in the following section.

5.2 Recommendations

In this section, recommendations for future research are given. First, concerning the application of ES for fuel efficiency optimization, the most important suggestions are summarized as follows.

Adaptive feedforward control: As mentioned in the previous section, the pursued robust performance is not yet obtained for fast engine operating point transients. To improve upon the current status, the proposed ES-based control designs can be generalized to an adaptive feedforward control approach, as a function of the operating point. Given that the engine speed is measured, and the torque can be estimated fairly accurate, the operating point is essentially a measured disturbance. Currently, this measurement is not used for ES; ES aims to provide the optimal inputs to the system directly. Instead, the parameterization of an adaptive feedforward controller can be optimized. This feedforward controller then provides the optimal inputs to the system, as a function of the current operating point. In Marinkov et al., 2014; Sharafi et al., 2016 such adaptive approaches are proposed in a general ES context. By adopting these type of ES approaches, the adaptation can be focused on providing robustness to real-world disturbances, which are typically slowly varying.

Dedicated extremum seeking for fuel-path control: The functionality of the proposed ESTC design in Chapter 4 is demonstrated in a simulation study using air-path actuation. However, from a practical point of view, the obtained result is rather intuitive; fully opening the EGR valve minimizes the pumping loss. A similar result is obtained in the experiments in Chapter 3, where the EGR valve is opened up to the constraint value. As such, focusing ES on fuel-path control may be more interesting in practice, since therein, the efficiency optimum is typically not obtained at an actuation constraint. In addition, the fuel-path offers a greater potential for optimization, since a higher number of actuation degrees-of-freedom (DOFs) is available, e.g., multi-pulse injection, injection rate shaping, and rail pressure variation.

Cycle-to-cycle fuel-path actuation, essentially results in discrete actuation. When the controlled parameters are also obtained on a cycle-to-cycle basis, using in-cylinder pressure measurements, the complete system is discrete from a control point-of-view. In this respect, explicitly considering the discrete behavior by using sampled-data ES, see, e.g., Khong et al., 2013, is expected to result in faster convergence.

Second, the following recommendations are made, concerning the presented fundamental results on ES.

On the extremum seeking tracking control design: The provided convergence analysis in Chapter 4 illustrates the intuition behind the design and the assumptions made. However, it is only a stepping stone towards a complete stability proof. Possibly, the input-based derivative estimation presented in Guay and Dochain, 2015 or Haring, 2016, Chapter 2 can be adopted, for which conclusive stability results are presented for application in normal ES, i.e., without tracking. Application of such alternative input-based derivative estimation does not restrict the intended functionality of the proposed ESTC design.

To verify the pursued robust fuel efficiency optimization, application in a real engine is recommended. The proposed ESTC offers similar advantages as economic MPC, however without typical drawbacks of MPC, i.e., challenging real-time implementation and the requirement for accurate parametric models. As such, ESTC can be interesting for other applications, where simultaneous tracking and cost optimization is desired.

System identification principles for extremum-seeking controller design: Chapter 2 presents an analysis of dither-based ES in the frequency domain. The clear equivalence to system identification is used to obtain the existence of a lower bound on the dither amplitude, and to motivate input-based ES; from an identification perspective, neglecting the influence of the optimizer output to the system input in dither-based ES, is not a valid assumption.

Another established system identification result states that identification of a system which is part of a closed loop, by correlating its input and output, results in a biased estimate when the system is subject to output noise. Therefore, operating input-based ES without external excitation (dither-free) to aid the identification, may result in inaccurate derivative estimates near the optimum where the optimizer action is small compared to the noise level.

In summary, knowledge that is available in the system identification field should considered during the design of ES controllers.

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List of publications

Peer-reviewed journal articles

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- Van der Weijst, R., van Loon, B., Heertjes, M., and Heemels, M. (2018). "Scheduled Controller Design for Systems With Varying Sensor Configurations: A Frequency-Domain Approach'.' *IEEE Transactions on Control Systems Technology*, volume 26, number 2, pages 523–534.

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About the author

Robert van der Weijst was born on December 31st 1989 in Eersel, the Netherlands. After finishing secondary education in 2008 at the Rythovius College in Eersel, the Netherlands, he studied Mechanical Engineering at Eindhoven University of Technology in Eindhoven, the Netherlands. He received the Bachelor of Science degree in 2012 and the Master of Science degree in 2014 (with great appreciation). During his master, he performed an internship at Florida Institute of Technology, Florida, United States, where he worked on linear parametervarying control of a flexible wing aircraft. His master thesis, entitled "A Scheduled Control Design for Motion Stages with Varying Sensor Configurations", was conducted in cooperation with ASML the Netherlands B.V., under supervision of Bas van Loon, Marcel Heertjes, and Maurice Heemels.

In April 2015, Robert started his PhD research in the Control Systems Technology group at Eindhoven University of Technology, under supervision of Thijs van Keulen, Nathan van de Wouw, and Frank Willems. The results of his PhD research are presented in this dissertation, and are focused on increasing the robustness of optimal fuel efficiency in diesel engines, by online optimization using extremum seeking.