# Convergent systems: analysis and synthesis

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**Summary.** In this paper we extend the notion of convergent systems defined by B.P. Demidovich and introduce the notions of uniformly, exponentially convergent and input-to-state convergent systems. Basic (interconnection) properties of such systems are established. Sufficient conditions for input-to-state convergence are presented. For a class of nonlinear systems we design (output) feedback controllers that make the closed-loop system input-to-state convergent. The conditions for such controller design are formulated in terms of LMIs.

**Key words:** Convergent systems, stability properties, asymptotic properties, interconnected systems, observers, output-feedback

## 1 Introduction

In many control problems it is required that controllers are designed in such a way that all solutions of the corresponding closed-loop system "forget" their initial conditions. Actually, this is one of the main tasks of a feedback to eliminate the dependency of solutions on initial conditions. In this case, all solutions converge to some steady-state solution which is determined only by the input of the closed-loop system. This input can be, for example, a command signal or a signal generated by a feedforward part of the controller or, as in the observer design problem, it can be the measured signal from the observed system. This "convergence" property of a system plays an important role in many nonlinear control problems including tracking, synchronization, observer design and the output regulation problem, see e.g. [13; 15; 16; 18] and references therein.

For asymptotically stable linear time invariant systems with inputs, this is a natural property. Indeed, due to linearity of the system every solution is globally asymptotically stable and, therefore, for a given input, all solutions of such a system "forget" their initial conditions and converge to each other. After transients, the dynamics of the system are determined only by the input.

For nonlinear systems, in general, global asymptotic stability of a system with zero input does not guarantee that all solutions of this system with a *non-zero* input "forget" their initial conditions and converge to each other. There are many examples of nonlinear globally asymptotically stable systems, which, being excited by a periodic input, have coexisting periodic solutions. Such periodic solutions do not converge to each other. This fact indicates that for nonlinear systems the convergent dynamics property requires additional conditions.

The property that all solutions of a system "forget" their initial conditions and converge to some limit- or steady-state solution has been addressed in a number of papers. In [17] this property was investigated for systems with right-hand sides which are periodic in time. In that work systems with a unique periodic globally asymptotically stable solution were called *convergent.* Later, the definition of convergent systems given by V.A. Pliss in [17] was extended by B.P. Demidovich in [3] (see also [11]) to the case of systems which are not necessarily periodic in time. According to [3], a system is called convergent if there exists a unique globally asymptotically stable solution which is bounded on the whole time axis. Obviously, if such solution does exist, all other solutions, regardless of their initial conditions, converge to this solution, which can be considered as a limit- or steady-state solution. In [2; 3] (see also [11]) B.P. Demidovich presented a simple sufficient condition for such a convergence property. With the development of absolute stability theory, V.A. Yakubovich showed in [20] that for a linear system with one scalar nonlinearity satisfying some incremental sector condition, the circle criterion guarantees the convergence property for this system with any nonlinearity satisfying this incremental sector condition. In parallel with this Russian line of research, the property of solutions converging to each other was addressed in the works of T. Yoshizawa [21; 22] and J.P. LaSalle [9]. In [9] this property of a system was called *extreme stability.* In [21] T. Yoshizawa provided Lyapunov and converse Lyapunov theorems for extreme stability.

Several decades after these publications, the interest in stability properties of solutions with respect to one another revived. In the mid-nineties, W. Lohmiller and J.-J.E. Slotine (see [10] and references therein) independently reobtained and extended the result of B.P. Demidovich. A different approach was pursued in the works by V. Fromion *et al,* [4–6]. In this approach a dynamical system is considered as an operator which maps some functional space of inputs to a functional space of outputs. If such operator is Lipschitz continuous (has a finite incremental gain), then, under certain observability and reachability conditions, all solutions of a state-space realization of this system converge to each other. In [1], D. Angeli developed a Lyapunov approach for studying both the global uniform asymptotic stability of all solutions of a system (in [1], this property is called incremental stability) and the so-called incremental input-to-state stability property, which is compatible with the input-to-state stability approach (see e.g. [19]). The drawback of the incremental stability and incremental input-to-state stability notions introduced in [1] is that they are not coordinate independent.

In this paper we extend the notion of convergent systems defined by B.P. Demidovich. More specifically, we introduce the notions of (uniformly, exponentially) convergent systems and input-to-state convergent systems in Section 2. These notions are coordinate independent, which distinguishes them from the other approaches mentioned above. In Section 3 we present results on basic properties of (interconnected) convergent systems. Sufficient conditions for exponential and input-to-state convergence properties are presented in Section 4. In Section 5 we present (output) feedback controllers that make the corresponding closed-loop system input-to-state convergent. Section 6 contains the conclusions.

## 2 Convergent systems

In this section we give definitions of convergent systems. These definitions extend the definition given in [3] (see also [11]). Consider the system

$$\dot{z} = F(z, t), \tag{1}$$

where $z \in \mathbb{R}^d$, $t \in \mathbb{R}$ and $F(z,t)$ is locally Lipschitz in $z$ and piecewise continuous in $t$.

**Definition 1.** *System (1) is said to be*

- convergent *if there exists a solution* $\bar{z}(t)$ *satisfying the following conditions*
  (i) $\bar{z}(t)$ *is defined and bounded for all* $t \in \mathbb{R}$,
  (ii) $\bar{z}(t)$ *is globally asymptotically stable.*
- uniformly convergent *if it is convergent and* $\bar{z}(t)$ *is globally uniformly asymptotically stable.*
- exponentially convergent *if it is convergent and* $\bar{z}(t)$ *is globally exponentially stable.*

The solution $\bar{z}(t)$ is called a *limit solution*. As follows from the definition of convergence, any solution of a convergent system "forgets" its initial condition and converges to some limit solution which is independent of the initial condition. In general, the limit solution $\bar{z}(t)$ may be non-unique. But for any two limit solutions $\bar{z}_1(t)$ and $\bar{z}_2(t)$ it holds that $|\bar{z}_1(t) - \bar{z}_2(t)| \to 0$ as $t \to +\infty$. At the same time, for *uniformly* convergent systems the limit solution is unique, as formulated below.

*Property 1.* If system (1) is uniformly convergent, then the limit solution $\bar{z}(t)$ is the only solution defined and bounded for all $t \in \mathbb{R}$.

*Proof.* Suppose there exists another solution $\tilde{z}(t)$ defined and bounded for all $t \in \mathbb{R}$. Let $R > 0$ be such that $|\tilde{z}(t) - \bar{z}(t)| < R$ for all $t \in \mathbb{R}$. Such $R$ exists since both $\tilde{z}(t)$ and $\bar{z}(t)$ are bounded for all $t \in \mathbb{R}$. Suppose at some instant $t_* \in \mathbb{R}$ the solutions $\bar{z}(t)$ and $\tilde{z}(t)$ satisfy $|\tilde{z}(t_*) - \bar{z}(t_*)| \geq \varepsilon > 0$ for some $\varepsilon > 0$. Since $\bar{z}(t)$ is globally uniformly asymptotically stable, there exists a number $T(\varepsilon, R) > 0$ such that if $|\tilde{z}(t_0) - \bar{z}(t_0)| < R$ for some $t_0 \in \mathbb{R}$ then

$$|\tilde{z}(t) - \bar{z}(t)| < \varepsilon, \quad \forall\ t \geq t_0 + T(\varepsilon, R). \tag{2}$$

Set $t_0 := t_* - T(\varepsilon, R)$. Then for $t = t_*$ inequality (2) implies $|\tilde{z}(t_*) - \bar{z}(t_*)| < \varepsilon$. Thus, we obtain a contradiction. Since $t_*$ has been chosen arbitrarily, this implies $\tilde{z}(t) \equiv \bar{z}(t)$.$\square$

The convergence property is an extension of stability properties of asymptotically stable linear time-invariant (LTI) systems. Recall that for a piecewise continuous vector-function $f(t)$, which is defined and bounded on $t \in \mathbb{R}$, the system $\dot{z} = Az + f(t)$ with a Hurwitz matrix $A$ has a unique solution $\bar{z}(t)$ which is defined and bounded on $t \in (-\infty, +\infty)$. It is given by the formula $\bar{z}(t) := \int_{-\infty}^{t} \exp(A(t-s)) f(s) ds$. This solution is globally exponentially stable with the rate of convergence depending only on the matrix $A$. Thus, an asymptotically stable LTI system excited by a bounded piecewise-continuous function $f(t)$ is globally exponentially convergent.

In the scope of control problems, time dependency of the right-hand side of system (1) is usually due to some input. This input may represent, for example, a disturbance or a feedforward control signal. Below we will consider convergence properties for systems with inputs. So, instead of systems of the form (1), we consider systems

$$\dot{z} = F(z, w) \tag{3}$$

with state $z \in \mathbb{R}^d$ and input $w \in \mathbb{R}^m$. The function $F(z, w)$ is locally Lipschitz in $z$ and continuous in $w$. In the sequel we will consider the class $\overline{\mathbb{PC}}_m$ of piecewise continuous inputs $w(t) : \mathbb{R} \to \mathbb{R}^m$ which are bounded for all $t \in \mathbb{R}$. Below we define the convergence property for systems with inputs.

**Definition 2.** *System (3) is said to be* (uniformly, exponentially) convergent *if for every input $w \in \overline{\mathbb{PC}}_m$ the system $\dot{z} = F(z, w(t))$ is (uniformly, exponentially) convergent. In order to emphasize the dependency on the input $w(t)$, the limit solution is denoted by $\bar{z}_w(t)$.*

The next property extends the uniform convergence property to the input-to-state stability (ISS) framework.

**Definition 3.** *System (3) is said to be input-to-state convergent if it is uniformly convergent and for every input $w \in \overline{\mathbb{PC}}_m$ system (3) is ISS with respect to the limit solution $\bar{z}_w(t)$, i.e. there exist a $\mathcal{KL}$-function $\beta(r, s)$ and a $\mathcal{K}_\infty$-function $\gamma(r)$ such that any solution $z(t)$ of system (3) corresponding to some input $\hat{w}(t) := w(t) + \Delta w(t)$ satisfies*

$$|z(t) - \bar{z}_w(t)| \leq \beta(|z(t_0) - \bar{z}_w(t_0)|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} |\Delta w(\tau)|). \tag{4}$$

*In general, the functions $\beta(r,s)$ and $\gamma(r)$ may depend on the particular input $w(t)$. If $\beta(r,s)$ and $\gamma(r)$ are independent of the input $w(t)$, then such system is called uniformly input-to-state convergent.*

Similar to the conventional ISS property, the property of input-to-state convergence is especially useful for studying convergence properties of interconnected systems as will be illuminated in the next section.

## 3 Basic properties of convergent systems

As follows from the previous section, the (uniform) convergence property and the input-to-state convergence property are extensions of stability properties of asymptotically stable LTI systems. In this section we present certain properties of convergent systems that are inherited from asymptotically stable LTI systems. Since all ingredients of the (uniform) convergence and the input-to-state convergence properties are invariant under smooth coordinate transformations (see Definitions 1, 3), we can formulate the following property.

*Property 2.* The uniform convergence property and input-to-state convergence are preserved under smooth coordinate transformations.

The next statement summarizes some properties of uniformly convergent systems excited by periodic or constant inputs.

*Property 3 ([3]).* Suppose system (3) with a given input $w(t)$ is uniformly convergent. If the input $w(t)$ is constant, the corresponding limit solution $\bar{z}_w(t)$ is also constant; if the input $w(t)$ is periodic with period $T$, then the corresponding limit solution $\bar{z}_w(t)$ is also periodic with the same period $T$.

*Proof.* Suppose the input $w(t)$ is periodic with period $T > 0$. Denote $\tilde{z}_w(t) := \bar{z}_w(t+T)$. Notice that $\tilde{z}_w(t)$ is a solution of system (3). Namely, by the definition of $\tilde{z}_w(t)$, it is a solution of the system

$$\dot{z} = F(z, w(t+T)) \equiv F(z, w(t)).$$

Moreover, $\tilde{z}(t)$ is bounded for all $t \in \mathbb{R}$ due to boundedness of the limit solution $\bar{z}(t)$. Therefore, by Property 1 it holds that $\tilde{z}(t) \equiv \bar{z}(t)$, i.e. the limit solution $\bar{z}(t)$ is $T$-periodic. A constant input $w(t) \equiv w_*$ is periodic for any period $T > 0$. Hence, the corresponding limit solution $\bar{z}_w(t)$ is also periodic with any period $T > 0$. This implies that $\bar{z}_w(t)$ is constant.□

If two inputs converge to each other, so do the corresponding limit solutions, as follows from the next property.

*Property 4.* Suppose system (3) is uniformly convergent and $F(z,w)$ is $C^1$. Then for any two inputs $w_1(t)$ and $w_2(t)$ satisfying $w_1(t) - w_2(t) \to 0$ as $t \to +\infty$, the corresponding limit solutions $\bar{z}_{w_1}(t)$ and $\bar{z}_{w_2}(t)$ satisfy $\bar{z}_{w_1}(t) - \bar{z}_{w_2}(t) \to 0$ as $t \to +\infty$.

*Proof.* See Appendix.

The next two properties relate to parallel and series connections of uniformly convergent systems.

*Property 5 (Parallel connection).* Consider the system

$$\begin{cases} \dot{z} = F(z, w), & z \in \mathbb{R}^d \\ \dot{y} = G(y, w), & y \in \mathbb{R}^q. \end{cases} \tag{5}$$

Suppose the $z$- and $y$-subsystems are uniformly convergent (input-to-state convergent). Then system (5) is uniformly convergent (input-to-state convergent).

*Proof.* The proof directly follows from the definitions of uniformly convergent and input-to-state convergent systems.□

*Property 6 (Series connection).* Consider the system

$$\begin{cases} \dot{z} = F(z, y, w), & z \in \mathbb{R}^d \\ \dot{y} = G(y, w), & y \in \mathbb{R}^q. \end{cases} \tag{6}$$

Suppose the $z$-subsystem with $(y, w)$ as input is input-to-state convergent and the $y$-subsystem with $w$ as input is input-to-state convergent. Then system (6) is input-to-state convergent.

*Proof.* See Appendix.

The next property deals with bidirectionally interconnected input-to-state convergent systems.

*Property 7.* Consider the system

$$\begin{cases} \dot{z} = F(z, y, w), & z \in \mathbb{R}^d \\ \dot{y} = G(z, y, w), & y \in \mathbb{R}^q. \end{cases} \tag{7}$$

Suppose the $z$-subsystem with $(y, w)$ as input is input-to-state convergent. Assume that there exists a class $\mathcal{KL}$ function $\beta_y(r, s)$ such that for any input $(z, w) \in \overline{\mathbb{PC}}_{d+m}$ any solution of the $y$-subsystem satisfies

$$|y(t)| \leq \beta_y(|y(t_0)|, t - t_0).$$

Then the interconnected system (7) is input-to-state convergent.

*Proof.* Denote $\bar{z}_w(t)$ to be the limit solution of the $z$-subsystem corresponding to $y = 0$ and to some $w \in \overline{\mathbb{PC}}_m$. Then $(\bar{z}_w(t), 0)$ is a solution of the interconnected system (7) which is defined and bounded for all $t \in \mathbb{R}$. Performing the change of coordinates $\tilde{z} = z - \bar{z}_w(t)$ and applying the small gain theorem for ISS systems from [7] we establish the property.□

*Remark 1.* Property 7 can be used for establishing the separation principle for input-to-state convergent systems as it will be done in Section 5. In that context system (7) represents a system in closed loop with a state-feedback controller and an observer generating state estimates for this controller. The $y$-subsystem corresponds to the observer error dynamics.$\lhd$

## 4 Sufficient conditions for convergence

In the previous sections we presented the definitions and basic properties of convergent systems. The next question to be addressed is: how to check whether a system exhibits these convergence properties? In this section we provide sufficient conditions for convergence for smooth systems.

A simple sufficient condition for the exponential convergence property for smooth systems was proposed in [2] (see also [11]). Here we give a different formulation of the result from [2] adapted for systems with inputs and extended to include the input-to-state convergence property.

**Theorem 1.** *Consider system (3) with the function $F(z,w)$ being $C^1$ with respect to $z$ and continuous with respect to $w$. Suppose there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that*

$$P\frac{\partial F}{\partial z}(z,w) + \frac{\partial F}{\partial z}^T(z,w)P \leq -Q, \quad \forall z \in \mathbb{R}^d, \quad w \in \mathbb{R}^m. \tag{8}$$

*Then system (3) is exponentially convergent and input-to-state convergent.*

*Proof.* A complete proof of this theorem is given in Appendix. It is based on the following technical lemma, which we will use in Section 5.

**Lemma 1 ([2; 11]).** *Condition (8) implies*

$$(z_1 - z_2)^T P(F(z_1,w) - F(z_2,w)) \leq -a(z_1 - z_2)^T P(z_1 - z_2). \tag{9}$$

*for all $z_1, z_2 \in \mathbb{R}^d$, $w \in \mathbb{R}^m$, and for some $a > 0$.*

We will refer to condition (8) as the Demidovich condition, after B.P. Demidovich, who applied this condition for studying convergence properties of dynamical systems [2; 3; 11]. In the sequel, we say that a system satisfies the Demidovich condition if the right-hand side of this system satisfies condition (8) for some matrices $P = P^T > 0$ and $Q = Q^T > 0$.

*Example 1.* Let us illustrate Theorem 1 with a simple example. Consider the system

$$\begin{aligned} \dot{z}_1 &= -z_1 + wz_2 + w \\ \dot{z}_2 &= -wz_1 - z_2. \end{aligned} \tag{10}$$

The Jacobian of the right-hand side of system (10) equals

$$J(z_1, z_2, w) = \begin{pmatrix} -1 & w \\ -w & -1 \end{pmatrix}.$$

Obviously, $J + J^T = -2I < 0$. Thus, the Demidovich condition (8) is satisfied for all $z_1$, $z_2$ and $w$ (with $P = I$ and $Q = 2I$). By Theorem 1, system (10) is input-to-state convergent.$\triangleleft$

The next example illustrates the differences between the input-to-state convergence and incremental ISS ($\delta$ISS) defined in [1].

*Example 2.* Consider the scalar system $\dot{z} = -z + w^3$. As follows from [1], this system is not $\delta$ISS. At the same time, by Theorem 1 this system is input-to-state convergent.

*Remark 2.* In some cases feasibility of the Demidovich condition (8) can be concluded from the feasibility of certain LMIs. Namely, suppose there exist matrices $\mathcal{A}_1, \ldots, \mathcal{A}_s$ such that

$$\frac{\partial F}{\partial z}(z, w) \in \mathrm{co}\{\mathcal{A}_1, \ldots, \mathcal{A}_s\}, \quad \forall \ z \in \mathbb{R}^d, \ w \in \mathbb{R}^m,$$

where $\mathrm{co}\{\mathcal{A}_1, \ldots, \mathcal{A}_s\}$ is the convex hull of matrices $\mathcal{A}_1, \ldots, \mathcal{A}_s$. If the LMIs

$$P\mathcal{A}_i + \mathcal{A}_i^T P < 0, \quad i = 1, \ldots, s \tag{11}$$

admit a common positive definite solution $P = P^T > 0$, then condition (8) is satisfied with this matrix $P$. Taking into account the existence of powerful LMI solvers, this is a useful tool for checking convergence properties.

In some cases, feasibility of the LMI (11) can be checked using frequency domain methods following from the Kalman-Yakubovich lemma (see [8; 20]). For example, one can use the circle criterion, as follows from the next lemma.

**Lemma 2 ([8; 20]).** *Consider a Hurwitz matrix $A \in \mathbb{R}^{d\times d}$, matrices $B \in \mathbb{R}^{d\times 1}$, $C \in \mathbb{R}^{1\times d}$ and some number $\gamma > 0$. Denote $\mathcal{A}_\gamma^- := A - \gamma BC$ and $\mathcal{A}_\gamma^+ := A + \gamma BC$. There exists $P = P^T > 0$ such that*

$$P\mathcal{A}_\gamma^- + (\mathcal{A}_\gamma^-)^T P < 0, \quad P\mathcal{A}_\gamma^+ + (\mathcal{A}_\gamma^+)^T P < 0 \tag{12}$$

*if and only if the inequality $\left|C(i\omega I - A)^{-1}B\right| < \frac{1}{\gamma}$ is satisfied for all $\omega \in \mathbb{R}$.*

This lemma allows to check input-to-state convergence for the so-called Lur'e systems, as shown in the following example.

*Example 3.* Consider the system

$$\begin{aligned} \dot{z} &= Az + B\varphi(y) + Ew \\ y &= Cz + Hw, \end{aligned} \tag{13}$$

with the Hurwitz matrix $A$, scalar output $y$ and scalar nonlinearity $\varphi(y) \in \mathbb{R}$. Suppose the nonlinearity $\varphi(y)$ is $C^1$ and it satisfies the condition $\left|\frac{\partial \varphi}{\partial y}(y)\right| \leq \gamma$ for all $y \in \mathbb{R}$. Then the Jacobian of the right-hand side of system (13), which is equal to $\frac{\partial F}{\partial z} = A + BC\frac{\partial \varphi}{\partial y}(y)$, satisfies $\frac{\partial F}{\partial z} \in \mathrm{co}\{A_\gamma^-, A_\gamma^+\}$ for all $y \in \mathbb{R}$. By Lemma 2, if the condition $\left|C(i\omega I - A)^{-1}B\right| < \frac{1}{\gamma}$ is satisfied for all $\omega \in \mathbb{R}$, then LMI (12) admits a common positive definite solution. Therefore, system (13) satisfies the Demidovich condition (8) for all $z \in \mathbb{R}^d$ and all $w \in \mathbb{R}^m$. By Theorem 1, such a system is exponentially convergent and input-to-state convergent.$\triangleleft$

By Property 6 a series connection of input-to-state convergent systems is again an input-to-state convergent system. Therefore we obtain the following corollary of Property 6 and Theorem 1: *a series connection of systems satisfying the Demidovich condition is an input-to-state convergent system.*

## 5 Controller design for convergent systems

The convergence property is desirable in many control problems because the steady-state dynamics of a convergent system are independent of the initial conditions. In this section we address the problem of how to achieve the convergence property in a control system by means of feedback. Consider control systems of the form

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ y &= h(x, w), \end{aligned} \tag{14}$$

with state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^k$, external input $w \in \mathbb{R}^m$ and output $y \in \mathbb{R}^l$. It is assumed that the functions $f(x, u, w)$ and $h(x, w)$ are $C^1$. In this setting the input $u$ corresponds to the feedback part of the controller. The external input $w$ includes external time-dependent inputs such as disturbances and feedforward control signals. Once the convergence property is achieved by a proper choice of feedback, the feedforward control signals can be used in order to shape the steady-state dynamics of the closed-loop system (see e.g. [14; 16]). We will focus on the problem of finding a feedback that makes the closed-loop system convergent and will not address the problem of shaping the steady-state dynamics by means of a feedforward controller.

Denote $\zeta := (x, u, w)$,

$$\mathcal{A}(\zeta) := \frac{\partial f}{\partial x}(x, u, w), \quad \mathcal{B}(\zeta) := \frac{\partial f}{\partial u}(x, u, w), \quad \mathcal{C}(\zeta) := \frac{\partial h}{\partial x}(x, w).$$

In the sequel we make the following assumption:
**A1** there exist matrices $\mathcal{A}_1, \ldots, \mathcal{A}_s$, $\mathcal{B}_1, \ldots, \mathcal{B}_s$ and $\mathcal{C}_1, \ldots, \mathcal{C}_s$ such that

$$[\mathcal{A}(\zeta)\, \mathcal{B}(\zeta)] \in \mathrm{co}\{[\mathcal{A}_1\, \mathcal{B}_1], \ldots, [\mathcal{A}_s\, \mathcal{B}_s]\}, \;\; [\mathcal{A}(\zeta)\, \mathcal{C}(\zeta)] \in \mathrm{co}\{[\mathcal{A}_1\, \mathcal{C}_1], \ldots, [\mathcal{A}_s\, \mathcal{C}_s]\}$$

for all $\zeta \in \mathbb{R}^{n+k+m}$.
The following lemma provides conditions under which there exists a state feedback rendering the corresponding closed-loop system input-to-state convergent.

**Lemma 3.** *Consider the system (14). Suppose the LMI*

$$\mathcal{P}_c = \mathcal{P}_c^T > 0, \quad \mathcal{A}_i\mathcal{P}_c + \mathcal{P}_c\mathcal{A}_i^T + \mathcal{B}_i\mathcal{Y} + \mathcal{Y}^T\mathcal{B}_i^T < 0, \quad i = 1, \ldots, s \tag{15}$$

*is feasible. Then the system*

$$\dot{x} = f(x, K(x+v), w), \tag{16}$$

*with* $K := \mathcal{Y}\mathcal{P}_c^{-1}$ *and* $(v, w)$ *as inputs is input-to-state convergent.*

*Proof.* Denote $F(x, v, w) := f(x, K(x+v), w)$. The Jacobian of the right-hand side of system (16) equals

$$\frac{\partial F}{\partial x}(x, v, w) := \frac{\partial f}{\partial x}(x, K(x+v), w) + \frac{\partial f}{\partial u}(x, K(x+v), w)K.$$

Due to assumption **A1**, $\frac{\partial F}{\partial x}(x, v, w) \in \mathrm{co}\{(\mathcal{A}_i + \mathcal{B}_iK), \; i = 1, \ldots, s\}$ for all $(x, v, w) \in \mathbb{R}^{n+n+m}$. Since the LMI (15) is feasible, for the matrix $K := \mathcal{Y}\mathcal{P}_c^{-1}$ it holds that

$$\mathcal{P}_c^{-1}(\mathcal{A}_i + \mathcal{B}_iK) + (\mathcal{A}_i + \mathcal{B}_iK)^T\mathcal{P}_c^{-1} < 0, \quad i = 1, \ldots, s.$$

Therefore, by Remark 2 the closed-loop system (16) satisfies the Demidovich condition with the matrix $P := \mathcal{P}_c^{-1} > 0$. By Theorem 1 system (16) with $(v, w)$ as inputs is input-to-state convergent.□

The next lemma shows how to design an observer based on the Demidovich condition.

**Lemma 4.** *Consider system (14). Suppose the LMI*

$$\mathcal{P}_o = \mathcal{P}_o^T > 0, \quad \mathcal{P}_o\mathcal{A}_i + \mathcal{A}_i^T\mathcal{P}_o + \mathcal{X}\mathcal{C}_i + \mathcal{C}_i^T\mathcal{X}^T < 0, \quad i = 1, \ldots, s \tag{17}$$

*is feasible. Then the system*

$$\dot{\hat{x}} = f(\hat{x}, u, w) + L(h(\hat{x}, w) - y), \quad \text{with} \;\; L := \mathcal{P}_o^{-1}\mathcal{X} \tag{18}$$

*is an observer for system (14) with a globally exponentially stable error dynamics. Moreover, the error dynamics*

$$\Delta\dot{x} = G(x + \Delta x, u, w) - G(x, u, w), \tag{19}$$

*where $G(x,u,w) := f(x,u,w) + Lh(x,w)$ is such that for any bounded $x(t)$ and $w(t)$ and any feedback $u = U(\Delta x, t)$ all solutions of system (19) satisfy*

$$|\Delta x(t)| \leq Ce^{-a(t-t_0)}|\Delta x(t_0)|, \tag{20}$$

*where the numbers $C > 0$ and $a > 0$ are independent of $x(t)$, $w(t)$ and $u = U(\Delta x, t)$.*

*Proof.* Let us first prove the second part of the lemma. The Jacobian of the right-hand side of system (19) equals

$$\frac{\partial G}{\partial \Delta x}(x + \Delta x, u, w) = \frac{\partial f}{\partial x}(x + \Delta x, u, w) + L\frac{\partial h}{\partial x}(x + \Delta x, w).$$

Due to Assumption **A1** it holds that

$$\frac{\partial G}{\partial \Delta x}(x + \Delta x, u, w) \in \mathrm{co}\{(\mathcal{A}_i + L\mathcal{C}_i),\ i = 1, \ldots, s\}$$

for all $x, u, w$ and $\Delta x$. Since the LMI (17) is feasible, for the matrix $L := \mathcal{P}_o^{-1}\mathcal{X}$ it holds that

$$\mathcal{P}_o(\mathcal{A}_i + L\mathcal{C}_i) + (\mathcal{A}_i + L\mathcal{C}_i)^T\mathcal{P}_o < 0, \quad i = 1, \ldots, s.$$

Therefore, by Remark 2 system (19) with inputs $x$, $u$ and $w$ satisfies the Demidovich condition with the matrix $P := \mathcal{P}_o > 0$ and some matrix $Q > 0$. Consider the function $V(\Delta x) := 1/2\Delta x^T P \Delta x$. By Lemma 1 the derivative of this function along solutions of system (19) satisfies

$$\frac{dV}{dt} = \Delta x^T P(G(x + \Delta x, u, w) - G(x, u, w)) \leq -2aV(\Delta x). \tag{21}$$

In inequality (21) the number $a > 0$ depends only on the matrices $P$ and $Q$ and does not depend on the particular values of $x$, $u$ and $w$. This inequality, in turn, implies that there exists $C > 0$ depending only on the matrix $P$ such that if the inputs $x(t)$ and $w(t)$ are defined for all $t \geq t_0$ then the solution $\Delta x(t)$ is also defined for all $t \geq t_0$ and satisfies (20). It remains to show that system (18) is an observer for system (14). Denote $\Delta x := \hat{x} - x(t)$. Since $x(t)$ is a solution of system (14), $\Delta x(t)$ satisfies equation (19). By the previous analysis, we obtain that $\Delta x(t)$ satisfies (20). Therefore, the observation error $\Delta x$ exponentially tends to zero.□

Lemmas 3 and 4 show how to design a state feedback controller that makes the closed-loop system input-to-state convergent and an observer for this system with an exponentially stable error dynamics. In fact, for such controllers and observers one can use the separation principle in order to design an output feedback controller that makes the closed-loop system input-to-state convergent. This statement follows from the next theorem.

**Theorem 2.** *Consider system (14). Suppose LMIs (15) and (17) are feasible. Denote* $K := \mathcal{Y}\mathcal{P}_c^{-1}$ *and* $L := \mathcal{P}_o^{-1}\mathcal{X}$*. Then system (14) in closed loop with the controller*

$$\dot{\hat{x}} = f(\hat{x}, u, w) + L(h(\hat{x}, w) - y), \tag{22}$$

$$u = K\hat{x} \tag{23}$$

*with* $w$ *as an input is input-to-state convergent.*

*Proof.* Denote $\Delta x := \hat{x} - x$. Then in the new coordinates $(x, \Delta x)$ the equations of the closed-loop system are

$$\dot{x} = f(x, K(x + \Delta x), w), \tag{24}$$

$$\Delta\dot{x} = G(x + \Delta x, u, w) - G(x, u, w), \tag{25}$$

$$u = K(x + \Delta x), \tag{26}$$

where $G(x, u, w) = f(x, u, w) + Lh(x, w)$. By the choice of $K$, system (24) with $(\Delta x, w)$ as inputs is input-to-state convergent (see Lemma 3). By the choice of the observer gain $L$, for any inputs $x(t)$, $w(t)$ and for the feedback $u = K(x(t) + \Delta x)$, any solution of system (25), (26) satisfies

$$|\Delta x(t)| \leq Ce^{-a(t-t_0)}|\Delta x(t_0)|, \tag{27}$$

where the numbers $C > 0$ and $a > 0$ are independent of $x(t)$ and $w(t)$ (see Lemma 4). Applying Property 7 (see Section 3), we obtain that the closed-loop system (24)-(26) is input-to-state convergent.□

Although the proposed controller and observer structures do not significantly differ from the ones proposed in literature, they achieve the new goal of rendering the closed-loop system convergent (as opposed to asymptotically stable). The output-feedback controller design presented in Theorem 2 relies on the separation principle which, in turn, is based on the input-to-state convergence of the system/state-feedback controller combination. This input-to-state convergence property serves as a counterpart of the input-to-state stability property often used to achieve separation of controller and observer designs in rendering the closed-loop system asymptotically stable (as opposed to convergent).

## 6 Conclusions

In this paper we have extended the notion of convergent systems defined by B.P. Demidovich and introduced the notions of (uniformly, exponentially) convergent systems as well as input-to-state convergent systems. These notions are coordinate independent, which makes them more convenient to use than the notions of incremental stability and incremental input-to-state stability.

We have presented basic properties of convergent systems and studied parallel, series and feedback interconnections of input-to-state convergent systems. These properties resemble the properties of asymptotically stable LTI systems. Due to this fact (input-to-state) convergent systems are convenient to deal with in many control and system analysis problems. We have presented a simple sufficient condition for the input-to-state convergence property. In certain cases this condition can be reduced to checking the feasibility of certain LMIs. Finally, for a class of nonlinear systems we have presented an (output) feedback controller that make the closed-loop system input-to-state convergent. The presented controller consists of a state-feedback controller that makes the closed-loop system input-to-state convergent and an observer with an exponentially stable error dynamics. For such controllers and observers the separation principle holds. This allows us to unite the obtained controller and observer. The conditions for such controller and observer designs are formulated in terms of LMIs.

The results presented in this paper are mostly for systems with smooth right-hand sides. Convergent systems with non-smooth and discontinuous right-hand sides are considered in [12; 13; 16]. Extensions of convergent systems to non-global settings, further convergence properties and controller design techniques as well as applications to the output regulation problem, controlled synchronization problem and nonlinear observer design problem can be found in [16].

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## Appendix

*Proof of Property 4.* Consider two inputs $w_1$ and $w_2 \in \overline{\mathbb{PC}}_m$ satisfying $w_1(t) - w_2(t) \to 0$ as $t \to +\infty$ and the corresponding limit solutions $\bar{z}_{w_1}(t)$ and $\bar{z}_{w_2}(t)$. By the definition of convergence, both $\bar{z}_{w_1}(t)$ and $\bar{z}_{w_2}(t)$ are defined and bounded for all $t \in \mathbb{R}$. Consider the system

$$\Delta\dot{z} = F(\bar{z}_{w_2}(t) + \Delta z, w_2(t) + \Delta w) - F(\bar{z}_{w_2}(t), w_2(t)). \tag{28}$$

This system describes the dynamics of $\Delta z = z(t) - \bar{z}_{w_2}(t)$, where $z(t)$ is some solution of system (3) with the input $w_2(t) + \Delta w(t)$. Since $F(z, w) \in C^1$, and $\bar{z}_{w_2}(t)$ and $w_2(t)$ are bounded, the partial derivatives

$$\frac{\partial F}{\partial z}(\bar{z}_{w_2}(t) + \Delta z, w_2(t) + \Delta w), \quad \frac{\partial F}{\partial w}(\bar{z}_{w_2}(t) + \Delta z, w_2(t) + \Delta w)$$

are bounded in some neighborhood of the origin $(\Delta z, \Delta w) = (0, 0)$, uniformly in $t \in \mathbb{R}$. Also, for $\Delta w \equiv 0$ system (28) has a uniformly globally asymptotically stable equilibrium $\Delta z = 0$. This implies (Lemma 5.4, [8]) that system (28) is locally ISS with respect to the input $\Delta w$. Therefore, there exist numbers $k_z > 0$ and $k_w > 0$ such that for any input $\Delta w(t)$ satisfying $|\Delta w(t)| \leq k_w$ for all $t \geq t_0$ and $\Delta w(t) \to 0$ as $t \to +\infty$, it holds that any solution $\Delta z(t)$ starting in $|\Delta z(t_0)| \leq k_z$ tends to zero, i.e. $\Delta z(t) \to 0$ as $t \to +\infty$.

Choose $t_0 \in \mathbb{R}$ such that $|w_1(t) - w_2(t)| \leq k_w$ for all $t \geq t_0$. Consider a solution of the system

$$\dot{z} = F(z, w_1(t)) \tag{29}$$

starting in a point $z(t_0)$ satisfying $|z(t_0) - \bar{z}_{w_2}(t_0)| \leq k_z$. By the reasoning presented above, $\Delta z(t) := z(t) - \bar{z}_{w_2}(t) \to 0$ as $t \to +\infty$. At the same time, $\bar{z}_{w_1}(t)$ is a uniformly globally asymptotically stable solution of system(29). Hence, $z(t) - \bar{z}_{w_1}(t) \to 0$ as $t \to +\infty$. Therefore, $\bar{z}_{w_2}(t) - \bar{z}_{w_1}(t) \to 0$ as $t \to +\infty$.□

*Proof of Property 6.* Consider some input $w \in \overline{\mathbb{PC}}_m$. Since the $y$-subsystem is input-to-state convergent, there exists a solution $\bar{y}_w(t)$ which is defined and bounded for all $t \in \mathbb{R}$. Since the $z$-subsystem with $(y, w)$ as inputs is input-to-state convergent, there exists a limit solution $\bar{z}_w(t)$ corresponding to the input $(\bar{y}_w(t), w(t))$. This $\bar{z}_w(t)$ is defined and bounded for all $t \in \mathbb{R}$.

Let $(z(t), y(t))$ be some solution of system (6) with some input $\tilde{w}(t)$. Denote $\Delta z := z - \bar{z}_w(t)$, $\Delta y := y - \bar{y}_w(t)$ and $\Delta w = \tilde{w} - w(t)$. Then $\Delta z$ and $\Delta y$ satisfy the equations

$$\Delta\dot{z} = F(\bar{z}_w(t) + \Delta z, \bar{y}_w(t) + \Delta y, w(t) + \Delta w) - F(\bar{z}_w(t), \bar{y}_w(t), w(t)) \tag{30}$$

$$\Delta\dot{y} = G(\bar{y}_w(t) + \Delta y, w(t) + \Delta w) - G(\bar{y}_w(t), w(t)). \tag{31}$$

Since both the $z$-subsystem and the $y$-subsystem of system (6) are input-to-state convergent, system (30) with $(\Delta y, \Delta w)$ as input is ISS and system (31) with $\Delta w$ as input is ISS. Therefore, the cascade interconnection of ISS systems (30), (31) is an ISS system (see e.g. [19]). In the original coordinates $(z, y)$ this means that system (6) is ISS with respect to the solution $(\bar{z}_w(t), \bar{y}_w(t))$. This implies that system (6) is input-to-state convergent.□

*Proof of Theorem 1.* The proof of exponential convergence can be found in [2; 11]. We only need to show that system (3) is input-to-state convergent. Consider some input $w(t)$ and the corresponding limit solution $\bar{z}_w(t)$. Let $z(t)$ be a solution of system (3) corresponding to some input $\hat{w}(t)$. Denote $\Delta z := z - \bar{z}_w(t)$ and $\Delta w := \hat{w} - w(t)$. Then $\Delta z$ satisfies the equation

$$\Delta\dot{z} = F(\bar{z}_w(t) + \Delta z, w(t) + \Delta w) - F(\bar{z}_w(t), w(t)). \tag{32}$$

We will show that system (32) with $\Delta w$ as input is ISS. Due to the arbitrary choice of $w(t)$, this fact implies that system (3) is input-to-state convergent.

Consider the function $V(\Delta z) = \frac{1}{2}(\Delta z)^T P \Delta z$. Its derivative along solutions of system (32) satisfies

$$\begin{aligned} \frac{dV}{dt} &= \Delta z^T P\{F(\bar{z}_w(t)+\Delta z, w(t)+\Delta w(t)) - F(\bar{z}_w(t), w(t))\} \quad (33)\\ &= \Delta z^T P\{F(\bar{z}_w(t)+\Delta z, w(t)+\Delta w(t)) - F(\bar{z}_w(t), w(t)+\Delta w(t))\} \\ &\quad +\Delta z^T P\{F(\bar{z}_w(t), w(t)+\Delta w(t)) - F(\bar{z}_w(t), w(t))\}. \end{aligned}$$

Applying Lemma 1 to the first component in (33), we obtain

$$\Delta z^T P\{F(\bar{z}_w(t)+\Delta z, w(t)+\Delta w(t)) - F(\bar{z}_w(t), w(t)+\Delta w(t))\} \le -a|\Delta z|_P^2, \quad (34)$$

where $|\Delta z|_P^2 := (\Delta z)^T P \Delta z$. Applying the Cauchy inequality to the second component in formula (33), we obtain the following estimate:

$$|\Delta z^T P\{F(\bar{z}_w(t), w(t)+\Delta w(t)) - F(\bar{z}_w(t), w(t))\}| \le |\Delta z|_P |\delta(t, \Delta w)|_P, \quad (35)$$

where

$$\delta(t, \Delta w) := F(\bar{z}_w(t), w(t)+\Delta w(t)) - F(\bar{z}_w(t), w(t)).$$

Since $F(z,w)$ is continuous and $\bar{z}_w(t)$ and $w(t)$ are bounded for all $t \in \mathbb{R}$, the function $\delta(t, \Delta w)$ is continuous in $\Delta w$ uniformly in $t \in \mathbb{R}$. This, in turn, implies that $\tilde{\rho}(r) := \sup_{t\in\mathbb{R}} \sup_{|\Delta w|\le r} |\delta(t,\Delta w)|_P$ is a continuous nondecreasing function. Define the function $\rho(r) := \tilde{\rho}(r) + r$. This function is continuous, strictly increasing and $\rho(0) = 0$. Thus, it is a class $\mathcal{K}$ function. Also, due to the definition of $\rho(r)$, we obtain the following estimate

$$|\delta(t, \Delta w)|_P \le \rho(|\Delta w|).$$

After substituting this estimate, together with estimates (35) and (34), in formula (33), we obtain

$$\frac{dV}{dt} \le -a|\Delta z|_P^2 + |\Delta z|_P \rho(|\Delta w|). \quad (36)$$

From this formula we obtain

$$\frac{dV}{dt} \le -\frac{a}{2}|\Delta z|_P^2, \quad \forall\ |\Delta z|_P \ge \frac{2}{a}\rho(|\Delta w|). \quad (37)$$

By the Lyapunov characterization of the ISS property (see e.g. [8], Theorem 5.2), we obtain that system (32) is input-to-state stable. This completes the proof of the theorem.□

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